Morse homology for the heat flow – Linear theory

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Consider the linear parabolic partial differential equation $\mathcal{D}_u\xi = 0$ which arises by linearizing the heat flow on the loop space of a Riemannian manifold M. The solutions are vector fields along infinite cylinders u in M. For these solutions we establish regularity and apriori estimates. We show that for nondegenerate asymptotic boundary conditions the solutions decay exponentially in L^2 in forward and backward time. In this case $\mathcal{D}_u$ viewed as linear operator from parabolic Sobolev space $\mathcal{W}^{1,p}$ to L^p is Fredholm whenever $p > 1$. We close with an L^p estimate for products of first order terms which is a crucial ingredient in the sequel [13] to prove regularity and the implicit function theorem. The results of the present text are the base to construct in [13] an algebraic chain complex whose homology represents the homology of the loop space.

1 Introduction

Let M be a Riemannian manifold. Denote by ∇ the Levi-Civita connection and by R the Riemannian curvature tensor. The **loop space** $\mathcal{L}M$ by definition is the space $C^\infty(S^1, M)$ of free loops in M. Here and throughout we identify $S^1 = \mathbb{R}/\mathbb{Z}$ and think of $x \in \mathcal{L}M$ as a smooth map $x : \mathbb{R} \to M$ which satisfies $x(t+1) = x(t)$. **Smooth** means C^∞ smooth. Fix a smooth map $u : \mathbb{R} \times S^1 \to M$ and a smooth function $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies axioms (V0)–(V3) in section 2. It is safe to think of $\mathcal{V}$ as being of the form $\mathcal{V}(x) := \int_0^1 V_t\,(x(t))\,dt$ for $V \in C^\infty(S^1 \times M, \mathbb{R})$ and $V_t(q) := V(t, q)$. In this case $\mathrm{grad}\mathcal{V}(x) = \nabla V_t(x)$ and $\mathcal{H}_\mathcal{V}(x)\xi = \nabla_\xi \nabla V_t(x)$ for $x \in \mathcal{L}M$.

In this paper we study the linear parabolic PDE

$$\mathcal{D}_u\xi := \nabla_s\xi - \nabla_t\nabla_t\xi - R(\xi, \partial_t u)\partial_t u - \mathcal{H}_\mathcal{V}(u)\xi = 0 \tag{1}$$

for vector fields ξ along u were the covariant Hessian $\mathcal{H}_\mathcal{V}$ of $\mathcal{V}$ is given by (9).

Equation (1) arises as follows. Consider the action

$$\mathcal{S}_\mathcal{V}(x) = \frac{1}{2}\int_0^1 |\dot{x}(t)|^2\, dt - \mathcal{V}(x) \tag{2}$$

for smooth loops $x : S^1 \to M$. Its critical points are those x that solve the ODE

$$\nabla_t \dot{x} = -\mathrm{grad}\mathcal{V}(x). \tag{3}$$

For $\mathcal{V} = 0$ these are the closed geodesics. The negative L^2 gradient equation for $\mathcal{S}_\mathcal{V}$ on $\mathcal{L}M$ takes the form of the **heat equation**

$$\partial_s u - \nabla_t \partial_t u - \mathrm{grad}\mathcal{V}(u) = 0 \tag{4}$$

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for smooth cylinders $u:\mathbb{R}\times S^1\to M$. Here $\mathrm{grad}\mathcal{V}(u)$ denotes the value of $\mathrm{grad}\mathcal{V}$ as defined by (8) on the loop $u_s:t\mapsto u(s,t)$. If one linearizes (4) at a solution u, then the definition of $\mathcal{D}_u$ in (1) becomes an identity; see [9, app A.2].

More precisely, the covariant Hessian of $\mathcal{S}_\mathcal{V}$ at a smooth loop $x:S^1\to M$ is the linear operator $A_x: W^{2,2}(S^1,x^*TM)\to L^2(S^1,x^*TM)$ given by

$$A_x\xi=-\nabla_t\nabla_t\xi-R(\xi,\dot x)\dot x-\mathcal{H}_\mathcal{V}(x)\xi. \tag{5}$$

This operator is self-adjoint with respect to the standard L^2 inner product and the number of negative eigenvalues is finite; see e.g. [10]. The latter is denoted by $\mathrm{ind}_\mathcal{V}(A_x)$ and called the **Morse index** of A_x. If x is a critical point of $\mathcal{S}_\mathcal{V}$ we define its Morse index by

$$\mathrm{ind}_\mathcal{V}(x):=\mathrm{ind}_\mathcal{V}(A_x)$$

and we call x **nondegenerate** if A_x is bijective. In the following $\mathcal{W}^{1,p}_u$ denotes *parabolic* Sobolev space; see (7) and appendix A. In this notation the linear operator $\mathcal{D}_u:\mathcal{W}^{1,p}_u\to\mathcal{L}^p_u$ defined by (1) is of the form

$$\mathcal{D}_u\xi=\nabla_s\xi+A_{u_s}\xi. \tag{6}$$

Here the spaces $\mathcal{L}^p_u$ and $\mathcal{W}^{1,p}_u$ are defined as the completions of the space of smooth compactly supported sections of the pullback tangent bundle $u^*TM\to\mathbb{R}\times S^1$ with respect to the norms

$$\begin{aligned}\|\xi\|_p&=\left(\int_{-\infty}^{\infty}\int_0^1|\xi|^p\,dtds\right)^{1/p},\\ \|\xi\|_{\mathcal{W}^{1,p}}&=\left(\int_{-\infty}^{\infty}\int_0^1|\xi|^p+|\nabla_s\xi|^p+|\nabla_t\nabla_t\xi|^p\,dtds\right)^{1/p}.\end{aligned} \tag{7}$$

Overview

In appendix A we briefly introduce relevant parabolic Sobolev spaces $\mathcal{W}^{k,p}$ and recall the well-known local regularity theorem A.1 which is a key tool in this text.

In section 2 we state the axioms for the abstract perturbations $\mathcal{V}$ used thoughout. They have been introduced in [8], because this class is rich enough such that transversality works; cf. [13].

Section 3 is the main part of this work. Here we investigate the solutions to the linear heat equation (1). Theorem 3.1 asserts regularity for weak solutions. In subsection 3.2 on apriori estimates we derive pointwise bounds of ξ, $\nabla_t\xi$, $\nabla_t\nabla_t\xi$, and $\nabla_s\xi$ in terms of the L^2 norm (theorem 3.3 and theorem 3.4). Section 3.3 then establishes exponential decay of these L^2 norms. The combination of these results is used in section 3.4 to prove theorem 3.13 which asserts that the operator $\mathcal{D}_u$ is Fredholm for a rather general class of smooth cylinders u in M. However, nondegeneracy of the asymptotic limits of u is crucial.

In section 4 we prove an L^p estimate which is crucial in the sequel [13] to deal with products of spatial first order terms. More precisely, lemma 4.1 is the key ingredient to prove [8, thm. A.3] on regularity of strong solutions to the heat equation (4) and to prove the quadratic estimate in [13] which enters the proof of the implicit function theorem [8, thm. A.5]. A further consequence, useful in [13], is boundedness of the action along finite energy solutions of the heat equation (4); see remark 4.2.

Outlook

Based on the results of the present text we construct in [13] an algebraic chain complex whose chain groups are generated by perturbed closed geodesics and whose boundary operator is defined by counting, modulo time shift, heat flow trajectories between geodesics of Morse index difference one. To see the connection to [13] observe that if u solves the (nonlinear) heat equation (4) then $\xi:=\partial_s u$ solves the linear equation (1), that is $\partial_s u\in\ker\mathcal{D}_u$. In a forthcoming paper we prove that the resulting homology theory is isomorphic to singular homology of the loop space. Due to the lack of a flow this is nonstandard and involves Conley theory on the loop space.

Notation 1.1 If $f=f(s,t)$ is a map, then f_s abbreviates the map $f(s,\cdot):t\mapsto f(s,t)$. In contrast partial derivatives are denoted by $\partial_s f$ and $\partial_t f$.

2 Perturbations

We introduce a class of abstract perturbations for which transversality in [13] works. The **abstract perturbations** take the form of smooth maps $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$. For $x \in \mathcal{L}M$ define the L^2-gradient $\mathrm{grad}\mathcal{V}(x) \in \Omega^0(S^1, x^*TM)$ of $\mathcal{V}$ by

$$\int_0^1 \langle \mathrm{grad}\mathcal{V}(u), \partial_s u \rangle \, dt = \frac{d}{ds}\mathcal{V}(u) \tag{8}$$

for every smooth path $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot)$. The **covariant Hessian of $\mathcal{V}$** at a loop $x : S^1 \to M$ is the operator $\mathcal{H}_{\mathcal{V}}(x)$ on $\Omega^0(S^1, x^*TM)$ defined by

$$\mathcal{H}_{\mathcal{V}}(u)\partial_s u := \nabla_s \mathrm{grad}\mathcal{V}(u) \tag{9}$$

for every smooth map $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot)$. The axiom (V1) below asserts that this Hessian is a zeroth order operator. We impose the following conditions on $\mathcal{V}$; here $|\cdot|$ denotes the pointwise absolute value at $(s,t) \in \mathbb{R} \times S^1$ and $\|\cdot\|_{L^p}$ denotes the L^p-norm over S^1 at time s. Although condition (V1) and the first part of (V2) are special cases of (V3) we state the axioms in the form below, because some of our results don't require all the conditions to hold.

(V0): $\mathcal{V}$ is continuous with respect to the C^0 topology on $\mathcal{L}M$. Moreover, there is a constant $C = C(\mathcal{V})$ such that

$$\sup_{x \in \mathcal{L}M} |\mathcal{V}(x)| + \sup_{x \in \mathcal{L}M} \|\mathrm{grad}\mathcal{V}(x)\|_{L^\infty(S^1)} \le C.$$

(V1): There is a constant $C = C(\mathcal{V})$ such that

$$\begin{aligned} |\nabla_s \mathrm{grad}\mathcal{V}(u)| &\le C\big(|\partial_s u| + \|\partial_s u\|_{L^1}\big), \\ |\nabla_t \mathrm{grad}\mathcal{V}(u)| &\le C\Big(1 + |\partial_t u|\Big) \end{aligned}$$

for every smooth map $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot)$ and every $(s,t) \in \mathbb{R} \times S^1$.

(V2): There is a constant $C = C(\mathcal{V})$ such that

$$\begin{aligned} |\nabla_s \nabla_s \mathrm{grad}\mathcal{V}(u)| &\le C\Big(|\nabla_s \partial_s u| + \|\nabla_s \partial_s u\|_{L^1} + \big(|\partial_s u| + \|\partial_s u\|_{L^2}\big)^2\Big), \\ |\nabla_t \nabla_s \mathrm{grad}\mathcal{V}(u)| &\le C\Big(|\nabla_t \partial_s u| + \big(1 + |\partial_t u|\big)\big(|\partial_s u| + \|\partial_s u\|_{L^1}\big)\Big), \end{aligned}$$

and

$$|\nabla_s \nabla_s \mathrm{grad}\mathcal{V}(u) - \mathcal{H}_{\mathcal{V}}(u)\nabla_s \partial_s u| \le C\big(|\partial_s u| + \|\partial_s u\|_{L^2}\big)^2$$

for every smooth map $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot)$ and every $(s,t) \in \mathbb{R} \times S^1$.

(V3): For any two integers $k > 0$ and $\ell \ge 0$ there is a constant $C = C(k, \ell, \mathcal{V})$ such that

$$\big|\nabla_t^\ell \nabla_s^k \mathrm{grad}\mathcal{V}(u)\big| \le C \sum_{k_j, \ell_j} \left(\prod_{\substack{j \\ \ell_j > 0}} \big|\nabla_t^{\ell_j} \nabla_s^{k_j} u\big| \right) \prod_{\substack{j \\ \ell_j = 0}} \left(\big|\nabla_s^{k_j} u\big| + \big\|\nabla_s^{k_j} u\big\|_{L^{p_j}} \right)$$

for every smooth map $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot)$ and every $(s,t) \in \mathbb{R} \times S^1$; here $p_j \ge 1$ and $\sum_{\ell_j = 0} 1/p_j = 1$; the sum runs over all partitions $k_1 + \cdots + k_m = k$ and $\ell_1 + \cdots + \ell_m \le \ell$ such that $k_j + \ell_j \ge 1$ for all j. For $k = 0$ the same inequality holds with an additional summand C on the right.

3 The linearized heat equation

Fix a smooth function $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3) and a smooth map $u : \mathbb{R} \times S^1 \to M$. This section deals with the linear parabolic PDE (1).

Fix $p > 1$. In section 3.1 we show that **strong solutions**, that is, vector fields ξ along u being locally of class $\mathcal{W}^{1,p}$ and satisfying (1) almost everywhere, are automatically smooth. More generally, for ξ of class L^p_{loc} theorem 3.1 (implicitly) defines the notion of **weak solution** and asserts that even weak solutions are smooth. In section 3.2 we derive pointwise estimates of ξ and certain partial derivatives of ξ in terms of the L^2 norm of ξ over small backward cylinders. In section 3.3 we establish asymptotic exponential decay of the slicewise L^2 norm $\|\xi_s\|_{L^2(S^1)}$ of a solution ξ whenever the covariant Hessian A_{u_s} given by (5) is asymptotically injective. Still assuming asymptotic injectivity we prove in section 3.4 that the operator $\mathcal{D}_u : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u$ defined by (1) is Fredholm.

3.1 Regularity

Theorem 3.1 (Local regularity of weak solutions) *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V3) *and constants $q > 1$ and $a < b$. Let $u : (a,b] \times S^1 \to M$ be a smooth map with bounded derivatives of all orders. Then the following is true. If η is a vector field along u of class L^q_{loc} such that*

$$\langle \eta, \mathcal{D}_u^* \xi \rangle = 0$$

for every smooth vector field ξ along u of compact support in $(a,b) \times S^1$, then η is smooth. Here the operator $\mathcal{D}_u^*$ is defined by the left hand side of (1) with ∇_s replaced by $-\nabla_s$ and $\langle \cdot, \cdot \rangle$ denotes integration over the pointwise inner products.

Remark 3.2 Theorem 3.1 remains true if we replace $\mathcal{D}_u^*$ by $\mathcal{D}_u$ and define u on $[a,b) \times S^1$. This follows by the variable substitution $s \mapsto -s$.

Proof. It suffices to prove the conclusion in a neighborhood of any point $z \in (a,b] \times S^1$. Shifting the s and t variables, if necessary, we may assume that $z \in \Omega_r = (-r^2, 0] \times (-r, r)$ for some sufficiently small $r > 0$. Now choose local coordinates on the manifold M around the point $u(z)$ and fix $r > 0$ sufficiently small such that $u(\overline{\Omega_r})$ is contained in the local coordinate patch. In these local coordinates the vector field η is represented by the map $(\eta^1, \ldots, \eta^n) : \Omega_r \to \mathbb{R}^n$ of class L^q_{loc} and the Riemannian metric g by the matrix with components g_{ij}. Throughout we use Einstein's sum convention. By induction we will prove that

$$v_\mu := g_{\mu j} \eta^j \in \bigcap_{m=1}^{\infty} \mathcal{W}^{m,q}_{loc}(\Omega_r), \qquad \mu = 1, \ldots, n.$$

But the intersection of spaces equals $C^\infty(\Omega_r)$; see e.g. [4, app. B.1]. Applying the inverse metric matrix yields $\eta^j = g^{j\mu} v_\mu \in C^\infty(\Omega_r)$ proving the theorem.

Step $m = 1$. Fix $\mu \in \{1, \ldots, n\}$ and consider vector fields of the form

$$\xi^{(\mu,\phi)} = (0, \ldots, 0, \phi, 0, \ldots, 0) : \Omega_r \to \mathbb{R}^n$$

where a function $\phi \in C_0^\infty(\operatorname{int} \Omega_r)$ occupies slot μ. Via extension by zero we view $\xi^{(\mu,\phi)}$ as a compactly supported smooth vector field along u. Now our assumption implies that $\langle \eta, \mathcal{D}_u^* \xi^{(\mu,\phi)} \rangle = 0$ for every $\phi \in C_0^\infty(\operatorname{int} \Omega_r)$. By straightforward calculation this is equivalent to

$$\int_{\Omega_r} v_\mu \left(-\partial_s \phi - \partial_t \partial_t \phi \right) = \int_{\Omega_r} f_\mu \phi - \int_{\Omega_r} h_\mu \, \partial_t \phi$$

for every $\phi \in C_0^\infty(\operatorname{int} \Omega_r)$, where $h_\mu = -2 v_k \Gamma^k_{i\mu} \, \partial_t u^i$ and

$$f_\mu = v_k \Big(\Gamma^k_{i\mu} \, \partial_s u^i + \frac{\partial \Gamma^k_{i\mu}}{\partial u^r} \, \partial_t u^r \, \partial_t u^i + \Gamma^k_{i\mu} \, \partial_t \partial_t u^i + \Gamma^k_{ij} \, \partial_t u^i \Gamma^j_{r\mu} \, \partial_t u^r + R^k_{\mu ij} \, \partial_t u^i \, \partial_t u^j + H^k_\mu \Big).$$

Here $R^k_{\ell ij}$ represents the Riemann curvature operator and H^k_ℓ the Hessian $\mathcal{H}_{\mathcal{V}}(u)$ in local coordinates. The Christoffel symbols associated to the Levi Civita connection ∇ are denoted by Γ^k_{ij}.

From now on the domain of all spaces will be Ω_r, unless specified differently. Observe that $v_\mu \in L^q_{loc} \subset L^1_{loc}$ by smoothness of the metric, compactness of M, and the fact that $\eta^\ell \in L^q_{loc}$ by assumption. It follows that h_μ and f_μ are in L^q_{loc}. Here we used in addition boundedness of the derivatives of u and axiom (V1). Hence $\partial_t v_\mu \in L^q_{loc}$ by theorem A.1 (b) and this implies that $\partial_t h_\mu \in L^q_{loc}$. Now integration by parts shows that

$$\int_{\Omega_r} v_\mu \left(-\partial_s\phi - \partial_t\partial_t\phi\right) = \int_{\Omega_r} \left(f_\mu + \partial_t h_\mu\right)\phi$$

for every $\phi \in C^\infty_0(\operatorname{int}\Omega_r)$ and therefore $v_\mu \in \mathcal{W}^{1,q}_{loc}$ by theorem A.1 (a).

Induction step $m \Rightarrow m+1$. Assume that $v_\mu \in \mathcal{W}^{m,q}_{loc}$. Then $f_\mu, h_\mu \in \mathcal{W}^{m,q}_{loc}$ by compactness of M, boundedness of the derivatives of u, and axiom (V3). Hence $\partial_t v_\mu \in \mathcal{W}^{m,q}_{loc}$ by theorem A.1 (b). But this implies that $\partial_t h_\mu$ is in $\mathcal{W}^{m,q}_{loc}$ and so is $f_\mu + \partial_t h_\mu$. Therefore $v_\mu \in \mathcal{W}^{m+1,q}_{loc}$ by theorem A.1 (a). □

3.2 Apriori estimates

Theorem 3.3 *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V2) *and a constant C_0. Then there is a constant $C = C(C_0, \mathcal{V}) > 0$ such that the following is true. Assume $u : \mathbb{R}\times S^1 \to M$ is a smooth map with $\|\partial_t u\|_\infty \le C_0$ and ξ is a smooth vector field along u satisfying the linear heat equation (1). Then*

$$|\xi(s,t)| \le C\, \|\xi\|_{L^2([s-\frac{1}{2},s]\times S^1)}$$

for every $(s,t) \in \mathbb{R}\times S^1$. If in addition $\|\partial_s u\|_\infty + \|\nabla_t\partial_t u\|_\infty \le C_0$, then

$$|\nabla_t\xi(s,t)| \le C\, \|\xi\|_{L^2([s-1,s]\times S^1)}$$

for every $(s,t) \in \mathbb{R}\times S^1$.

Theorem 3.4 *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V2) *and a constant C_0. Then there is a constant $C = C(C_0, \mathcal{V}) > 0$ such that the following is true. Assume $u : \mathbb{R}\times S^1 \to M$ is a smooth map with*

$$\|\partial_t u\|_\infty + \|\partial_s u\|_\infty + \|\nabla_t\partial_t u\|_\infty + \|\nabla_t\partial_s u\|_\infty + \|\nabla_t\nabla_t\partial_t u\|_\infty \le C_0$$

and ξ is a smooth vector field along u solving the linear heat equation (1). Then

$$|\nabla_t\nabla_t\xi(s,t)| + |\nabla_s\xi(s,t)| \le C\, \|\xi\|_{L^2([s-2,s]\times S^1)}$$

for every $(s,t) \in \mathbb{R}\times S^1$.

Remark 3.5 If in theorem 3.3 or theorem 3.4 the vector field ξ solves $\mathcal{D}^*_u\xi = 0$, then $\eta(s,t) := \xi(-s,t)$ solves (1). The apriori estimates for η then translate into apriori estimates for ξ. For example, it follows that

$$|\xi(s,t)| \le C\, \|\xi\|_{L^2([s,s+\frac{1}{2}]\times S^1)}$$

for every $(s,t) \in \mathbb{R}\times S^1$ and similarly for the higher order derivatives.

The proof of theorem 3.3 and theorem 3.4 is based on the following mean value inequality which we recall from [8], since it is used many times. Consider the **parabolic domain** defined for $r > 0$ by $P_r := (-r^2, 0)\times(-r, r)$.

Lemma 3.6 ([8, lemma B.1]) *There is a constant $c_1 > 0$ such that the following holds for all $r \in (0,1]$ and $a \ge 0$. If $w : P_r \to \mathbb{R}$, $(s,t) \mapsto w(s,t)$, is C^1 in the s-variable and C^2 in the t-variable such that $(\partial_t\partial_t - \partial_s)w \ge -aw$ and $w \ge 0$, then*

$$w(0) \le \frac{c_1 e^{ar^2}}{r^3}\int_{P_r} w.$$

Corollary 3.7 *Let c_1 be the constant of lemma 3.6 and fix two constants $r \in (0,1]$ and $\mu \geq 0$. Then the following is true. If $F : [-r^2, 0] \to \mathbb{R}$ is a C^1 function such that $-F' + \mu F \geq 0$ and $F \geq 0$, then*

$$F(0) \leq \frac{2c_1 e^{\mu r^2}}{r^2} \int_{-r^2}^{0} F(s)\, ds.$$

Proof. Lemma 3.6 with $w(s,t) := F(s)$. □

Corollary 3.8 (to [8, le. B.4]) *Fix constants $r, R > 0$ and three nonnegative functions $U, F, G : [-(R+r)^2, 0] \to [0,\infty)$ such that U is C^1 and F, G are continuous. If $-U' \geq G - F$, then*

$$\int_{-R^2}^{0} G(s)\, ds \leq \frac{R+r}{R} \left(\int_{-(R+r)^2}^{0} F(s)\, ds + \left(\frac{4}{r^2} + \frac{1}{Rr} \right) \int_{-(R+r)^2}^{0} U(s)\, ds \right).$$

Proof. [8, le. B.4] with $u(s,t) = U(s)$, $f(s,t) = F(s)$, $g(s,t) = G(s)$. □

Proof of theorem 3.3. We prove the theorem in three steps. The idea is to prove in step 1 the desired pointwise estimate in its integrated form (slicewise estimate). In steps 2 and 3 this is then used to prove the pointwise estimates. Note that in step 3 we provide an estimate which is not used in the current proof, but later on in the proof of theorem 3.4. Occasionaly we denote $\xi(s,\cdot)$ by ξ_s.

Step 1. *There is a constant $C_1 = C_1(C_0, \mathcal{V}) > 0$ such that*

$$\int_0^1 |\xi(s,t)|^2 \; dt + \int_{s-\frac{1}{16}}^{s} \int_0^1 |\nabla_t \xi(s,t)|^2 \; dtds \leq C_1 \|\xi\|^2_{L^2([s-\frac{1}{4}, s] \times S^1)}$$

for every $s \in \mathbb{R}$.

Define the functions $f, g : \mathbb{R} \times S^1 \to \mathbb{R}$ and $F, G : \mathbb{R} \to \mathbb{R}$ by

$$2f := |\xi|^2, \quad 2g := |\nabla_t \xi|^2, \quad F(s) := \int_0^1 f(s,t)\, dt, \quad G(s) := \int_0^1 g(s,t)\, dt,$$

and abbreviate $L := \partial_t \partial_t - \partial_s$ and $\mathcal{L} := \nabla_t \nabla_t - \nabla_s$. Then

$$Lf = 2g + U, \qquad U := \langle \xi, \mathcal{L}\xi \rangle. \tag{10}$$

Assume that U satisfies the pointwise inequality

$$|U| \leq \mu f + \frac{1}{2} \|\xi_s\|_2^2 \tag{11}$$

for a suitable constant $\mu = \mu(C_0, \mathcal{V}) > 0$. Hence $Lf + \mu f + F \geq 2g$ by (10) and integration over the interval $0 \leq t \leq 1$ shows that $-F' + (\mu+1)F \geq 2G$. Step 1 then follows by Corollary 3.7 with $r = \frac{1}{2}$ and corollary 3.8 with $R = r = \frac{1}{4}$.

It remains to prove (11). Since ξ solves the linear heat equation (1), using the assumption $\|\partial_t u\|_\infty \leq C_0$ and axiom (V1) with constant c_1 we obtain that

$$\begin{aligned}
|U| &= |\langle \xi, \nabla_t \nabla_t \xi - \nabla_s \xi \rangle| \\
&= |\langle \xi, R(\xi, \partial_t u)\partial_t u + \mathcal{H}_{\mathcal{V}}(u)\xi \rangle| \\
&\leq \|R\|_\infty \|\partial_t u\|_\infty^2 |\xi|^2 + c_1 |\xi| \left(|\xi| + \|\xi_s\|_1 \right) \\
&\leq \left(2C_0^2 \|R\|_\infty + 2c_1 + {c_1}^2 \right) \frac{1}{2} |\xi|^2 + \frac{1}{2} \|\xi_s\|_2^2 .
\end{aligned}$$

Step 2. *We prove the estimate for $|\xi|$ in theorem 3.3.*

Note that $Lf \geq -|U|$ by (11). Hence the estimate (11) for $|U|$ and the slicewise estimate for ξ_s provided by step 1 prove the pointwise inequality

$$Lf \geq -\mu f - 2C_1 \left\|\xi\right\|^2_{L^2([s-\frac{1}{4},s]\times S^1)}$$

for all s and t. Fix (s_0, t_0) and set $a = a(s_0) := \frac{2C_1}{\mu} \left\|\xi\right\|^2_{L^2([s_0-\frac{1}{2},s_0]\times S^1)}$. Then

$$L\,(f + a) \geq -\mu\,(f + a)$$

for all t and $s \in [s_0 - \frac{1}{4}, s_0]$. Hence lemma 3.6 with $r = \frac{1}{2}$ applies to the function $w(s,t) := f(s_0+s, t_0+t)+a$ and we obtain that

$$\begin{aligned} f(s_0,t_0) &\leq 8c_1 e^{\mu/4} \int_{-\frac{1}{4}}^{0}\int_0^1 \left(f(s_0+s,t_0+t)+a\right)\,dtds \\ &\leq 8c_1 e^{\mu/4}\left(\frac{1}{2}+\frac{C_1}{2\mu}\right)\left\|\xi\right\|^2_{L^2([s_0-\frac{1}{2},s_0]\times S^1)}. \end{aligned}$$

Since $s_0 \in \mathbb{R}$ and $t_0 \in S^1$ were chosen arbitrarily, this proves step 2.

Step 3. *We prove the estimate for $|\nabla_t\xi|$ in theorem 3.3.*

Define functions $f_1, g_1 : \mathbb{R}\times S^1 \to \mathbb{R}$ by

$$2f_1 := |\nabla_t\xi|^2, \qquad 2g_1 := |\nabla_t\nabla_t\xi|^2,$$

and $F_1, G_1 : \mathbb{R}\to\mathbb{R}$ by $F_1(s) := \int_0^1 f_1(s,t)\,dt$ and $G_1(s) := \int_0^1 g_1(s,t)\,dt$. Then

$$Lf_1 = 2g_1 + U_t, \qquad U_t := \langle\nabla_t\xi, \mathcal{L}\nabla_t\xi\rangle. \tag{12}$$

Since ξ solves the linear heat equation (1), it follows that

$$\begin{aligned} \mathcal{L}\nabla_t\xi &= \nabla_t\left(\nabla_t\nabla_t\xi - \nabla_s\xi\right) - [\nabla_s,\nabla_t]\xi \\ &= \nabla_t\left(-R(\xi,\partial_t u)\partial_t u - \mathcal{H}_{\mathcal{V}}(u)\xi\right) - R(\partial_s u,\partial_t u)\xi \\ &= -\left(\nabla_t R\right)(\xi,\partial_t u)\partial_t u - R(\nabla_t\xi,\partial_t u)\partial_t u - R(\xi,\nabla_t\partial_t u)\partial_t u \\ &\quad - R(\xi,\partial_t u)\nabla_t\partial_t u - \nabla_t\mathcal{H}_{\mathcal{V}}(u)\xi - R(\partial_s u,\partial_t u)\xi. \end{aligned}$$

Now take the pointwise inner product of this identity and $\nabla_t\xi$ and estimate the resulting six terms separately using the L^∞ boundedness assumption of the various derivatives of u. For instance, term five satisfies the estimate

$$\left|\langle\nabla_t\xi, \nabla_t\mathcal{H}_{\mathcal{V}}(u)\xi\rangle\right| \leq c_2\,|\nabla_t\xi|\left(|\nabla_t\xi| + (1+|\partial_t u|)\left(|\xi| + \left\|\xi_s\right\|_1\right)\right)$$

by the second inequality of axiom (V2) with constant c_2. It follows that U_t satisfies the pointwise inequality

$$|U_t| \leq \mu f_1 + \mu\left|\xi\right|^2 + \mu\left\|\xi_s\right\|_2^2$$

for a suitable constant $\mu = \mu(C_0, \mathcal{V}) > 0$. Hence

$$Lf_1 \geq 2g_1 - \mu f_1 - \mu\left|\xi\right|^2 - \mu\left\|\xi_s\right\|_2^2$$

pointwise for all s and t. By step 1 and step 2 this implies the pointwise estimate

$$Lf_1 \geq -\mu f_1 - \mu\left\|\xi\right\|^2_{L^2([s-\frac{1}{2},s]\times S^1)}$$

for all s and t. Here we have chosen a larger value for the constant μ. Fix $(s_0,t_0)\in\mathbb{R}\times S^1$ and set $a = a(s_0) := \left\|\xi\right\|^2_{L^2([s_0-1,s_0]\times S^1)}$. Then

$$L\,(f_1 + a) \geq -\mu\,(f_1 + a)$$

for all t and $s \in [s_0 - \frac{1}{2}, s_0]$. Hence lemma 3.6 with $r = \frac{1}{2}$ applies to the function $w(s,t) := f_1(s_0+s, t_0+t)+a$ and proves the desired estimate, namely

$$\begin{aligned} f_1(s_0,t_0) &\leq 8c_1 e^{\mu/4} \int_{-\frac{1}{4}}^{0} \int_0^1 \left(f_1(s_0+s,t_0+t)+a\right)\, dtds \\ &= 8c_1 e^{\mu/4} \left(\frac{1}{2}\int_{s_0-\frac{1}{4}}^{s_0} \int_0^1 \left|\nabla_t \xi(s,t)\right|^2 dtds + \frac{a}{4}\right) \\ &\leq 8c_1 e^{\mu/4} \left(2\left\|\xi\right\|^2_{L^2([s_0-\frac{1}{2},s_0]\times S^1)} + \frac{1}{4}\left\|\xi\right\|^2_{L^2([s_0-1,s_0]\times S^1)}\right) \end{aligned}$$

for all $s_0 \in \mathbb{R}$ and $t_0 \in S^1$. The final inequality uses the estimate of step 1. This concludes the proof of step 3 and theorem 3.3. □

The proof of theorem 3.4 uses the same techniques. We refer to [12, thm. 3.4] for details.

3.3 Exponential decay

Given a smooth loop $x : S^1 \to M$ the linear operator A_x on $L^2(S^1, x^*TM)$ with dense domain $W^{2,2}$ is defined by (5). With respect to the L^2 inner product $\langle\cdot,\cdot\rangle$ this operator is self-adjoint; see e.g. [10] for the case of geometric perturbations V_t and use [12, le. 3.14] on symmetry of the Hessian $\mathcal{H}_{\mathcal{V}}$ in the general case.

Theorem 3.9 (Backward exponential L^2 decay) *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V2) *and a constant $c_0 > 0$. Then there exist positive constants δ, ρ, C such that the following holds. Let $x : S^1 \to M$ be a smooth loop such that A_x given by (5) is injective and $\|\partial_t x\|_2 + \|\nabla_t \partial_t x\|_2 \leq c_0$. Assume $u : (-\infty, 0] \times S^1 \to M$ is a smooth map and $T_0 > 0$ is a constant such that*

$$u_s = \exp_x \eta_s, \quad \|\eta_s\|_{W^{2,2}} \leq \delta, \quad \|\partial_s u_s\|_2 + \|\nabla_s \partial_t u_s\|_2 \leq \delta,$$

whenever $s \leq -T_0$. Assume further that ξ is a smooth vector field along u such that the function $s \mapsto \|\xi_s\|_2$ is bounded by a constant $c = c(\xi)$ and ξ solves one of two equations

$$\pm\nabla_s \xi - \nabla_t \nabla_t \xi - R(\xi, \partial_t u)\partial_t u - \mathcal{H}_{\mathcal{V}}(u)\xi = 0. \tag{13}$$

Then

$$\|\xi_s\|_2^2 \leq e^{\rho(s+T_0)} \|\xi_{-T_0}\|_2^2 \leq c^2 e^{\rho(s+T_0)}$$

and

$$\|\xi\|^2_{L^2((-\infty,s]\times S^1)} \leq \tfrac{C^2}{\rho} e^{\rho(s+T_0)} \|\xi\|^2_{L^2([-T_0-1,-T_0]\times S^1)}$$

for every $s \leq -T_0$.

Note the weak assumption (L^2 versus L^∞) on the s-derivatives of $\partial_t u_s$ and its base component u_s. To prove theorem 3.9 we need two lemmas.

Remark 3.10 (Forward exponential L^2 decay) If the domain of u is the forward half cylinder $[0,\infty)\times S^1$ and the vector field ξ along u solves $\pm(13)$, then theorem 3.9 applies to $v(\sigma,t) := u(-\sigma,t)$ and $\eta(\sigma,t) := \xi(-\sigma,t)$, since η solves $\mp(13)$. The estimates obtained for η provide estimates for ξ, for instance

$$\|\xi\|^2_{L^2([\sigma,\infty)\times S^1)} \leq \tfrac{C^2}{\rho} e^{\rho(-\sigma+T_0)} \|\xi\|^2_{L^2([T_0,T_0+1]\times S^1)}$$

for every $\sigma \geq T_0$.

Lemma 3.11 (Stability of injectivity) *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V2) *and a constant $c_0 > 1$. Then there are constants $\mu, \delta_0 > 0$ such that the following holds. If x and γ are smooth loops in M such that the operator A_x is injective and*

$$\gamma = \exp_x(\eta), \qquad \|\eta\|_{W^{2,2}} \leq \delta_0, \qquad \|\partial_t x\|_2 + \|\nabla_t \partial_t x\|_2 \leq c_0,$$

then

$$\|\xi\|_2 + \|\nabla_t \xi\|_2 + \|\nabla_t \nabla_t \xi\|_2 \le \mu \|A_\gamma \xi\|_2$$

for every $\xi \in \Omega^0(S^1, \gamma^* TM)$.

Proof. By self-adjointness and injectivity the operator A_x is bijective. Hence it admits a bounded inverse by the open mapping theorem. This proves the estimate in the case $\gamma = x$ for some positive constant, say $\mu_0 = \mu_0(\mathcal{V}, c_0) > 1$. Since bijectivity is preserved under small perturbations (with respect to the operator norm), the result for general x follows from continuous dependence of the operator family on η with respect to the $W^{2,2}$ topology. More precisely, given a smooth vector field ξ along γ, define $X = \Phi^{-1}\xi$ where $\Phi = \Phi(x, \eta)$ denotes parallel transport along the geodesic $[0,1] \ni \tau \mapsto \exp_x(\tau\eta)$. Recall that Φ is pointwise an isometry, then straightforward calculation shows that

$$\|\xi\|_2 + \|\nabla_t \xi\|_2 + \|\nabla_t \nabla_t \xi\|_2 \le c c_0^2 \mu_0 \left\|\Phi A_x \Phi^{-1} \xi\right\|_2$$

where the constant $c > 1$ depends only on the closed Riemannian manifold M and the constant c_1 associated to the Sobolev embedding $W^{1,2} \hookrightarrow C^0$. Now

$$\left\|\Phi A_x \Phi^{-1}\xi - A_\gamma \xi\right\|_2 \le C \|\eta\|_{W^{2,2}} \|\xi\|_{W^{1,2}} \le \delta_0 C \|\xi\|_{W^{1,2}}$$

by straightforward calculation, where the constant $C > 1$ depends on $\|R\|_\infty$, c_0, c_1, δ_0, and the constant in axiom (V2) and where we estimated the term quadratic in $\nabla_t \eta$ by $\|\nabla_t \eta\|_\infty^2 \le c_1^2 \|\eta\|_{W^{2,2}}^2$. The second inequality uses the assumption on η. Now combine both estimates and choose $\delta_0 > 0$ sufficiently small to obtain the assertion of the lemma with $\mu = 2cc_0^2\mu_0$. □

The following lemma 3.12 is well-known; see e.g. [12, le. 3.13] for details.

Lemma 3.12 *Let $f \ge 0$ be a C^2 function on the interval $(-\infty, -T_0]$. If f is bounded by a constant c and satisfies the differential inequality $f'' \ge \rho^2 f$ for some constant $\rho \ge 0$, then $f(s) \le e^{\rho(s+T_0)} f(-T_0)$ for every $s \le -T_0$.*

To prove theorem 3.9 it is useful to denote $\exp_u(\xi)$ by $E(u, \xi)$ and define linear maps, for $\xi \in T_u M$ and $i, j \in \{1, 2\}$, by

$$E_i(u,\xi) : T_u M \to T_{exp_u \xi} M, \qquad E_{ij}(u,\xi) : T_u M \times T_u M \to T_{exp_u \xi} M.$$

If $u : \mathbb{R} \to M$ is a smooth curve and ξ, η are smooth vector fields along u, then the maps E_i and E_{ij} are characterized by the identities

$$\begin{aligned} \frac{d}{ds} \exp_u(\xi) &= E_1(u,\xi)\partial_s u + E_2(u,\xi)\nabla_s \xi \\ \nabla_s \left(E_1(u,\xi)\eta\right) &= E_{11}(u,\xi)\left(\eta, \partial_s u\right) + E_{12}(u,\xi)\left(\eta, \nabla_s \xi\right) + E_1(u,\xi)\nabla_s \eta \\ \nabla_s \left(E_2(u,\xi)\eta\right) &= E_{21}(u,\xi)\left(\eta, \partial_s u\right) + E_{22}(u,\xi)\left(\eta, \nabla_s \xi\right) + E_2(u,\xi)\nabla_s \eta. \end{aligned} \tag{14}$$

These maps satisfy the identities

$$E_{11}(u,0) = E_{12}(u,0) = E_{22}(u,0) = 0, \qquad E_1(u,0) = E_2(u,0) = \mathbb{1}. \tag{15}$$

Proof of theorem 3.9. Fix c_0 and $\mathcal{V}$ and let C be the constant of theorem 3.3 and μ and δ_0 be the constants of lemma 3.11 with this choice. Set $\delta := \delta_0$ and suppose u, x, T_0, ξ satisfy the assumptions of the theorem. Then lemma 3.11 for $\gamma = u_s$ and vector fields $\eta = \eta_s$ and $\xi = \xi_s$ asserts that

$$\|\xi_s\|_2^2 + \|\nabla_t \xi_s\|_2^2 + \|\nabla_t \nabla_t \xi_s\|_2^2 \le \mu^2 \|A_{u_s} \xi_s\|_2^2 = \mu^2 \|\nabla_s \xi_s\|_2^2 \tag{16}$$

whenever $s \le -T_0$. The last step uses the consequence $\nabla_s \xi_s = \mp A_{u_s} \xi_s$ of (5) and (13). From now on we assume that $s \le -T_0$. Observe that

$$\begin{aligned} \partial_t u_s &= E_1(x, \eta_s)\partial_t x + E_2(x, \eta_s)\nabla_t \eta_s \\ \nabla_t \partial_t u_s &= E_{11}(x,\eta_s)\left(\partial_t x, \partial_t x\right) + 2E_{12}(x,\eta_s)\left(\partial_t x, \nabla_t \eta_s\right) + E_1(x,\eta_s)\nabla_t \partial_t x \\ &\quad + E_{22}(x,\eta_s)\left(\nabla_t \eta_s, \nabla_t \eta_s\right) + E_2(x,\eta_s)\nabla_t \nabla_t \eta_s. \end{aligned}$$

By the identities (15) we can choose $\delta > 0$ smaller, if necessary, such that

$$\|\partial_t u_s\|_2 \leq \|E_1(x,\eta_s)\|_\infty \|\partial_t x\|_2 + \|E_2(x,\eta_s)\|_\infty \|\nabla_t \eta_s\|_2 \leq 2c_0.$$

and, similarly, that $\|\nabla_t \partial_t u_s\|_2 \leq 2c_0$.

Claim. *Consider the function $F(s) := \frac{1}{2}\|\xi_s\|_2^2 = \frac{1}{2}\int_0^1 |\xi(s,t)|^2\, dt$. Then there is a sufficiently small constant $\delta > 0$ such that $F''(s) \geq \frac{1}{\mu^2}F(s)$ whenever $s \leq -T_0$.*

Before proving the claim we show how it implies the conclusions of theorem 3.9. Set $\rho = \rho(c_0,\mathcal{V}) := \frac{1}{\mu}$, then $F'' \geq \rho^2 F$ on $(-\infty, T_0]$. Hence lemma 3.12 proves the first conclusion of theorem 3.9. Use this conclusion, the fact that $\|\cdot\|_2 \leq \|\cdot\|_\infty$ on the domain S^1, and theorem 3.3 with constant $C = C(c_0,\mathcal{V})$ to obtain that

$$\|\xi_s\|_2^2 \leq e^{\rho(s+T_0)}\|\xi_{-T_0}\|_\infty^2 \leq C^2 e^{\rho(s+T_0)}\|\xi\|^2_{L^2([-T_0-1,-T_0]\times S^1)}$$

whenever $s \leq -T_0$. Fix $\sigma \leq -T_0$ and integrate this estimate over $s \in (-\infty,\sigma]$. This proves the final conclusion of theorem 3.9.

It remains to prove the claim. In the following calculation we drop the subindex s for simplicity and denote the $L^2(S^1)$ inner product by $\langle\cdot,\cdot\rangle$. By straightforward computation it follows that $F''(s) = \|\nabla_s\xi_s\|_2^2 + \langle \xi, \nabla_s\nabla_s\xi\rangle$ and

$$\begin{aligned}
\langle \xi, \nabla_s\nabla_s\xi\rangle &= \pm\big\langle \xi, \nabla_s\left(\nabla_t\nabla_t\xi + R(\xi,\partial_t u)\partial_t u + \mathcal{H}_\mathcal{V}(u)\xi\right)\big\rangle \\
&= \pm\big\langle \xi, [\nabla_s,\nabla_t\nabla_t]\xi + \nabla_t\nabla_t\nabla_s\xi + \nabla_s\left(R(\xi,\partial_t u)\partial_t u + \mathcal{H}_\mathcal{V}(u)\xi\right)\big\rangle \\
&= \pm\big\langle \xi, \nabla_t[\nabla_s,\nabla_t]\xi + [\nabla_s,\nabla_t]\nabla_t\xi + \nabla_s\left(R(\xi,\partial_t u)\partial_t u + \mathcal{H}_\mathcal{V}(u)\xi\right)\big\rangle \\
&\quad \pm \big\langle \nabla_t\nabla_t\xi, \nabla_s\xi\big\rangle \\
&= \pm\big\langle \pm\nabla_s\xi - R(\xi,\partial_t u)\partial_t u - \mathcal{H}_\mathcal{V}(u)\xi, \nabla_s\xi\big\rangle \\
&\quad \pm \big\langle \xi, \big(\nabla_t R\big)(\partial_s u,\partial_t u)\xi + R(\nabla_t\partial_s u,\partial_t u)\xi + R(\partial_s u,\nabla_t\partial_t u)\xi \\
&\qquad + 2R(\partial_s u,\partial_t u)\nabla_t\xi + \big(\nabla_s R\big)(\xi,\partial_t u)\partial_t u + R(\nabla_s\xi,\partial_t u)\partial_t u \\
&\qquad + R(\xi,\nabla_s\partial_t u)\partial_t u + R(\xi,\partial_t u)\nabla_s\partial_t u + \nabla_s\mathcal{H}_\mathcal{V}(u)\xi\big\rangle \\
&= \|\nabla_s\xi\|_2^2 \pm \big\langle \xi, \nabla_s\mathcal{H}_\mathcal{V}(u)\xi - \mathcal{H}_\mathcal{V}(u)\nabla_s\xi\big\rangle \\
&\quad \pm \big\langle \xi, \big(\nabla_t R\big)(\partial_s u,\partial_t u)\xi + 2R(\xi,\partial_t u)\nabla_t\partial_s u + R(\partial_s u,\nabla_t\partial_t u)\xi \\
&\qquad + 2R(\partial_s u,\partial_t u)\nabla_t\xi + \big(\nabla_s R\big)(\xi,\partial_t u)\partial_t u\big\rangle.
\end{aligned}$$

To obtain the first and the fourth step we replaced ξ according to (13). The third step is by integration by parts. In the final step we used twice the first Bianchi identity and [12, le. 3.14] on symmetry of the Hessian. Note that the term $\nabla_t\partial_s u$ forces us to assume $W^{1,2}$ and not only L^∞ smallness of $\partial_s u_s$.

Abbreviate $\|\cdot\|_{1,2} := \|\cdot\|_{W^{1,2}(S^1)}$ and assume from now on that $s \leq -T_0$. Recall that $\|\partial_t u_s\|_\infty \leq c_1\|\partial_t u_s\|_{1,2} \leq 4c_0c_1$ where c_1 is the Sobolev constant of the embedding $W^{1,2}(S^1) \hookrightarrow C^0(S^1)$. Then the former two identities imply that

$$\begin{aligned}
F''(s) &\geq 2\|\nabla_s\xi_s\|_2^2 - C_1\left(\|\partial_s u_s\|_\infty + \|\nabla_t\partial_s u_s\|_2\right)\left(\|\xi_s\|_\infty^2 + \|\xi_s\|_\infty\|\nabla_t\xi\|_2\right) \\
&\geq 2\|\nabla_s\xi_s\|_2^2 - C_2\|\partial_s u_s\|_{1,2}\|\xi_s\|_{1,2}^2
\end{aligned}$$

for positive constants $C_1 = C_1(c_0,c_1,\mathcal{V},\|R\|_{C^2})$ and $C_2 = C_2(c_1,C_1)$. Choose $\delta > 0$ again smaller, if necessary, namely such that $\delta < 1/(2\mu^2C_2)$. Hence $\|\partial_s u_s\|_{1,2} \leq \delta < \frac{1}{2\mu^2C_2}$ where the first inequality is by assumption. Therefore

$$F''(s) \geq 2\|\nabla_s\xi_s\|_2^2 - \frac{1}{2\mu^2}\|\xi_s\|_{1,2}^2 \geq \|\nabla_s\xi_s\|_2^2$$

where the second inequality is by (16). But $\|\nabla_s\xi_s\|_2^2 \geq \frac{1}{\mu^2}\|\xi_s\|_2^2 = \frac{2}{\mu^2}F(s)$ again by (16) and definition of F. This proves the claim and theorem 3.9. □

3.4 The Fredholm operator

Theorem 3.13 (Fredholm) *Fix a perturbation $\mathcal{V}:\mathcal{L}M\to\mathbb{R}$ that satisfies* (V0)–(V3)*, a constant $p>1$, and two nondegenerate critical points $x^\pm$ of $\mathcal{S}_\mathcal{V}$. Assume $u:\mathbb{R}\times S^1\to M$ is a smooth map such that*

$$u_s=\exp_{x^\pm}(\eta_s^\pm),\quad \left\|\eta_s^\pm\right\|_{W^{2,2}}\to 0,\quad \left\|\partial_s u_s\right\|_{W^{1,2}}\to 0,\quad \text{as } s\to\pm\infty,$$

that $\|\nabla_t\partial_t u\|_\infty+\|\nabla_t\partial_s u\|_\infty+\|\nabla_t\nabla_t\partial_t u\|_\infty<\infty$, and that $\|\nabla_t\nabla_t\partial_s u_s\|_2$ is bounded, uniformly in $s\in\mathbb{R}$. Then the operator $\mathcal{D}_u:\mathcal{W}_u^{1,p}\to\mathcal{L}_u^p$ is Fredholm and

$$\operatorname{index}\mathcal{D}_u=\operatorname{ind}_\mathcal{V}(x^-)-\operatorname{ind}_\mathcal{V}(x^+).$$

Moreover, the formal adjoint operator $\mathcal{D}_u^*=-\nabla_s+A_{u_s}:\mathcal{W}_u^{1,p}\to\mathcal{L}_u^p$ is Fredholm with $\operatorname{index}\mathcal{D}_u^*=-\operatorname{index}\mathcal{D}_u$.

Boundedness in L^∞ of the three derivatives is required to apply theorem 3.4 in the proof of proposition 3.15. For the assumption on $\nabla_t\nabla_t\partial_s u_s$ see the footnote below. Recall that the linear operator $\mathcal{D}_u:\mathcal{W}_u^{1,p}\to\mathcal{L}_u^p$ is given by (6). From now on and throughout this section we assume that $p>1$ and u and $x^\pm$ satisfy the assumptions of theorem 3.13. We set $x:=x^-$ and $y:=x^+$. The goal of this section is to prove theorem 3.13. By definition a **Fredholm operator** is a bounded linear operator with closed range and finite dimensional kernel and cokernel. The difference of these dimensions is called the **Fredholm index** of $\mathcal{D}_u$ denoted by $\operatorname{index}\mathcal{D}_u$. Furthermore, observe that the **formal adjoint operator** $\mathcal{D}_u^*:\mathcal{W}_u^{1,p}\to\mathcal{L}_u^p$ with respect to the L^2-inner product has the form

$$\mathcal{D}_u^*\xi=-\nabla_s\xi-\nabla_t\nabla_t\xi-R(\xi,\partial_t u)\partial_t u-\mathcal{H}_\mathcal{V}(u)\xi. \tag{17}$$

We proceed as follows. In the case $p=2$ we show that our situation matches the assumptions of [6] and this proves the Fredholm property for $p=2$. Then we reduce the case $p>1$ to the case $p=2$. Here key steps are to prove closedness of the range and independence of kernel and cokernel of p. The latter argument is based on exponential L^2 decay (theorem 3.9) and local regularity (theorem 3.1).

For easy reference we derive some consequences from the assumptions of theorem 3.13. Consider the constant $a:=\max\{\mathcal{S}_\mathcal{V}(x),\mathcal{S}_\mathcal{V}(y)\}$ and the constant $C_0>0$ in axiom (V0). Then (2) and (3) imply that

$$\left\|\partial_t x\right\|_2^2=2a+2\mathcal{V}(x)\le 2(a+C_0),\quad \left\|\nabla_t\partial_t x\right\|_2=\left\|\operatorname{grad}\mathcal{V}(x)\right\|_2\le C_0, \tag{18}$$

and similarly for y. Hence by the Sobolev embedding $W^{1,2}(S^1)\hookrightarrow C^0(S^1)$ with constant c_1, the fact that $\partial_t u_s$ converges asymptotically to $\partial_t x^\pm$ in $W^{1,2}(S^1)$, and smoothness of u there is a constant $c_2=c_2(a,C_0,u)$ such that

$$\left\|\partial_t u\right\|_\infty=\sup_{s\in\mathbb{R}}\left\|\partial_t u_s\right\|_\infty\le c_1\sup_{s\in\mathbb{R}}\left\|\partial_t u_s\right\|_{W^{1,2}}\le c_2. \tag{19}$$

Similarly, since $\partial_s u_s$ converges asymptotically to zero in $W^{1,2}(S^1)$ it holds that

$$\left\|\partial_s u\right\|_\infty=\sup_{s\in\mathbb{R}}\left\|\partial_s u_s\right\|_\infty\le c_1\sup_{s\in\mathbb{R}}\left\|\partial_s u_s\right\|_{W^{1,2}}\le c_3. \tag{20}$$

for some constant $c_3=c_3(u)$.

Remark 3.14 (a) The map u in theorem 3.13 satisfies the assumptions of the local regularity theorem 3.1; since u is smooth on the whole cylinder, all its derivatives are bounded on any given compact subset $Q\supset(a,b]\times S^1$. By (19) and (20) the assumptions of the apriori estimates theorem 3.3 and theorem 3.4 are satisfied. (b) The map u and x, y in theorem 3.13 satisfy the assumptions of the exponential L^2 decay results theorem 3.9 and remark 3.10; for x and y use (18) and for u use that $\partial_s u_s$ converges asymptotically to zero in $W^{1,2}(S^1)$.

Fredholm property and index for $p=2$

To prove that $\mathcal{D}_u$ is Fredholm it is useful to represent $\mathcal{D}_u$ with respect to an orthonormal frame along u. Since M is not necessarily orientable, we define

$$\sigma=\sigma(u):=\begin{cases}+1, & \text{if } u^*TM\to\mathbb{R}\times S^1 \text{ is trivial}\\ -1, & \text{else}\end{cases}$$

and $E_\sigma := \operatorname{diag}(\sigma, 1, \ldots, 1) \in \mathbb{R}^{n\times n}$. The orthogonal group $\mathrm{O}(n)$ has two connected components, one contains $E_1 = \mathbb{1}$ and the other one E_{-1}. Hence there exists a (smooth) orthonormal trivialization $\phi = \phi_\sigma : \mathbb{R}\times[0,1]\times\mathbb{R}^n \to u^*TM$ such that $\phi(s,t) = \phi(s,0)E_\sigma$ for all $s\in\mathbb{R}$ and $t\in[\frac{3}{4},1]$. The vector space of smooth sections of u^*TM is isomorphic to the space C^∞_σ of all maps $\vec{v}\in C^\infty(\mathbb{R}\times[0,1],\mathbb{R}^n)$ such that $\vec{v}(s,t) = E_\sigma\vec{v}(s,0)$ for all $s\in\mathbb{R}$ and $t\in[\frac{3}{4},1]$.

Denote by W the closure of C^∞_σ with respect to the Sobolev $W^{2,2}$ norm and by H its closure with respect to the L^2 norm. Then $\mathcal{D}_u : \mathcal{W}^{1,2}_u \to \mathcal{L}^2_u$ given by (6) is represented by the Atiyah-Patodi-Singer type operator

$$D_{A+C} := \phi^{-1}\mathcal{D}_u\phi = \frac{d}{ds} + A(s) + C(s) \tag{21}$$

from $\mathcal{W}^{1,2} := L^2(\mathbb{R},W)\cap W^{1,2}(\mathbb{R},H)$ to $L^2(\mathbb{R},H)$. Here $A(s)$ is the family of symmetric second order operators on H with dense domain W given by

$$A(s) = -\frac{d^2}{dt^2} - B(s,t) - Q(s,t)$$

where $B = (\partial_t P) + 2P\partial_t + P^2$ and $Q = \phi^{-1}R(\phi,\partial_t u)\partial_t u + \phi^{-1}\mathcal{H}_\mathcal{V}(u)\phi$. The families of skew-symmetric matrices $P(s,t)$ and $C(s,t)$ are determined by

$$\phi^{-1}\nabla_t\phi = \partial_t + P, \qquad \phi^{-1}\nabla_s\phi = \partial_s + C.$$

Observe that $\lim_{s\to\pm\infty} C(s,t) = 0$, uniformly in t, since $\partial_s u_s$ converges asymptotically to zero in $C^0(S^1)$ by assumption of theorem 3.13. Hence $C(s)$ converges asymptotically to zero in $\mathcal{L}(\mathbb{R}^n)$. Therefore the linear operator $C : \mathcal{W}^{1,2}\to L^2$ is a compact perturbation of D_A by [6, lem. 3.18]. But the Fredholm property and the Fredholm index are invariant under compact perturbations and so it remains to prove that D_A is a Fredholm operator and compute its index. By [6, thm. A] it suffices to verify the following properties.

(i) The inclusion of Hilbert spaces $W\hookrightarrow H$ is compact with dense image.

(ii) The operator $A(s) : H\to H$ with dense domain W is unbounded and self-adjoint for every s.

(iii) The norm of W is equivalent to the graph norm of $A(s)$ for every s.

(iv) The map $\mathbb{R}\to\mathcal{L}(W,H) : s\mapsto A(s)$ is of class C^1 with respect to the weak operator topology.

(v) There exist invertible operators $A^\pm\in\mathcal{L}(W,H)$ which are the limits of $A(s)$ in the norm topology, as s tends to $\pm\infty$.

Statements (i) and (ii) follow by the Sobolev embedding theorem, the well-known fact that the 1-dimensional Laplacian $-d^2/dt^2$ on $[0,1]$ with periodic boundary conditions is self-adjoint, and by the Kato-Rellich Theorem using that the perturbation $B+Q$ is of relative bound zero; see [5]. To prove (iii) one has to establish that the W norm is bounded above by a constant times the graph norm and vice versa. The first inequality uses the elliptic estimate for the operator $A(s)$ and the second one follows since $\|\partial_t u_s\|_\infty$ and $\|\nabla_t\partial_t u_s\|_2$ are bounded by (19) and since the Hessian $\mathcal{H}_\mathcal{V}(u_s)$ is a bounded linear operator on $L^2(S^1, {u_s}^*TM)$ by axiom (V1). To prove (iv) we need to show that, given any $\xi\in W$ and $\eta\in H$, the map $s\mapsto\langle\eta, A(s)\xi\rangle$ is in $C^1(\mathbb{R},\mathbb{R})$. This follows by the bounds in (19) and (20), by the final estimate in axiom (V2), and the apparently unnatural[1] assumption in theorem 3.13 that $\nabla_t\nabla_t\partial_s u_s$ be uniformly L^2 bounded. Statement (v) is true, since the critical points $x^\pm$ are nondegenerate and u_s converges in $W^{2,2}(S^1)$ to $x^\pm$, as $s\to\pm\infty$.

Now (i–v) are precisely the assumptions of theorem A in [6] which therefore asserts that the operator $D_A : \mathcal{W}^{1,2}\to L^2$ is Fredholm and its index is given by the spectral flow of the operator family $A(s)$. Since the spectral flow represents the net change in the number of negative eigenvalues of $A(s)$ as s runs from $-\infty$ to ∞, it is equal to the Morse index difference $\operatorname{ind}(A^-) - \operatorname{ind}(A^+)$. To see this observe that $\operatorname{ind}(A^+)$ equals $\operatorname{ind}(A^-)$ plus the number of eigenvalues changing from positive to negative minus the number of those changing sign in

[1] If in [6, thm.A], hence in (iv), *continuously differentiable* could be replaced by *continuous*, then the assumption on $\|\nabla_t\nabla_t\partial_s u_s\|_2$ can be dropped in theorem 3.13.

the opposite direction. Finally, the Fredholm indices of D_A and D_{A+C} are equal, since $\{D_{A+\tau C}\}_{\tau\in[0,1]}$ is an interpolating family of Fredholm operators. This proves theorem 3.13 for $\mathcal{D}_u : \mathcal{W}^{1,2}_u \to \mathcal{L}^2_u$.

Observe that the formal adjoint operator $\mathcal{D}^*_u : \mathcal{W}^{1,2}_u \to \mathcal{L}^2_u$ is represented by $-D_{-A-C}$. Since A satisfies (i-v), so does $-A$. Thus D_{-A} is a Fredholm operator, again by [6, thm. A], and its index is given by minus the spectral flow of the operator family $A = A(s)$. Thus $-D_{-A}$ is Fredholm of the same index. But $-D_{-A}$ and $-D_{-A-C}$ are homotopic through the family of Fredholm operators $\{-D_{-A-\tau C}\}_{\tau\in[0,1]}$. This proves theorem 3.13 for $p = 2$.

Fredholm property and index for $p > 1$

Still under the assumptions of theorem 3.13 consider the vector spaces given by

$$X_0^{(*)} := \Big\{\xi \in C^\infty(\mathbb{R}\times S^1, u^*TM) \mid \mathcal{D}_u^{(*)}\xi = 0, \exists c,\delta > 0\ \forall s\in\mathbb{R} : \\ \|\xi_s\|_\infty + \|\nabla_t\xi_s\|_\infty + \|\nabla_t\nabla_t\xi_s\|_\infty + \|\nabla_s\xi_s\|_\infty \le ce^{-\delta|s|}\Big\}.$$

Proposition 3.15 *Let $p > 1$, then*

$$\ker\left[\mathcal{D}_u : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u\right] = X_0, \qquad \ker\left[\mathcal{D}^*_u : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u\right] = X_0^*.$$

P r o o f. It suffices to prove the first statement. The other one then follows by reflection $s \mapsto -s$. The inclusion $\supset$ is trivial. To prove the inclusion $\subset$ assume that $\xi \in \mathcal{W}^{1,p}$ solves $\mathcal{D}_u\xi = 0$ almost everywhere. Recall remark 3.14. Being a local property smoothness of ξ follows from theorem 3.1 for $\eta = \xi$ using integration by parts. Exponential decay in $L^\infty(S^1)$ of ξ and $\nabla_s\xi$ follows by combining the apriori estimates theorem 3.3 and theorem 3.4 with the L^2 exponential decay results theorem 3.9 and remark 3.10. The final step is to use that by smoothness of ξ its L^2 norm over any compact subset of $\mathbb{R}\times S^1$ is finite. Note that the exponential decay results require nondegeneracy of the critical points $x^\pm$ and boundedness of the map $s \mapsto \|\xi_s\|_2$. Hence it remains to verify the latter. Consider the constant $c_2 = c_2(a, C_0, u)$ in (19) and let $C = C(c_2, \mathcal{V})$ be the corresponding constant in theorem 3.3. Then for $s \in \mathbb{R}$ we obtain that

$$\|\xi_s\|_2 \le \|\xi_s\|_\infty \le C\,\|\xi\|_{L^2(Z_s)}, \qquad Z_s := [s-\tfrac{1}{2}, s]\times S^1.$$

Now there are three cases. The case $p = 2$ is trivial. If $p > 2$, define $q > 2$ by $\frac{1}{q} + \frac{1}{p} = \frac{1}{2}$ and apply Hölder's inequality to $\|1\cdot\xi\|_{L^2(Z_s)}$ to conclude that

$$\|\xi_s\|_2 \le C\left(\tfrac{1}{2}\right)^{\frac{1}{q}}\|\xi\|_{L^p(Z_s)} \le C\left(\tfrac{1}{2}\right)^{\frac{p-2}{2p}}\|\xi\|_p$$

for every $s \in \mathbb{R}$. If $1 < p < 2$, apply the Sobolev embedding $W^{1,p}(Z_s) \hookrightarrow L^2(Z_s)$ with Sobolev constant $c_p > 0$ to obtain that

$$\|\xi_s\|_2 \le C\,\|\xi\|_{L^2(Z_s)} \le c_pC\,\|\xi\|_{W^{1,p}(Z_s)} \le c_pC\,\|\xi\|_{\mathcal{W}^{1,p}}$$

for every $s \in \mathbb{R}$. □

Proposition 3.16 *Assume $u : \mathbb{R}\times S^1 \to M$ is a smooth map such that $\|\partial_s u\|_\infty$, $\|\partial_t u\|_\infty$, and $\|\nabla_t\partial_t u\|_\infty$ are finite and $\lim_{s\to\pm\infty} u(s,t)$ exists, uniformly in t. Then, for every $p > 1$, there is a constant $c = c(p, u, M)$ such that*

$$\|\nabla_s\xi\|_p + \|\nabla_t\xi\|_p + \|\nabla_t\nabla_t\xi\|_p \le c\left(\|\nabla_s\xi - \nabla_t\nabla_t\xi\|_p + \|\xi\|_p\right) \tag{22}$$

for every smooth compactly supported vector field ξ along u. Estimate (22) remains valid for $-\nabla_s$ replacing ∇_s. Estimate (22) also remains valid if u is defined on the backward halfcylinder $(-\infty, 0]\times S^1$.

P r o o f. The proof of (22) for $\mathbb{R}\times S^1$ and for $(-\infty, 0]\times S^1$ is based on the parabolic analogue of the Calderon-Zygmund estimate [8, thm. C.2] for $\mathbb{R}^2$ and $\mathbb{H}^-$, respectively, via a covering argument. Details for $\mathbb{R}\times S^1$ are provided by [8, prop. D.2]. Lemma D.4 in [8] allows to add the term $\nabla_t\xi$ to the left hand side of (22). The underlying reason is periodicity in the t variable. The statement for $-\nabla_s$ follows by reflection $s \mapsto -s$. □

Proposition 3.17 *The range of* $\mathcal{D}_u, \mathcal{D}_u^*: \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p$ *is closed whenever* $p > 1$.

Proof. The structure of proof is standard; see e.g. [7, sec. 2]. We sketch the two key steps for $\mathcal{D}_u$. Step one is the linear estimate

$$\|\xi\|_{\mathcal{W}^{1,p}} \le c_p \left(\|\mathcal{D}_u \xi\|_p + \|\xi\|_p \right) \tag{23}$$

for compactly supported smooth vector fields ξ along u. This follows immediately from proposition 3.16, [8, lemma D.4], the L^∞ bound for $\partial_t u$ in (19) and axiom (V1). Step two is to prove bijectivity of $\mathcal{D}_u$ in the case of the constant cylinder $u(s,t) = x(t)$, whenever x is a nondegenerate critical point of $\mathcal{S}_\mathcal{V}$. A proof for $p \ge 2$ in the related case of half cylinders is given in [12, thm. 8.5]. The case $1 < p \le 2$ follows by duality; cf. [7, exc. 2.5]. Both steps are then combined by a cutoff function argument; see [7, thm 2.2]. □

Proposition 3.17 enables us to define the cokernels of $\mathcal{D}_u : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p$ and $\mathcal{D}_u^* : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p$ as Banach space quotients. Namely, for $p > 1$ set

$$\operatorname{coker} \mathcal{D}_u := \frac{\mathcal{L}_u^p}{\operatorname{im} \mathcal{D}_u}, \qquad \operatorname{coker} \mathcal{D}_u^* := \frac{\mathcal{L}_u^p}{\operatorname{im} \mathcal{D}_u^*}.$$

The following result shows that also the cokernels are independent of p.

Proposition 3.18 *Let* $p > 1$, *then*

$$\operatorname{coker} \left[\mathcal{D}_u : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p \right] = X_0^*, \qquad \operatorname{coker} \left[\mathcal{D}_u^* : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p \right] = X_0.$$

Proof. We prove the second identity. The other one follows by reflection $s \mapsto -s$. We identify the cokernel of $\mathcal{D}_u^*$ with the annihilator of the image of $\mathcal{D}_u^*$ given by

$$(\operatorname{im} \mathcal{D}_u^*)^\perp := \left\{ \eta \in \mathcal{L}_u^q \mid \langle \eta, \mathcal{D}_u^* \xi \rangle = 0 \text{ for all } \xi \in \mathcal{W}_u^{1,p} \right\}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, hence $\mathcal{L}_u^q = (\mathcal{L}_u^p)^*$. It remains to prove that $(\operatorname{im} \mathcal{D}_u^*)^\perp = X_0$. The inclusion $\supset$ is trivial. To prove the inclusion $\subset$ assume that $\eta \in (\operatorname{im} \mathcal{D}_u^*)^\perp$. Hence η is smooth by theorem 3.1. Integration by parts shows that $\mathcal{D}_u \eta = 0$. Exponential decay follows by combining theorem 3.3 and theorem 3.4 with theorem 3.9 and remark 3.10 as explained in the proof of proposition 3.15. However, since we do not yet know that $\eta \in \mathcal{W}_u^{1,q}$, we continue the final estimate in the proof of proposition 3.15 using (23) and $\mathcal{D}_u \eta = 0$ together with a cutoff function argument to obtain that $\|\eta_s\|_2 \le c_q C \|\eta\|_{\mathcal{W}_u^{1,q}(Z_s)} \le 3 c_q C C_q \|\eta\|_q$. □

Proof of theorem 3.13. The range of $\mathcal{D}_u : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p$ is closed by proposition 3.17. Moreover, by proposition 3.15 and proposition 3.18 the kernel and the cokernel of $\mathcal{D}_u : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p$ are given by X_0 and X_0^*, respectively. But these vector spaces do not depend on $p > 1$. Apply the result for $p = 2$. □

4 A product estimate

Lemma 4.1 (Product estimate) *Let* N *be a Riemannian manifold with Levi-Civita connection* ∇ *and Riemannian curvature tensor* R. *Fix constants* $p \ge 2$ *and* $c_0 > 0$. *Then there is a constant* $C = C(p, c_0, \|R\|_\infty)$ *such that the following holds. Assume* $u : (a,b] \times S^1 \to N$ *is a smooth map such that* $\|\partial_s u\|_\infty + \|\partial_t u\|_\infty \le c_0$, *then*

$$\left(\int_a^b \int_0^1 \left(|\nabla_t \xi| \, |\nabla_t X| \right)^p \, dt ds \right)^{1/p} \le C \, \|\xi\|_{\mathcal{W}^{1,p}} \left(\|\nabla_t X\|_p + \|\nabla_t \nabla_t X\|_p \right)$$

for all smooth compactly supported vector fields ξ *and* X *along* u.

Remark 4.2 (a) Lemma 4.1 continues to hold for smooth maps u that are defined on the whole cylinder $\mathbb{R} \times S^1$. In this case the (compact) supports of ξ and X are contained in an interval of the form $(a,b]$.
(b) In the proof step 2 for $p = 2$ leads to the remarkable fact that along finite energy solutions of (4) the action is automatically bounded; cf. [13, cor. 2.10].

Proof of lemma 4.1. The proof has three steps. Step 2 requires $p \geq 2$. Abbreviate $I = (a, b]$ and for $q, r \in [1, \infty]$ consider the norm

$$\|\xi\|_{q;r} := \|\xi\|_{L^q(I, L^r(S^1))} .$$

STEP 1. *Fix reals $\alpha \geq 1$ and $q, r, q', r' \in [\alpha, \infty]$ such that $\frac{1}{q} + \frac{1}{r} = \frac{1}{\alpha}$ and $\frac{1}{q'} + \frac{1}{r'} = \frac{1}{\alpha}$. Then for all functions $f, g \in C^\infty(I \times S^1)$ it holds that*

$$\|fg\|_\alpha \leq \|f\|_{q';q} \|g\|_{r';r} .$$

Let $f_s(t) := f(s,t)$. Apply Hölder's inequality twice to obtain that

$$\|fg\|^\alpha_{L^\alpha(I\times S^1)} = \int_a^b \|f_s g_s\|^\alpha_{L^\alpha(S^1)} \, ds \leq \int_a^b \left(\|f_s\|_{L^q(S^1)} \|g_s\|_{L^r(S^1)}\right)^\alpha ds \leq \|u\|^\alpha_{L^{q'}(I)} \|v\|^\alpha_{L^{r'}(I)}$$

where $u(s) := \|f_s\|_{L^q(S^1)}$ and $v(s) := \|g_s\|_{L^r(S^1)}$. This proves step 1.

STEP 2. *Given p, c_0, and u as in the hypothesis of the lemma, then there is a constant $c = c(p, c_0)$ such that*

$$\|\nabla_t \xi\|_{\infty;p} \leq c \|\xi\|_{\mathcal{W}^{1,p}}$$

for every smooth compactly supported vector field ξ along $u : I \times S^1 \to N$.

The proof uses the *generalized Young inequality*: Given reals $a, b, c \geq 0$ and $1 < \alpha, \beta, \gamma < \infty$ such that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$, then $abc \leq \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta} + \frac{c^\gamma}{\gamma}$. Abbreviate $\xi(s,t)$ by ξ, then integration by parts shows that

$$\begin{aligned}
&\frac{d}{ds}\int_0^1 |\nabla_t \xi(s,t)|^p \, dt \\
&= p\int_0^1 |\nabla_t\xi|^{p-2} \langle \nabla_t\xi, \nabla_t\nabla_s\xi + [\nabla_s, \nabla_t]\xi\rangle \, dt \\
&= -p\int_0^1 \left(\frac{d}{dt}|\nabla_t\xi|^{p-2}\right)\langle\nabla_t\xi, \nabla_s\xi\rangle \, dt - p\int_0^1 |\nabla_t\xi|^{p-2}\langle\nabla_t\nabla_t\xi, \nabla_s\xi\rangle \, dt \\
&\quad + p\int_0^1 |\nabla_t\xi|^{p-2}\langle\nabla_t\xi, R(\partial_s u, \partial_t u)\xi\rangle \, dt \\
&= -p(p-2)\int_0^1 |\nabla_t\xi|^{p-4}\langle\nabla_t\xi, \nabla_t\nabla_t\xi\rangle\langle\nabla_t\xi, \nabla_s\xi\rangle \, dt \\
&\quad - p\int_0^1 |\nabla_t\xi|^{p-2}\left(\langle\nabla_t\nabla_t\xi, \nabla_s\xi\rangle - \langle\nabla_t\xi, R(\partial_s u, \partial_t u)\xi\rangle\right) dt.
\end{aligned}$$

Take the absolute value of the right hand side, apply the generalized Young inequality in the case[2] $p > 2$ with $\alpha = p/(p-2)$, $\beta = p$, $\gamma = p$, and the standard Young inequality with $\alpha = p/(p-1)$, $\beta = p$ to obtain the inequality

$$\begin{aligned}
&\frac{d}{ds}\int_0^1 |\nabla_t \xi(s,t)|^p \, dt \\
&\leq p(p-1)\int_0^1 |\nabla_t\xi|^{p-2}|\nabla_t\nabla_t\xi|\cdot|\nabla_s\xi| \, dt + pc_0^2\|R\|_\infty\int_0^1 |\nabla_t\xi|^{p-1}|\xi| \, dt \\
&\leq p(p-1)\int_0^1 \left(\frac{p-2}{p}|\nabla_t\xi|^p + \frac{1}{p}|\nabla_t\nabla_t\xi|^p + \frac{1}{p}|\nabla_s\xi|^p\right) dt \\
&\quad + pc_0^2\|R\|_\infty\int_0^1 \left(\frac{p-1}{p}|\nabla_t\xi|^p + \frac{1}{p}|\xi|^p\right) dt \\
&\leq C_1\left(\|\xi_s\|^p_{L^p(S^1)} + \|\nabla_s\xi_s\|^p_{L^p(S^1)} + \|\nabla_t\nabla_t\xi_s\|^p_{L^p(S^1)}\right).
\end{aligned}$$

[2] The case $p = 2$ is taken care of by the standard Young inequality.

Here $C_1 > 0$ is a constant depending only on p, c_0, and $\|R\|_\infty$ and $\xi_s(t) := \xi(s,t)$. Note that we used [8, lemma D.4] to estimate the terms involving $\nabla_t \xi_s$. Now fix $\sigma \in (a,b]$ and integrate this inequality over $s \in (a,\sigma]$ to obtain the estimate

$$\|\nabla_t \xi_\sigma\|^p_{L^p(S^1)} \leq c\,\|\xi\|^p_{\mathcal{W}^{1,p}((a,b]\times S^1)}\,.$$

Here we used compactness of the support of ξ and monotonicity of the integral. Since the right hand side is independent of σ the proof of step 2 is complete.

STEP 3. *We prove the lemma.*

Consider the functions $f(s,t) := |\nabla_t \xi(s,t)|$ and $g(s,t) := |\nabla_t X(s,t)|$. Then by step 1 with $\alpha = q = r'$ equal to p and with $r = q' = \infty$ we obtain that

$$\int_a^b \int_0^1 \left(|\nabla_t \xi(s,t)|\,|\nabla_t X(s,t)|\right)^p \,dtds = \|fg\|_p^p \leq \|\nabla_t \xi\|^p_{\infty;p}\,\|\nabla_t X\|^p_{p;\infty}\,.$$

Now apply step 2 to the first factor. For the second one we exploit the fact that, since the slices $s \times S^1$ of our domain are compact, there is the Sobolev embedding $W^{1,p}(S^1) \hookrightarrow L^\infty(S^1)$ with constant $\mu = \mu(p) > 0$. It follows that

$$\begin{aligned}\int_a^b \|\nabla_t X_s\|^p_{L^\infty(S^1)}\,ds &\leq \int_a^b \mu^p\,\|\nabla_t X_s\|^p_{W^{1,p}(S^1)}\,ds\\ &= \mu^p \int_a^b \|\nabla_t X_s\|^p_{L^p(S^1)} + \|\nabla_t\nabla_t X_s\|^p_{L^p(S^1)}\,ds.\end{aligned}$$

This concludes the proof of lemma 4.1. □

A Local regularity

By $\mathbb{H}^-$ we denote the **closed lower half plane**, that is, the set of pairs of reals (s,t) with $s \leq 0$. In this section all maps are real-valued and the domains of the various Banach spaces which appear are understood to be open subsets Ω of either $\mathbb{R}^2$ or $\mathbb{H}^-$. To deal with the heat equation it is useful to consider the anisotropic Sobolev spaces $W_p^{k,2k}$. We call them **parabolic Sobolev spaces** and denote them by $\mathcal{W}^{k,p}$. For constants $p \geq 1$ and integers $k \geq 0$ these spaces are defined as follows. Set $\mathcal{W}^{0,p} = L^p$ and denote by $\mathcal{W}^{1,p}$ the set of all $u \in L^p$ which admit weak derivatives $\partial_s u$, $\partial_t u$, and $\partial_t\partial_t u$ in L^p. For $k \geq 2$ define $\mathcal{W}^{k,p} := \{u \in \mathcal{W}^{1,p} \mid \partial_s u, \partial_t u, \partial_t\partial_t u \in \mathcal{W}^{k-1,p}\}$ where the derivatives are again meant in the weak sense. The norm

$$\|u\|_{\mathcal{W}^{k,p}} := \left(\int\int \sum_{2\nu+\mu\leq 2k} |\partial_s^\nu \partial_t^\mu u(s,t)|^p \,dtds\right)^{1/p} \tag{24}$$

gives $\mathcal{W}^{k,p}$ the structure of a Banach space. Here ν and μ are nonnegative integers. Note the difference to (standard) Sobolev space $W^{k,p}$ where the norm is given by $\|u\|^p_{k,p} := \sum_{\nu+\mu\leq k}\|\partial_s^\nu\partial_t^\mu u\|_p^p$. A **rectangular domain** is a set of the form $I \times J$ where I and J are bounded intervals. For rectangular (more generally, Lipschitz) domains Ω the parabolic Sobolev spaces $\mathcal{W}^{k,p}$ can be identified with the closure of $C^\infty(\overline{\Omega})$ with respect to the $\mathcal{W}^{k,p}$ norm; see e.g. [4, app. B.1].

Theorem A.1 (Local regularity) *Fix a constant $1 < q < \infty$, an integer $k \geq 0$, and an open subset $\Omega \subset \mathbb{H}^-$. Then the following is true.*

(a) *If $u \in L^1_{loc}(\Omega)$ and $f \in \mathcal{W}^{k,q}_{loc}(\Omega)$ satisfy*

$$\int_\Omega u\,(-\partial_s\phi - \partial_t\partial_t\phi) = \int_\Omega f\phi \tag{25}$$

for every $\phi \in C_0^\infty(\operatorname{int}\Omega)$, then $u \in \mathcal{W}^{k+1,q}_{loc}(\Omega)$. Here $\operatorname{int}\Omega$ denotes the interior of the set Ω.

(b) *If* $u \in L^1_{loc}(\Omega)$ *and* $f, h \in \mathcal{W}^{k,q}_{loc}(\Omega)$ *satisfy*

$$\int_\Omega u\,(-\partial_s\phi - \partial_t\partial_t\phi) = \int_\Omega f\phi - \int_\Omega h\,\partial_t\phi \tag{26}$$

for every $\phi \in C_0^\infty(\operatorname{int}\Omega)$*, then* u *and* $\partial_t u$ *are in* $\mathcal{W}^{k,q}_{loc}(\Omega)$.

Part (b) of the theorem will be used to prove the regularity theorem 3.1. While theorem A.1 is well-known it is hard to find in the literature in the $1+1$ dimensional setting at hand. For an elementary, though lengthy, proof via standard techniques we refer to [11, 12]. That proof is based on parabolic analogues of the Calderon–Zygmund inequality and the Weyl lemma[3].

Lemma A.2 (Parabolic Weyl lemma) *Let* $\Omega \subset \mathbb{H}^-$ *be an open subset. If* $u \in L^1_{loc}(\Omega)$ *satisfies*

$$\int_\Omega u\,(-\partial_s\phi - \partial_t\partial_t\phi) = 0 \tag{27}$$

for every $\phi \in C_0^\infty(\operatorname{int}\Omega)$*, then* $u \in C^\infty(\Omega)$ *and* $\partial_s u - \partial_t\partial_t u = 0$ *on* Ω.

P r o o f. The proof is based on approximating u via convolution by a family of smooth solutions u_ε converging to u in L^1; see [12] for more details. The point is that convolution is carried out over *individual time slices* for almost all times s using mollifiers defined on $\mathbb{R}$. On the other hand, given any integer $k \geq 0$, standard local C^k estimates for smooth solutions of the linear homogeneous heat equation in terms of the L^1 norm apply; see [2, sec. 2.3 thm. 9]. They provide C^k bounds on compact sets in terms of $\|u_\varepsilon\|_1$. But $\|u_\varepsilon\|_1 \leq \|u\|_1$ by Young's convolution inequality. Hence these C^k bounds are uniform in ε and therefore, by Arzela-Ascoli, the family u_ε converges in $C^{k-1}_{loc}(\Omega)$ to a map v. It follows that $u = v$ by uniqueness of the limit. As this is true for every k and, moreover, every point is contained in a compact subset of Ω it follows that $u \in C^\infty(\Omega)$. Integration by parts then proves the identity

$$\partial_s u - \partial_t\partial_t u = 0 \tag{28}$$

on the interior of Ω. It continues to hold on Ω, since u is C^∞ smooth on Ω. □

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[3] The naming Weyl lemma is standard, but also incorrect; see [1].