The heat flow and the homology of the loop space

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Abstract

We use the heat flow on the loop space of a closed Riemannian manifold to construct an algebraic chain complex. The chain groups are generated by perturbed closed geodesics. The boundary operator is defined in the spirit of Floer theory by counting, modulo time shift, heat flow trajectories that converge asymptotically to nondegenerate closed geodesics of Morse index difference one.

Contents

1 Introduction

Let M be a closed Riemannian manifold and denote by ∇ the Levi-Civita connection and by $\mathcal{L}M$ the *loop space*, that is the space of free loops $C^\infty(S^1, M)$. For $x : S^1 \to M$ consider the action functional

$$\mathcal{S}_V(x) = \int_0^1 \left(\frac{1}{2} |\dot{x}(t)|^2 - V(t, x(t)) \right) dt.$$

Here and throughout we identify $S^1 = \mathbb{R}/\mathbb{Z}$ and think of $x \in \mathcal{L}M$ as a smooth map $x : \mathbb{R} \to M$ which satisfies $x(t+1) = x(t)$. *Smooth* means C^∞ smooth. The potential is a smooth function $V : S^1 \times M \to \mathbb{R}$ and we set $V_t(q) := V(t, q)$. The critical points of $\mathcal{S}_V$ are the 1-periodic solutions of the ODE

$$\nabla_t \dot{x} = -\nabla V_t(x), \tag{1}$$

where ∇V_t denotes the gradient and $\nabla_t \dot{x}$ denotes the covariant derivative, with respect to the Levi-Civita connection, of the vector field $\dot{x} := \frac{d}{dt} x$ along the loop x in direction $\dot{x}$. By $\mathcal{P} = \mathcal{P}(V)$ we denote the set of 1-periodic solutions of (1). In the case $V = 0$ these are the closed geodesics.

From now on we assume that $\mathcal{S}_V$ is a *Morse function* on the loop space, i.e. the 1-periodic solutions of (1) are all nondegenerate. We proved in [W02] that this holds for a generic potential V. In this case the set

$$\mathcal{P}^a(V) := \{x \in \mathcal{P}(V) \mid \mathcal{S}_V(x) \le a\}$$

is finite for every real number a. Now consider the $\mathbb{Z}$-module

$$C_*^a = C_*^a(V) := \bigoplus_{x \in \mathcal{P}^a(V)} \mathbb{Z}x.$$

It is graded by the Morse indices of the closed geodesics. Moreover, this module carries a boundary operator whenever $\mathcal{S}_V$ is Morse–Smale. To define the boundary operator consider the (negative) L^2 gradient flow lines of $\mathcal{S}_V$ on the loop space. These are solutions $u : \mathbb{R} \times S^1 \to M$ of the *heat equation*

$$\partial_s u - \nabla_t \partial_t u - \nabla V_t(u) = 0 \tag{2}$$

satisfying

$$\lim_{s \to \pm\infty} u(s, t) = x^\pm(t), \qquad \lim_{s \to \pm\infty} \partial_s u(s, t) = 0, \tag{3}$$

where $x^\pm \in \mathcal{P}(V)$. The limits are uniform in t together with the first partial t-derivative, that is in $C^1(S^1)$; see remark 1.4. The space of solutions of (2) and (3) will be denoted by $\mathcal{M}(x^-, x^+; V)$. Call $\mathcal{S}_V$ *Morse–Smale* if the operator $\mathcal{D}_u$ obtained by linearizing (2) is onto as a linear operator between appropriate Banach spaces (see (12) below) and this is true for all $u \in \mathcal{M}(x^-, x^+; V)$ and $x^\pm \in \mathcal{P}(V)$. Note that Morse–Smale implies Morse. Under the Morse–Smale hypothesis the space $\mathcal{M}(x^-, x^+; V)$ is a smooth manifold whose dimension is equal to the difference of the Morse indices of the closed geodesics $x^{\pm}$. In the case of index difference one it follows that the quotient $\mathcal{M}(x^-, x^+; V)/\mathbb{R}$ by the (free) time shift action is a finite set. Counting these elements with appropriate signs defines a boundary operator on $C_*^a(V)$. Call the homology $\mathrm{HM}_*^a(\mathcal{L}M, \mathcal{S}_V)$ of this chain complex *Morse homology* or *heat flow homology*. It is naturally isomorphic to the singular homology of the free loop space for every regular value a of $\mathcal{S}_V$:

$$\mathrm{HM}_*^a(\mathcal{L}M, \mathcal{S}_V) \simeq \mathrm{H}_*(\mathcal{L}^a M; \mathbb{Z}), \qquad \mathcal{L}^a M := \{x \in \mathcal{L}M \mid \mathcal{S}_V(x) \leq a\}.$$

It is an open question if $\mathcal{S}_V$ is Morse–Smale for a generic potential V_t. In section 1.1 below we introduce a class of *abstract* perturbations $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ for which we can establish transversality. In contrast we call the potentials V_t *geometric* perturbations. The isomorphism above follows from the corresponding isomorphism for abstract perturbations by approximation.

The construction of the Morse complex in finite dimensions goes back to Thom [T49], Smale [Sm60, Sm61], and Milnor [M65]. It was rediscovered by Witten [Wi82] and extended to infinite dimensions by Floer [F89a, F89b]. We refer to [AM05] for an extensive historical account.

1.1 Perturbations

We introduce a class of abstract perturbations of equation (6) for which the analysis works. Later in section 7.1 we extract a countable subset and construct a separable Banach space of perturbations for which transversality works. The abstract perturbations take the form of smooth maps $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$. For $x \in \mathcal{L}M$ let $\mathrm{grad}\mathcal{V}(x) \in \Omega^0(S^1, x^*TM)$ denote the L^2-gradient of $\mathcal{V}$; it is defined by

$$\int_0^1 \langle \mathrm{grad}\mathcal{V}(u), \partial_s u \rangle \, dt = \frac{d}{ds}\mathcal{V}(u)$$

for every smooth path $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot)$. The *covariant Hessian of* $\mathcal{V}$ at a loop $x : S^1 \to M$ is the operator

$$\mathcal{H}_{\mathcal{V}}(x) : \Omega^0(S^1, x^*TM) \to \Omega^0(S^1, x^*TM)$$

defined by

$$\mathcal{H}_{\mathcal{V}}(u)\partial_s u := \nabla_s \mathrm{grad}\mathcal{V}(u) \tag{4}$$

for every smooth map $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot)$. The axiom (V1) below asserts that this Hessian is a zeroth order operator. We impose the following conditions on $\mathcal{V}$; here $|\cdot|$ denotes the pointwise absolute value at $(s,t) \in \mathbb{R} \times S^1$ and $\|\cdot\|_{L^p}$ denotes the L^p-norm over S^1 at time s. Although condition (V1) and the first part of (V2) are special cases of (V3) we state the axioms in the form below, because some of our results don't require all the conditions to hold.

(V0) $\mathcal{V}$ is continuous with respect to the C^0 topology on $\mathcal{L}M$. Moreover, there is a constant $C = C(\mathcal{V})$ such that

$$\sup_{x \in \mathcal{L}M} |\mathcal{V}(x)| + \sup_{x \in \mathcal{L}M} \|\mathrm{grad}\mathcal{V}(x)\|_{L^\infty(S^1)} \leq C.$$

(V1) There is a constant $C = C(\mathcal{V})$ such that

$$|\nabla_s \mathrm{grad}\mathcal{V}(u)| \le C\big(|\partial_s u| + \|\partial_s u\|_{L^1}\big),$$
$$|\nabla_t \mathrm{grad}\mathcal{V}(u)| \le C\Big(1 + |\partial_t u|\Big)$$

for every smooth map $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s,\cdot)$ and every $(s,t) \in \mathbb{R} \times S^1$.

(V2) There is a constant $C = C(\mathcal{V})$ such that

$$|\nabla_s \nabla_s \mathrm{grad}\mathcal{V}(u)| \le C\Big(|\nabla_s \partial_s u| + \|\nabla_s \partial_s u\|_{L^1} + \big(|\partial_s u| + \|\partial_s u\|_{L^2}\big)^2\Big),$$
$$|\nabla_t \nabla_s \mathrm{grad}\mathcal{V}(u)| \le C\Big(|\nabla_t \partial_s u| + \big(1 + |\partial_t u|\big)\big(|\partial_s u| + \|\partial_s u\|_{L^1}\big)\Big),$$

and

$$|\nabla_s \nabla_s \mathrm{grad}\mathcal{V}(u) - \mathcal{H}_{\mathcal{V}}(u)\nabla_s \partial_s u| \le C\big(|\partial_s u| + \|\partial_s u\|_{L^2}\big)^2$$

for every smooth map $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s,\cdot)$ and every $(s,t) \in \mathbb{R} \times S^1$.

(V3) For any two integers $k > 0$ and $\ell \ge 0$ there is a constant $C = C(k, \ell, \mathcal{V})$ such that

$$\left|\nabla_t^\ell \nabla_s^k \mathrm{grad}\mathcal{V}(u)\right| \le C \sum_{k_j,\ell_j} \left(\prod_{\substack{j \\ \ell_j > 0}} \left|\nabla_t^{\ell_j} \nabla_s^{k_j} u\right| \right) \prod_{\substack{j \\ \ell_j = 0}} \left(|\nabla_s^{k_j} u| + \left\|\nabla_s^{k_j} u\right\|_{L^{p_j}} \right)$$

for every smooth map $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s,\cdot)$ and every $(s,t) \in \mathbb{R} \times S^1$; here $p_j \ge 1$ and $\sum_{\ell_j = 0} 1/p_j = 1$; the sum runs over all partitions $k_1 + \dots + k_m = k$ and $\ell_1 + \dots + \ell_m \le \ell$ such that $k_j + \ell_j \ge 1$ for all j. For $k = 0$ the same inequality holds with an additional summand C on the right.

Remark 1.1. In (V0) the L^∞ bound for $\mathrm{grad}\,\mathcal{V}$ is imposed, since occasionally we need L^p bounds for fixed but arbitrary p. Continuity of $\mathcal{V}$ with respect to the C^0 topology is used to prove [SW03, lem. 10.2] and proposition 4.14.

Remark 1.2. Each geometric potential V provides an abstract perturbation $\mathcal{V}$ such that for smooth loops x and smooth vector fields ξ along x we have

$$\mathcal{V}(x) := \int_0^1 V_t(x(t))\, dt, \qquad \mathrm{grad}\mathcal{V}(x) = \nabla V_t(x), \qquad \mathcal{H}_{\mathcal{V}}(x)\xi = \nabla_\xi \nabla V_t(x).$$

Remark 1.3. To prove transversality in section 7 we use perturbations[1] of the form

$$\mathcal{V}(x) := \rho\left(\|x - x_0\|_{L^2}^2\right) \int_0^1 V_t(x(t))\, dt,$$

where $\rho : \mathbb{R} \to [0,1]$ is a smooth cutoff function and $x_0 : S^1 \to M$ is a smooth loop. Any such perturbation satisfies (V0)–(V3). Here compactness of M is crucial, in particular, finiteness of the diameter of M.

[1]Here and throughout the difference $x - x_0$ of two loops denotes the difference in some ambient Euclidean space into which M is (isometrically) embedded. Note that cutting off with respect to the L^2 norm – as opposed to the L^∞ norm – prevents us from expressing the difference in terms of the exponential map.

1.2 Main results

There are two main purposes of this text. One is to construct the Morse chain complex for the action functional on the loop space. The other is to provide proofs of the results announced and used in [SW03] to calculate the adiabatic limit of the Floer complex of the cotangent bundle. More precisely, in [SW03] we proved in joint work with D. Salamon that the connecting orbits of the heat flow are the adiabatic limit of Floer connecting orbits in the cotangent bundle with respect to the Hamiltonian given by kinetic plus potential energy. The key idea is to appropriately rescale the metric. Both purposes are achieved simultaneously by theorems 1.5–1.13.

We enlist the main results. From now on we replace the potential V by an abstract perturbation $\mathcal{V}$ satisfying (V0)–(V3). Then the action is given by

$$\mathcal{S}_{\mathcal{V}}(x) = \frac{1}{2}\int_0^1 |\dot{x}(t)|^2 \, dt - \mathcal{V}(x) \tag{5}$$

for smooth loops $x : S^1 \to M$ and the heat equation has the form

$$\partial_s u - \nabla_t \partial_t u - \mathrm{grad}\mathcal{V}(u) = 0 \tag{6}$$

for smooth maps $u : \mathbb{R} \times S^1 \to M$, $(s,t) \mapsto u(s,t)$. Here $\mathrm{grad}\mathcal{V}(u)$ denotes the value of $\mathrm{grad}\mathcal{V}$ on the loop $u_s : t \mapsto u(s,t)$. The relevant set $\mathcal{P}(\mathcal{V})$ of critical points of $\mathcal{S}_{\mathcal{V}}$ consists of the (smooth) loops $x : S^1 \to M$ that satisfy the ODE

$$\nabla_t \dot{x} = -\mathrm{grad}\mathcal{V}(x). \tag{7}$$

The subset $\mathcal{P}^a(\mathcal{V})$ consists of all critical points x with $\mathcal{S}_{\mathcal{V}}(x) \le a$. For two nondegenerate critical points $x^\pm \in \mathcal{P}(\mathcal{V})$ we denote by $\mathcal{M}(x^-, x^+; \mathcal{V})$ the set of all solutions u of (6) such that

$$\lim_{s\to\pm\infty} u(s,t) = x^\pm(t), \qquad \lim_{s\to\pm\infty} \partial_s u(s,t) = 0. \tag{8}$$

The limits are uniform in t together with the first partial t-derivative. These solutions are called *connecting orbits*. The energy of such a solution is given by

$$E(u) = \int_{-\infty}^{\infty}\int_0^1 |\partial_s u|^2 \, dtds = \mathcal{S}_{\mathcal{V}}(x^-) - \mathcal{S}_{\mathcal{V}}(x^+). \tag{9}$$

Remark 1.4 (Asymptotic limits)**.** In (3) and (8) we require convergence in $C^1(S^1)$ as opposed to $C^0(S^1)$ which is standard in elliptic Floer theory. We need our stronger assumption in theorem 3.10 to establish exponential decay. Actually $W^{1,2}(S^1)$ convergence already works. Compare [SW03] where the asymptotic C^0 limits of (u,v) and $(\partial_s u, \nabla_s v)$ are required to be $(x^\pm, \partial_t x^\pm)$ and zero, respectively. Now v corresponds to $\partial_t u$ in the adiabatic limit studied in [SW03] and we arrive at our choice of topology for the asymptotic limits.

Theorem 1.5 (Regularity)**.** *Fix a constant $p > 2$ and a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V3)*. Let $u : \mathbb{R} \times S^1 \to M$ be a continuous function of class $\mathcal{W}^{1,p}_{loc}$, that is $u, \partial_t u, \nabla_t \partial_t u, \partial_s u$ are locally L^p integrable. Assume further that u solves the heat equation (6) almost everywhere. Then u is smooth.*

Remark 1.6. It seems unlikely that the assumption $u \in \mathcal{W}^{1,p}_{loc}$ can be weakened to $u \in W^{1,p}_{loc}$, as announced in [SW03], unless we also weaken $p > 2$ to $p > 3$; see remark 2.19. However, in the applications of theorem 1.5 in [SW03] and in the present text the stronger assumption $u \in \mathcal{W}^{1,p}_{loc}$ is satisfied.

Theorem 1.7 (Apriori estimates)**.** *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V1) *and a constant c_0. Then there is a positive constant $C = C(c_0, \mathcal{V})$ such that the following holds. If $u : \mathbb{R} \times S^1 \to M$ is a smooth solution of (6) such that $\mathcal{S}_\mathcal{V}(u(s,\cdot)) \le c_0$ for every $s \in \mathbb{R}$ then*

$$\|\partial_t u\|_\infty + \|\nabla_t \partial_t u\|_\infty + \|\partial_s u\|_\infty + \|\nabla_t \partial_s u\|_\infty + \|\nabla_s \partial_s u\|_\infty \le C.$$

Theorem 1.8 (Exponential decay)**.** *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V3) *and assume $\mathcal{S}_\mathcal{V}$ is Morse.*

(F) *Let $u : [0,\infty) \times S^1 \to M$ be a smooth solution of (6). Then there are positive constants ρ and $c_0, c_1, c_2, \ldots$ such that*

$$\|\partial_s u\|_{C^k([T,\infty)\times S^1)} \le c_k e^{-\rho T}$$

for every $T \ge 1$. Moreover, there is a periodic orbit $x \in \mathcal{P}(\mathcal{V})$ such that $u(s,\cdot)$ converges to x in $C^2(S^1)$ as $s \to \infty$.

(B) *Let $u : (-\infty, 0] \times S^1 \to M$ be a smooth solution of (6) with finite energy. Then there are positive constants ρ and $c_0, c_1, c_2, \ldots$ such that*

$$\|\partial_s u\|_{C^k((-\infty,-T]\times S^1)} \le c_k e^{-\rho T}$$

for every $T \ge 1$. Moreover, there is a periodic orbit $x \in \mathcal{P}(\mathcal{V})$ such that $u(s,\cdot)$ converges to x in $C^2(S^1)$ as $s \to -\infty$.

The *covariant Hessian of* $\mathcal{S}_\mathcal{V}$ at a loop $x : S^1 \to M$ is the linear operator $A_x : W^{2,2}(S^1, x^*TM) \to L^2(S^1, x^*TM)$ given by

$$A_x \xi := -\nabla_t \nabla_t \xi - R(\xi, \dot{x})\dot{x} - \mathcal{H}_\mathcal{V}(x)\xi \tag{10}$$

where R denotes the Riemannian curvature tensor and the Hessian $\mathcal{H}_\mathcal{V}$ is defined by (4). This operator is self-adjoint with respect to the standard L^2 inner product. The number of negative eigenvalues is finite. It is denoted by $\mathrm{ind}_\mathcal{V}(A_x)$ and called the Morse index of A_x. If x is a critical point of $\mathcal{S}_\mathcal{V}$ we define its *Morse index* by $\mathrm{ind}_\mathcal{V}(x) := \mathrm{ind}_\mathcal{V}(A_x)$ and we call x *nondegenerate* if A_x is bijective. In this notation the linearized operator $\mathcal{D}_u : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u$ is given by

$$\mathcal{D}_u \xi := \nabla_s \xi + A_{u_s} \xi \tag{11}$$

where $u_s(t) := u(s,t)$ and the spaces $\mathcal{W}_u = \mathcal{W}^{1,p}_u$ and $\mathcal{L}_u = \mathcal{L}^p_u$ are defined as the completions of the space of smooth compactly supported sections of the pullback tangent bundle $u^*TM \to \mathbb{R} \times S^1$ with respect to the norms

$$\begin{aligned} \|\xi\|_\mathcal{L} &= \left(\int_{-\infty}^{\infty} \int_0^1 |\xi|^p \, dt ds \right)^{1/p}, \\ \|\xi\|_\mathcal{W} &= \left(\int_{-\infty}^{\infty} \int_0^1 |\xi|^p + |\nabla_s \xi|^p + |\nabla_t \nabla_t \xi|^p \, dt ds \right)^{1/p}. \end{aligned} \tag{12}$$

Theorem 1.9 (Fredholm)**.** *Fix a perturbation* $\mathcal{V}:\mathcal{L}M\to\mathbb{R}$ *that satisfies* (V0)–(V3)*, a constant* $p>1$*, and two nondegenerate critical points* $x^\pm\in\mathcal{P}(\mathcal{V})$*. Assume* $u:\mathbb{R}\times S^1\to M$ *is a smooth map such that* $\|\nabla_t\nabla_t\partial_s u_s\|_2$ *is bounded, uniformly in* $s\in\mathbb{R}$*, and*

$$u_s=\exp_{x^\pm}(\eta_s^\pm),\quad \left\|\eta_s^\pm\right\|_{W^{2,2}}\to 0,\quad \|\partial_s u_s\|_{W^{1,2}}\to 0,\quad \text{as } s\to\pm\infty.$$

Then the operator $\mathcal{D}_u:\mathcal{W}_u^{1,p}\to\mathcal{L}_u^p$ *is Fredholm and*

$$\operatorname{index}\mathcal{D}_u=\operatorname{ind}_{\mathcal{V}}(x^-)-\operatorname{ind}_{\mathcal{V}}(x^+).$$

Moreover, the formal adjoint operator $\mathcal{D}_u^*=-\nabla_s+A_{u_s}:\mathcal{W}_u^{1,p}\to\mathcal{L}_u^p$ *is Fredholm with* $\operatorname{index}\mathcal{D}_u^*=-\operatorname{index}\mathcal{D}_u$*.*

Concerning the funny assumption on $\nabla_t\nabla_t\partial_s u_s$ see the footnote in section 3.4.

Theorem 1.10 (Implicit function theorem)**.** *Fix a perturbation* $\mathcal{V}:\mathcal{L}M\to\mathbb{R}$ *that satisfies* (V0)–(V3)*. Assume* $x^\pm$ *are nondegenerate critical points of* $\mathcal{S}_{\mathcal{V}}$ *and* $\mathcal{D}_u$ *is onto for every* $u\in\mathcal{M}(x^-,x^+;\mathcal{V})$*. Then* $\mathcal{M}(x^-,x^+;\mathcal{V})$ *is a smooth manifold of dimension* $\operatorname{ind}_{\mathcal{V}}(x^-)-\operatorname{ind}_{\mathcal{V}}(x^+)$*.*

Proposition 1.11 (Finite set)**.** *Fix a perturbation* $\mathcal{V}:\mathcal{L}M\to\mathbb{R}$ *that satisfies* (V0)–(V3) *and assume* $\mathcal{S}_{\mathcal{V}}$ *is* Morse–Smale below level *a in the sense that every* $u\in\mathcal{M}(x^-,x^+;\mathcal{V})$ *is* regular *(i.e. the Fredholm operator* $\mathcal{D}_u$ *is surjective), for every pair* $x^\pm\in\mathcal{P}^a(\mathcal{V})$*. Then the quotient space*

$$\widehat{\mathcal{M}}(x^-,x^+;\mathcal{V}):=\mathcal{M}(x^-,x^+;\mathcal{V})/\mathbb{R}$$

is a finite set for every such pair of Morse index difference one. Here the (free) action of $\mathbb{R}$ *is given by time shift* $(\sigma,u)\mapsto u(\sigma+\cdot,\cdot)$*.*

Theorem 1.12 (Refined implicit function theorem)**.** *Fix a perturbation* $\mathcal{V}:\mathcal{L}M\to\mathbb{R}$ *that satisfies* (V0)–(V3) *and a pair of nondegenerate critical points* $x^\pm\in\mathcal{P}(\mathcal{V})$ *with* $\mathcal{S}_{\mathcal{V}}(x^+)<\mathcal{S}_{\mathcal{V}}(x^-)$ *and Morse index difference one. Then, for every* $p>2$ *and every large constant* $c_0>1$*, there are positive constants* δ_0 *and* c *such that the following holds. Assume* $\mathcal{S}_{\mathcal{V}}$ *is Morse–Smale below level* $2c_0^2$*. Assume further that* $u:\mathbb{R}\times S^1\to M$ *is a smooth map such that* $u(s,\cdot)$ *converges in* $W^{1,2}(S^1)$ *to* $x^\pm$*, as* $s\to\pm\infty$*, and such that*

$$|\partial_s u(s,t)|\le\frac{c_0}{1+s^2},\qquad |\partial_t u(s,t)|\le c_0,\qquad |\nabla_t\partial_t u(s,t)|\le c_0,$$

for all $(s,t)\in\mathbb{R}\times S^1$ *and*

$$\|\partial_s u-\nabla_t\partial_t u-\operatorname{grad}\mathcal{V}(u)\|_p\le\delta_0.$$

Then there exist elements $u_*\in\mathcal{M}(x^-,x^+;\mathcal{V})$ *and* $\xi\in\operatorname{im}\mathcal{D}_{u_*}^*\cap\mathcal{W}$ *satisfying*

$$u=\exp_{u_*}(\xi),\qquad \|\xi\|_{\mathcal{W}}\le c\,\|\partial_s u-\nabla_t\partial_t u-\operatorname{grad}\mathcal{V}(u)\|_p\,.$$

In the previous theorem "c_0 large" means that the constant c_0 should be larger than the constant C_0 in axiom (V0).

Theorem 1.13 (Transversality)**.** *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V3) *and assume $\mathcal{S}_\mathcal{V}$ is Morse. Then for every regular value a there is a Banach manifold $\mathcal{O}^a = \mathcal{O}^a(\mathcal{V})$ of perturbations supported on the sublevel set $\{\mathcal{S}_\mathcal{V} \le a\}$ and satisfying* (V0)–(V3) *such that the following is true. For every $v \in \mathcal{O}^a$ the functionals $\mathcal{S}_\mathcal{V}$ and $\mathcal{S}_{\mathcal{V}+v}$ have the same critical points on $\mathcal{L}M$ and the same sublevel set with respect to a. Moreover, there is a residual subset $\mathcal{O}^a_{reg} \subset \mathcal{O}^a$ such that the perturbed functional $\mathcal{S}_{\mathcal{V}+v}$ is Morse–Smale below level a whenever $v \in \mathcal{O}^a_{reg}$.*

Contrary to what we expected in view of excellent previous work by Abbondandolo and Majer [AM05, AM06] on C^1 flows on Banach manifolds, we do not get the natural isomorphism between heat flow homology and singular homology of the loop space as a simple byproduct. The reason is that the heat equation generates only a C^1 *semi*flow on the loop space. To overcome the problems we propose to generalize the notion of Conley index pairs to our infinite dimensional situation. In section 9.2 we give a detailed sketch of how to prove the isomorphism in theorem 1.14. Full details will be provided in a forthcoming paper. In this sense theorem 1.14 should be understood as an announcement.

Theorem 1.14. *Let $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ be a perturbation that satisfies* (V0)–(V3)*, let c be a regular value of $\mathcal{S}_\mathcal{V}$, and assume that $\mathcal{S}_\mathcal{V}$ is Morse–Smale below level c. Then, for every regular value $a < c$ of $\mathcal{S}_\mathcal{V}$ and every principal ideal domain R, there is a natural isomorphism*

$$\mathrm{HM}^a_*(\mathcal{L}M, \mathcal{S}_\mathcal{V}; R) \cong \mathrm{H}_*(\mathcal{L}^a M; R), \qquad \mathcal{L}^a M := \{x \in \mathcal{L}M \mid \mathcal{S}_\mathcal{V}(x) \le a\}.$$

If M is not simply connected then there is a separate isomorphism for each component of the loop space. The isomorphism commutes with the homomorphisms $\mathrm{HM}^a_*(\mathcal{L}M, \mathcal{S}_\mathcal{V}) \to \mathrm{HM}^b_*(\mathcal{L}M, \mathcal{S}_\mathcal{V})$ and $\mathrm{H}_*(\mathcal{L}^a M) \to \mathrm{H}_*(\mathcal{L}^b M)$ for $b \in (a, c)$.

Overview

In chapter 2 we prove local regularity and provide interior estimates for the linear heat equation $\partial_s u - \partial_t \partial_t u = 0$ for real-valued maps u defined on the lower half plane $\mathbb{H}^-$ or on cylindrical sets. The main result is theorem 2.1 on regularity of solutions to a perturbed heat equation. The proof is by parabolic bootstrapping and uses a subtle product estimate which is provided in section 2.3. The quadratic estimates required in our proof of the refined implicit function theorem 1.10 are also based on that product estimate.

In chapter 3 we study the solutions to the linearized version of the heat equation (6). In other words, the kernel of the operator $\mathcal{D}_u$ given by (11). In theorem 3.1 we show that these solutions are smooth, even if they are only weak solutions. In section 3.2 we derive pointwise bounds in terms of the L^2 norm. Section 3.3 then establishes exponential decay of these L^2 norms. The combination of these results is used in section 3.4 to prove that the operator

$\mathcal{D}_u$ is Fredholm for a rather general class of smooth cylinders u in M with nondegenerate asymptotic limits $x^{\pm} \in \mathcal{P}(\mathcal{V})$. The main result is theorem 1.9.

In chapter 4 we study the solutions u to the (nonlinear) heat equation (6). Since $\partial_s u$ solves the linearized equation the results of chapter 3 apply. In section 4.1 we prove smoothness of $\mathcal{W}^{1,p}_{loc}$ solutions and a compactness result for sequences with uniformly bounded gradient in appropriate norms. In sections 4.2–4.4 the following assumption is crucial. Fix a positive constant c_0. Then all solutions u of (6) with

$$\sup_{s\in\mathbb{R}} \mathcal{S}_{\mathcal{V}}(u_s) \le c_0$$

admit a uniform apriori estimate for $\|\partial_t u\|_\infty$ (theorem 4.5), uniform energy bounds (lemma 4.8), uniform gradient bounds (theorem 4.9), and uniform L^2 exponential decay (theorem 4.10). In section 4.5 we study compactness of the moduli spaces $\mathcal{M}(x^-, x^+; \mathcal{V})$ in case that $\mathcal{S}_{\mathcal{V}} : \mathcal{L}M \to \mathbb{R}$ is a Morse function.

Chapter 5 deals with implicit function type theorems. Here, in addition to the Morse condition, the Morse–Smale condition enters: To prove that the moduli spaces are smooth manifolds we not only need nondegeneracy of the asymptotic boundary data, that is the critical points $x^{\pm}$, but in addition surjectivity of the linearized operators. Under these assumptions we prove (proposition 1.11) that modulo time shift there are only finitely many heat flow lines from x^- to x^+ in case of Morse index difference one. Here the compactness results of section 4.5 enter. Furthermore, we prove the refined implicit function theorem 1.12, a major technical tool in [SW03]. Here the product estimate provided by lemma 2.14 is the crucial ingredient to obtain the required quadratic estimates. Furthermore, the choice of the sublevel set on which $\mathcal{S}_{\mathcal{V}}$ needs to be Morse–Smale requires care. The reason is that one starts out only with an *approximate* solution u along which the action is not necessarily decreasing. However, the assumptions guarantee that all loops u_s are contained in the sublevel set $\{\mathcal{S}_{\mathcal{V}} \le 2c_0^2\}$.

In chapter 6 we prove unique continuation for the heat equation (6) and its linearization. The proof is based on an extension of a result by Agmon and Nirenberg. In contrast to forward unique continuation the result on backward unique continuation is surprising at first sight. Of course, there is an assumption. Namely, the action along the two semi-infinite backward trajectories u, v which coincide at time $s = 0$ must be bounded. In this case we obtain that $u = v$.

In chapter 7 we construct a separable Banach space Y of abstract perturbations that satisfy axioms (V0)–(V3). Assume $\mathcal{S}_{\mathcal{V}}$ is Morse and a is a regular value. Then we define a Banach submanifold $\mathcal{O}^a(\mathcal{V})$ of admissible perturbations v supported in $\{\mathcal{S}_{\mathcal{V}} \le a\}$. They have the property that $\mathcal{S}_{\mathcal{V}}$ and $\mathcal{S}_{\mathcal{V}+v}$ do have the same critical points on the *whole* loop space $\mathcal{L}M$ and, moreover, their sublevel sets with respect to a coincide. The proof that there is a residual subset $\mathcal{O}^a_{reg}(\mathcal{V})$ of regular perturbations for which $\mathcal{S}_{\mathcal{V}+v}$ is Morse-Smale below level a requires unique continuation for the linearized heat equation and the fact that the action is strictly decreasing along nonconstant heat flow trajectories.

In chapter 8 we define Morse homology in terms of the heat flow. In section 8.1 we define the unstable manifold of a critical point x of the action functional $\mathcal{S}_\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ as the set of endpoints at time zero of all backward halfcylinders solving the heat equation (6) and emanating from x at $-\infty$. The main result is theorem 8.1 saying that if the critical point x is nondegenerate, then this is a contractible submanifold of the loop space and its dimension equals the Morse index of x. Section 8.2 puts together all results to construct the Morse complex for the negative L^2 gradient of the action functional on the loop space.

The aim of chapter 9 is to relate heat flow homology and singular homology of the loop space. The geometric idea is that only the unstable manifolds are relevant in homology, since all other loops move under the heat flow into a neighborhood of the unstable manifolds. To make this precise we solve in section 9.1 the forward time Cauchy problem for the heat equation (6) for initial values in the Hilbert manifold $\Lambda M = W^{1,2}(S^1, M)$. As a result we obtain a C^1 semiflow $\varphi : (0,\infty) \times \Lambda M \to \Lambda M$ which extends continuously to zero. In section 9.2 we sketch the proof of theorem 1.14.

Notation. If $f = f(s,t)$ is a map or more generally a section, then f_s abbreviates the map $f(s,\cdot) : t \mapsto f(s,t)$. In contrast partial derivatives are denoted by $\partial_s f$ and $\partial_t f$. By $\Omega^0(B,E)$ we denote the set of smooth sections of a vector bundle $E \to B$.

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2 Parabolic regularity

By $\mathbb{H}^-$ we denote the closed lower half plane, that is the set of pairs of reals (s,t) with $s \le 0$. In this section, unless specified differently, all maps are real-valued and the domains of the various Banach spaces which appear are understood to be either open subsets Ω of $\mathbb{R}^2$ or $\mathbb{H}^-$ or (cylindrical subsets of) the cylinder $Z = \mathbb{R}\times S^1$. To deal with the heat equation it is useful to consider the anisotropic Sobolev spaces $W_p^{k,2k}$. We call them *parabolic Sobolev spaces* and denote them by $\mathcal{W}^{k,p}$. For constants $p \ge 1$ and integers $k \ge 0$ these spaces are defined as follows. Set $\mathcal{W}^{0,p} = L^p$ and denote by $\mathcal{W}^{1,p}$ the set of all $u \in L^p$ which admit weak derivatives $\partial_s u$, $\partial_t u$, and $\partial_t\partial_t u$ in L^p. For $k \ge 2$ define

$$\mathcal{W}^{k,p} = \{u \in \mathcal{W}^{1,p} \mid \partial_s u, \partial_t u, \partial_t\partial_t u \in \mathcal{W}^{k-1,p}\}$$

where the derivatives are again meant in the weak sense. The norm

$$\|u\|_{\mathcal{W}^{k,p}} := \left(\int\int \sum_{2\nu+\mu\le 2k} |\partial_s^\nu \partial_t^\mu u(s,t)|^p \; dtds\right)^{1/p} \tag{13}$$

gives $\mathcal{W}^{k,p}$ the structure of a Banach space. Here ν and μ are nonnegative integers. For $k=1$ we obtain that

$$\|u\|_{\mathcal{W}^{1,p}}^p = \|u\|_p^p + \|\partial_s u\|_p^p + \|\partial_t u\|_p^p + \|\partial_t\partial_t u\|_p^p$$

and occasionally we abbreviate $\mathcal{W} = \mathcal{W}^{1,p}$. Note the difference to (standard) Sobolev space $W^{k,p}$ where the norm is given by

$$\|u\|_{k,p}^p = \sum_{\nu+\mu\le k} \|\partial_s^\nu \partial_t^\mu u\|_p^p .$$

A *rectangular domain* is a set of the form $I \times J$ where I and J are bounded intervals. For rectangular (or more generally Lipschitz) domains Ω the parabolic Sobolev spaces $\mathcal{W}^{k,p}$ can be identified with the closure of $C^\infty(\overline{\Omega})$ with respect to the $\mathcal{W}^{k,p}$ norm; see e.g. [MS04, appendix B.1]. Similarly, we define the $\mathcal{C}^k$ (or $\mathcal{W}^{k,\infty}$) norm by

$$\|u\|_{\mathcal{C}^k} := \sum_{2\nu+\mu\le 2k} \|\partial_s^\nu \partial_t^\mu u\|_\infty . \tag{14}$$

Again $\mathcal{C}^0 = C^0$ and

$$\|u\|_{\mathcal{C}^1} = \|u\|_\infty + \|\partial_s u\|_\infty + \|\partial_t u\|_\infty + \|\partial_t\partial_t u\|_\infty .$$

Throughout we use the notation

$$Z_T = (-T,0] \times S^1.$$

The main result in this section is the following.

Theorem 2.1 (Parabolic regularity)**.** *Fix constants $p > 2$, $\mu_0 > 1$, and $T > 0$. Fix a closed smooth submanifold $M \hookrightarrow \mathbb{R}^N$ and a smooth family of vector-valued symmetric bilinear forms $\Gamma : M \to \mathbb{R}^{N\times N\times N}$. Assume that $F : Z_T \to \mathbb{R}^N$ is a map of class L^p and $u : Z_T \to \mathbb{R}^N$ is a $\mathcal{W}^{1,p}$ map taking values in M with $\|u\|_{\mathcal{W}^{1,p}} \le \mu_0$ and such that the perturbed heat equation*

$$\partial_s u - \partial_t \partial_t u = \Gamma(u)\,(\partial_t u, \partial_t u) + F \tag{15}$$

is satisfied almost everywhere. Then the following is true. For every integer $k \ge 1$ such that $F \in \mathcal{W}^{k,p}(Z_T)$ and every $T' \in (0,T)$ there is a constant c_k depending on k, p, μ_0, $T - T'$, $\|\Gamma\|_{C^{2k+2}(M)}$, and $\|F\|_{\mathcal{W}^{k,p}(Z_T)}$ such that

$$\|u\|_{\mathcal{W}^{k+1,p}(Z_{T'})} \le c_k.$$

The theorem shows that if F is smooth, then u is smooth on domains which extend slightly less into the past. The refined version proposition 2.18 of this result is used in section 4.1 to prove regularity and compactness properties of solutions to the nonlinear heat equation (6). The proof of theorem 2.1 in section 2.4 is by parabolic bootstrapping. The main technical tool is the following.

Theorem 2.2 (Interior regularity)**.** *Fix constants $1 < q < \infty$ and $T > 0$ and an integer $k \ge 0$. Then the following is true.*

a) *If $u \in L^1_{loc}(Z_T)$ and $f \in \mathcal{W}^{k,q}_{loc}(Z_T)$ satisfy*

$$\int_{Z_T} u\,(-\partial_s\phi - \partial_t\partial_t\phi) = \int_{Z_T} f\phi \tag{16}$$

for every $\phi \in C_0^\infty((-T,0)\times S^1)$, then $u \in \mathcal{W}^{k+1,q}_{loc}(Z_T)$.

b) *For every $0 < T' < T$ there is a constant $c = c(k,q,T-T')$ such that*

$$\|u\|_{\mathcal{W}^{k+1,q}(Z_{T'})} \le c\left(\|\partial_s u - \partial_t\partial_t u\|_{\mathcal{W}^{k,q}(Z_T)} + \|u\|_{L^q(Z_T)}\right)$$

for every $u \in C^\infty(\overline{Z_T})$.

The prove part a) it is useful to prove in a first step the case where $f = 0$ and Ω is an open subset of the lower half plane. The corresponding statement for the Laplace operator is called *Weyl lemma*.

2.1 The parabolic Weyl lemma

Lemma 2.3. *Let $\Omega \subset \mathbb{H}^-$ be an open subset. If $u \in L^1_{loc}(\Omega)$ satisfies*

$$\int_\Omega u\,(-\partial_s\phi - \partial_t\partial_t\phi) = 0 \tag{17}$$

for every $\phi \in C_0^\infty(\operatorname{int}\Omega)$, then $u \in C^\infty(\Omega)$ and $\partial_s u - \partial_t\partial_t u = 0$ on Ω.

The proof of lemma 2.3 is based on approximating u via convolution by a family of smooth solutions u_ε converging to u in L^1. The point is that convolution is carried out over *individual time slices* for almost all times s using mollifiers defined on $\mathbb{R}$. (It is also possible to carry over the proof of the original Weyl lemma for the Laplacian using mollifiers supported in $\mathbb{R}^2$. However, this leads to restrictions and is explained in a separate section below.) On the other hand, given any integer $k \geq 0$, standard local C^k estimates for smooth solutions of the linear homogeneous heat equation in terms of the L^1 norm apply; see [Ev98, sec. 2.3 thm. 9]. They provide C^k bounds on compact sets in terms of $\|u_\varepsilon\|_1$. Now by Young's convolution inequality $\|u_\varepsilon\|_1 \leq \|u\|_1$. Hence these bounds are uniform in ε. Therefore by Arzela-Ascoli the family u_ε converges in $C^{k-1}_{loc}(\Omega)$ to a map v. Hence $u = v$ by uniqueness of the limit. As this is true for every k and, moreover, every point is contained in a compact subset of Ω it follows that $u \in C^\infty(\Omega)$. Integration by parts then shows that

$$\partial_s u - \partial_t \partial_t u = 0 \tag{18}$$

on the interior of Ω. Since u is C^∞ on Ω this identity continues to hold on Ω.

Proof of lemma 2.3. Every point of Ω is contained in (some translation of) a parabolic set $(-r^2, 0] \times (-r, r)$ whose closure is contained in Ω for some sufficiently small $r > 0$. Hence we may assume without loss of generality that $\Omega = (-r^2, 0] \times (-r, r)$ and $u \in L^1(\Omega)$. We prove the lemma in nine steps.

1) We introduce appropriate mollifiers: Fix a smooth function $\rho : \mathbb{R} \to [0, 1]$ which is compactly supported in the interval $(-1, 1)$ and satisfies $\int_\mathbb{R} \rho = 1$. For $\varepsilon > 0$ consider the mollifier $\rho_\varepsilon(t) := \frac{1}{\varepsilon}\rho\left(\frac{t}{\varepsilon}\right)$. It is compactly supported in the interval $(-\varepsilon, \varepsilon)$ and satisfies $\int_\mathbb{R} \rho_\varepsilon = 1$.

2) For almost every $s \in \mathbb{R}$ we define the family $\{\rho_\varepsilon * u_s\}_{\varepsilon>0} \subset C_0^\infty(\mathbb{R})$ and calculate the L^1 norm of its derivatives: From now on we extend u by zero to $\mathbb{R}^2 \setminus \Omega$ and denote the extension again by u. Then $u \in L^1(\mathbb{R}^2)$ and

$$u_s := u(s, \cdot) \in L^1(\mathbb{R})$$

for almost every $s \in \mathbb{R}$. For such s and $\varepsilon > 0$ consider the convolution

$$(\rho_\varepsilon * u_s)(t) := \int_\mathbb{R} \rho_\varepsilon(t - \tau) u_s(\tau)\, d\tau.$$

In this case $\rho_\varepsilon * u_s \in C_0^\infty(\mathbb{R})$,

$$\|\rho_\varepsilon * u_s - u_s\|_{L^1(\mathbb{R})} \to 0, \quad \text{as } \varepsilon \to 0,$$

and $\rho_\varepsilon * u_s$ converges to u_s, as $\varepsilon \to 0$, pointwise almost everywhere on $\mathbb{R}$; see [Jo98, app. A]. Moreover, by Young's convolution inequality we obtain that

$$\|\rho_\varepsilon * u_s\|_{L^1(\mathbb{R})} \leq \|\rho_\varepsilon\|_{L^1(\mathbb{R})} \|u_s\|_{L^1(\mathbb{R})} = \|u_s\|_{L^1(\mathbb{R})}$$

and, more generally, that

$$\left\| \frac{d^k}{dt^k} \left(\rho_\varepsilon * u_s \right) \right\|_{L^1(\mathbb{R})} = \left\| \left(\rho_\varepsilon^{(k)} * u_s \right) \right\|_{L^1(\mathbb{R})} \leq \left\| \rho_\varepsilon^{(k)} \right\|_{L^1(\mathbb{R})} \left\| u_s \right\|_{L^1(\mathbb{R})}$$
$$= \frac{\left\| \rho^{(k)} \right\|_{L^1(\mathbb{R})}}{\varepsilon^k} \left\| u_s \right\|_{L^1(\mathbb{R})}$$

for every positive integer k. Here $\rho^{(k)}$ denotes the k-th derivative of ρ.

3) We prove that for $\varepsilon > 0$ the function defined by

$$u_\varepsilon : \mathbb{R}^2 \to \mathbb{R}, \quad (s,t) \mapsto (\rho_\varepsilon * u_s)(t)$$

is integrable and u_ε converges to u in $L^1(\mathbb{R}^2)$, as $\varepsilon \to 0$. Indeed by step 2)

$$\left\| u_\varepsilon \right\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}} \left\| \rho_\varepsilon * u_s \right\|_{L^1(\mathbb{R})} ds \leq \int_{\mathbb{R}} \left\| u_s \right\|_{L^1(\mathbb{R})} ds = \left\| u \right\|_{L^1(\Omega)} .$$

Now define the family of functions $\{f_\varepsilon : \mathbb{R} \to \mathbb{R}\}_{\varepsilon > 0}$ for almost every s by

$$f_\varepsilon(s) := \left\| \rho_\varepsilon * u_s - u_s \right\|_{L^1(\mathbb{R})} .$$

By the former estimate these functions are integrable

$$\left\| f_\varepsilon \right\|_{L^1(\mathbb{R})} = \left\| u_\varepsilon - u \right\|_{L^1(\mathbb{R}^2)} \leq 2 \left\| u \right\|_{L^1(\Omega)} .$$

Moreover, they are dominated almost everywhere by an integrable function g. Namely, by step 2

$$|f_\varepsilon(s)| \leq 2 \left\| u_s \right\|_{L^1(\mathbb{R})} =: g(s), \qquad \left\| g \right\|_{L^1(\mathbb{R})} = 2 \left\| u \right\|_{L^1(\Omega)} .$$

Again step 2) shows that $f_\varepsilon \to 0$, as $\varepsilon \to 0$, for almost every s. Hence by the dominated convergence theorem it follows that

$$\begin{aligned} \lim_{\varepsilon \to 0} \left\| u_\varepsilon - u \right\|_{L^1(\mathbb{R}^2)} &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \left\| \rho_\varepsilon * u_s - u_s \right\|_{L^1(\mathbb{R})} \, ds \\ &= \int_{\mathbb{R}} \left(\lim_{\varepsilon \to 0} f_\varepsilon \right)(s) \, ds \\ &= 0. \end{aligned}$$

4) The function $u_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ defined in 3) admits integrable weak t-derivatives of all orders. Fix $\varepsilon > 0$ and a positive integer k, then

$$\begin{aligned} \int_{\mathbb{R}^2} u_\varepsilon \, \partial_t^k \psi \, dt \, ds &= \int_{\mathbb{R}^2} (\rho_\varepsilon * u_s) \, \partial_t^k \psi \, dt \, ds \\ &= (-1)^k \int_{\mathbb{R}^2} (\rho_\varepsilon^{(k)} * u_s) \, \psi \, dt \, ds \end{aligned}$$

for every $\psi \in C_0^\infty(\mathbb{R}^2)$. Here $\rho_\varepsilon^{(k)}$ denotes the k-th derivative. Moreover, the first step is by definition of u_ε and the second step by integration by parts followed by commuting differentiation and convolution. Next observe that the function $\rho_\varepsilon^{(k)} * u_s$ is integrable. Indeed step 2) shows that

$$\int_{\mathbb{R}} \left\| \rho_\varepsilon^{(k)} * u_s \right\|_{L^1(\mathbb{R})} ds \le \frac{c_k}{\varepsilon^k} \left\| u \right\|_{L^1(\Omega)}$$

with constant $c_k = c_k(\rho) = \|\partial_t^k \rho\|_{L^1(\mathbb{R})}$. Hence the weak t derivatives of the function $u_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ are integrable and given by

$$\partial_t^k u_\varepsilon(s,t) = (\rho_\varepsilon^{(k)} * u_s)(t).$$

5) Fix $\varepsilon > 0$. Define the subset $\Omega_\varepsilon = (-r^2, 0] \times (-r+\varepsilon, r-\varepsilon) \subset \Omega$. We prove by induction that for every $k \ge 1$ the weak derivative $\partial_s^k u_\varepsilon$ exists in $L^1(\Omega_\varepsilon)$ and equals $\partial_t^{2k} u_\varepsilon$ almost everywhere on Ω_ε. Here assumption (17) enters.
Step $k=1$. Straightforward calculation shows that

$$\begin{aligned}
\int_\Omega \psi\, \partial_t \partial_t u_\varepsilon &= \int_{\mathbb{R}^2} \psi(s,t) \left(\int_{\mathbb{R}} \partial_t \partial_t \rho_\varepsilon(t-\tau) u_s(\tau)\, d\tau \right) ds\, dt \\
&= \int_{\mathbb{R}^3} \psi(s,t)\, u(s,\tau)\, \partial_\tau \partial_\tau \rho_\varepsilon(t-\tau)\, d\tau\, ds\, dt \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} u(s,\tau)\, \partial_\tau \partial_\tau \Big(\rho_\varepsilon(t-\tau) \psi(s,t) \Big)\, d\tau\, ds \right) dt \\
&= -\int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} u(s,\tau)\, \partial_s \Big(\rho_\varepsilon(t-\tau) \psi(s,t) \Big)\, d\tau\, ds \right) dt \\
&= -\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \rho_\varepsilon(t-\tau) u_s(\tau)\, d\tau \right) \partial_s \psi(s,t)\, ds\, dt \\
&= -\int_\Omega u_\varepsilon \partial_s \psi
\end{aligned}$$

for every test function $\psi \in C_0^\infty(\operatorname{int} \Omega_\varepsilon)$. This identity means that on $\operatorname{int} \Omega_\varepsilon$, hence on Ω_ε, the weak derivative $\partial_s u_\varepsilon$ exists and equals $\partial_t \partial_t u_\varepsilon$ which is integrable by 4). To prove the identity note that the first and the final step are by definition of u_ε in 3). To obtain the second step we changed the order of integration and applied the chain rule. Steps three and five are obvious. To obtain step four we used assumption (17) and the fact that

$$\phi_t(s,\tau) := \rho_\varepsilon(t-\tau)\psi(s,t)$$

lies in $C_0^\infty(\operatorname{int} \Omega)$ for every $t \in \mathbb{R}$. To prove this assume that $\phi_t(s,\tau) \neq 0$. This means firstly that $\rho_\varepsilon(t-\tau) \neq 0$, hence $\tau \in [-\varepsilon + t, \varepsilon + t]$, and secondly that $\psi(s,t) \neq 0$. Now fix a sufficiently small constant $\delta = \delta(\varepsilon) > 0$ such that

$$\operatorname{supp} \psi \subset [-r^2 + \delta, -\delta] \times [-r + \varepsilon + \delta, r - \varepsilon - \delta] \subset \operatorname{int} \Omega_\varepsilon.$$

It follows that

$$\begin{aligned}
(s,\tau) &\in [-r^2 + \delta, -\delta] \times [-\varepsilon + (-r + \varepsilon + \delta), \varepsilon + (r - \varepsilon - \delta)] \\
&= [-r^2 + \delta, -\delta] \times [-r + \delta, r - \delta] \subset \operatorname{int} \Omega.
\end{aligned}$$

Induction step $k \Rightarrow k+1$. The calculation follows the same steps as above. We only indicate the minor differences. Assume that case k is true, then

$$\begin{aligned}\int_\Omega \psi\, \partial_t^{2k+2} u_\varepsilon &= (-1)^{k+1} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} u(s,\tau)\, \partial_s^{k+1} \Bigl(\rho_\varepsilon(t-\tau)\psi(s,t) \Bigr)\, d\tau\, ds \right) dt \\ &= (-1)^{k+1} \int_{\mathbb{R}^2} u_\varepsilon(s,t)\, \partial_s^{k+1} \psi(s,t)\, ds\, dt \\ &= -\int_\Omega \left(\partial_s^k u_\varepsilon \right) \partial_s \psi\end{aligned}$$

for every test function $\psi \in C_0^\infty(\operatorname{int}\Omega_\varepsilon)$. Note that to obtain the first step we applied $k+1$ times assumption (17) using that ϕ_t and therefore also its derivatives are in $C_0^\infty(\operatorname{int}\Omega)$. In the final step we used the induction hypothesis to integrate by parts k times the s variable.

6) The function u_ε is smooth on the closure of Ω_ε. Fix $\varepsilon > 0$ and positive integers m and ℓ. Then $\partial_t^m \partial_s^\ell u_\varepsilon$ equals $\partial_t^{m+2\ell} u_\varepsilon$ almost everywhere on Ω_ε by 5) and the latter function is integrable by 4). This proves that

$$u_\varepsilon \in \bigcap_{k=1}^\infty W^{k,1}(\Omega_\varepsilon) = C^\infty(\overline{\Omega}_\varepsilon).$$

Moreover, by 5) with $k=1$, each u_ε *solves the linear heat equation* (18) on $\overline{\Omega}_\varepsilon$.

7) From now on fix a compact subset $Q \subset \Omega$. We prove that for every positive integer k the family u_ε is uniformly bounded in the Banach space $C^k(Q)$ by a constant $\mu_k = \mu_k(Q)$. To see this consider the **compact parabolic rectangle** of radius r, height r^2, and top center point $(s,t) \in Q$ given by

$$P_r(s,t) := [s - r^2, s] \times [t - r, t + r].$$

By compactness of Q there is a constant $\varepsilon_0 = \varepsilon_0(Q) > 0$ such that $Q \subset \Omega_{\varepsilon_0}$ and, moreover, there is a constant $\rho = \rho(\varepsilon_0, Q) > 0$ such that

$$P_{2\rho}(s,t) \subset \Omega_{\varepsilon_0}$$

for every point $(s,t) \in Q$. By step 6) each function u_ε with $\varepsilon \in (0, \varepsilon_0)$ is a smooth solution of the linear homogeneous heat equation (18) on the domain Ω_ε and therefore on Ω_{ε_0}. Now given a point $(\sigma,\tau) \in Q$ and a pair of nonnegative integers m, ℓ there is by [Ev98, sec. 2.3 thm. 9] a constant $c_{m,\ell}(\sigma,\tau)$ such that

$$\max_{P_{\frac{\rho}{2}}(\sigma,\tau)} \left| \partial_t^m \partial_s^\ell v \right| \le \frac{c_{m,\ell}(\sigma,\tau)}{\rho^{m+2\ell+3}} \left\| v \right\|_{L^1(P_\rho(\sigma,\tau))}$$

for all smooth solutions v of the heat equation (18) in $P_{2\rho}(\sigma,\tau)$. By compactness of Q there are finitely many sets $P_{\rho/2}(\sigma_\nu, \tau_\nu)$ covering Q. Then the corresponding estimates for $v = u_\varepsilon$ and $m, \ell = 0, 1, \ldots, k$ imply that

$$\left\| u_\varepsilon \right\|_{C^k(Q)} \le \alpha \left\| u_\varepsilon \right\|_{L^1(\mathbb{R}^2)} \le \alpha \left\| u \right\|_{L^1(\Omega)}$$

for every $\varepsilon \in (0, \varepsilon_0)$ and where the constant $\alpha > 0$ depends only on the compact set Q (since ρ eventually depends on Q only). Inequality two uses step 3).

8) We prove that $u \in C^\infty(Q)$. In the setting of step 7) the Arzela-Ascoli theorem for each k together with choosing a diagonal subsequence yields existence of a sequence $\varepsilon_k \to 0$, as $k \to \infty$, and a smooth function $\hat{u}$ defined on Q such that $u_{\varepsilon_k} \to \hat{u}$ in $C^\infty(Q)$, as $k \to \infty$. On the other hand, the sequence u_{ε_k} converges to u in $L^1(Q)$ by step 3). Hence $u = \hat{u}$ by uniqueness of limits.

9) We prove lemma 2.3. The function u is smooth on Ω, because every point of Ω is contained in a compact subset Q on which u is smooth by step 8). To prove the identity $\partial_s u - \partial_t \partial_t u = 0$ on Ω assume by contradiction that this identity is violated at a point (s_*, t_*) of Ω. There are two cases.
If (s_*, t_*) is in the interior of Ω, then by smoothness of u there is a sufficiently small open neighborhood U of (s_*, t_*) in Ω and a function $\phi \in C_0^\infty(U, [0,1])$ with $\phi(s_*, t_*) = 1$ such that assumption (17) fails. For instance, if $c > 0$ is the value of the function $\partial_s u - \partial_t \partial_t u$ at the point (s_*, t_*), let U be the subset of Ω on which $\partial_s u - \partial_t \partial_t u > c/2$.
If (s_*, t_*) is in the boundary $0 \times (-r, r)$ of Ω, the former argument works for an interior point of Ω sufficiently close to (s_*, t_*). Existence of such an interior point uses again smoothness of u on Ω. This proves the parabolic Weyl lemma. $\square$

The heat ball approach

This subsection is supplementary. We give an alternative proof of the parabolic Weyl lemma 2.3 along the lines of the proof of the original Weyl lemma for the Laplacian. However, we will face two restrictions. Firstly, the set Ω should be open in $\mathbb{R}^2$ and, secondly, the function u should be locally L^q integrable over Ω for some $q > 3$.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^2$ be an open subset and $q > 3$. If $u \in L^q_{loc}(\Omega)$ satisfies*

$$\int_\Omega u\,(-\partial_s \phi - \partial_t \partial_t \phi) = 0 \tag{19}$$

for every $\phi \in C_0^\infty(\Omega)$, then u is a temperature on Ω.

Remark 2.5. To overcome the restriction $q > 3$ we tried in a first step to show that $u \in L^1_{loc}$ implies $u \in L^\infty_{loc}$ under the assumptions of lemma 2.3. This resulted in steps 1)–6) of the proof of lemma 2.3 and step 7) with $k = 1$. However, since step 7) actually works for every integer $k \geq 1$, it follows directly that u is smooth and lemma 2.4 became unecessary for this purpose.

The proof of the original Weyl lemma for the Laplacian is based on the fact that harmonic functions are characterized by their mean value property over balls or spheres; see e.g. [GT77, Jo98]. There is a similar statement for solutions of the heat equation. However, in the corresponding *parabolic* mean value equalities a weight other than one appears and this eventually leads to the restriction $q > 3$. More precisely, the weight is t^2/s^2. This function is L^p integrable over heat balls about the origin whenever $p = \frac{q}{q-1} \in (1, \frac{3}{2})$. A further difference is that balls and spheres over which the means are taken are replaced by heat balls and their boundaries, respectively. The parabolic mean value property with respect to boundaries is due to Fulks [Fu66] and with respect to heat balls it is due to Watson [Wa73]. Here we use that $\Omega \subset \mathbb{R}^2$ is open.

Recall that the *fundamental solution to the heat equation* is given by

$$\Phi(s,t) := \begin{cases} \frac{1}{\sqrt{4\pi s}} \, e^{-\frac{t^2}{4s}} & , \, s > 0, \, t \in \mathbb{R}, \\ 0 & , \, s < 0, \, t \in \mathbb{R}. \end{cases} \tag{20}$$

For $r > 0$ we denote by $E_r = E_r(0,0)$ the area enclosed by the level set which is determined by the identity

$$\Phi(-s,-t) = \frac{1}{2r\sqrt{\pi}}.$$

This level set is parametrized by

$$t(s) = \pm\sqrt{2s \ln \frac{-s}{r^2}}, \quad s \in (-r^2, 0).$$

Think of it as resembling an ellipse in the plane such that the origin is located at the 'north pole'. For general base points $(s,t) \in \mathbb{R}^2$ the sets $\boldsymbol{E_r(s,t)}$ are defined by translation and they are called **heat balls of "radius" $\boldsymbol{r}$**.

Definition 2.6. Following Watson [Wa73] we call a function u defined on an open subset $\Omega \subset \mathbb{R}^2$ a *temperature* if $\partial_t \partial_t u$ and $\partial_s u$ are continuous functions on Ω and $\partial_s u - \partial_t \partial_t u = 0$ pointwise on Ω.

Temperatures are automatically in $C^\infty(\Omega)$; see e.g. [Ev98, sec. 2.3 thm. 8].

Theorem 2.7 ([Wa73] § 10 cor. 1)**.** *Let u be a continuous function on an open subset $\Omega \subset \mathbb{R}^2$. Then the following are equivalent.*

(a) *The function u is a temperature.*

(b) *At every point $(s,t) \in \Omega$ the weighted mean value equality for u holds*

$$u(s,t) = \frac{1}{8\sqrt{\pi} \cdot r} \int_{E_r(s,t)} \frac{(t-\tau)^2}{(s-\sigma)^2} \, u(\sigma,\tau) \, d\tau \, d\sigma$$

whenever $\overline{E_r(s,t)} \subset \Omega$.

Proof of lemma 2.4. First we sketch the proof. The main idea is to mollify the given weak solution u to obtain a family $\{u_r\}$ of smooth functions converging in L^1, hence almost everywhere, to u. Here we mollify over 2-dimensional domains as opposed to the slicewise mollification used in the proof of lemma 2.3. Now assumption (19) implies that each function u_r is a temperature on a slightly smaller set $\Omega_r \subset \Omega$. Hence u_r satisfies the weighted mean value equality in

Watson's theorem 2.7. On the other hand, the family $\{u_r\}$ is uniformly bounded – here we use the assumption $q > 3$ – and equicontinuous. Hence by Arzela-Ascoli it converges in C^0 to a continuous function v as $r \to 0$. But we already know that $\{u_r\}$ converges almost everywhere to u. Hence $v = u$. Since the functions u_r satisfy the mean value equality, so does their C^0 limit u, and therefore u is a temperature again by theorem 2.7.

In seven steps we provide the details in case that Ω is bounded and u is L^q integrable over Ω. In step eight we prove the general case.

(1) We introduce appropriate mollifiers. Fix a smooth function ρ which is compactly supported in the unit heat ball $E = E_1(0,0)$ and satisfies $\int_E \rho = 1$. For $r > 0$ consider the rescaled function

$$\rho_r(s,t) := \frac{1}{r^3}\,\rho\left(\frac{s}{r^2}, \frac{t}{r}\right).$$

It is compactly supported in E_r and satisfies $\int_{E_r} \rho_r = 1$. Denote by $\boldsymbol{\Omega_r}$ the set of all points $(s,t) \in \Omega$ such that the closure of the heat ball $E_r(s,t)$ is contained in Ω.

(2) We define the family $\{u_r\}$. First we extend u by zero to $\mathbb{R}^2 \setminus \Omega$ and denote the extended map again by u. Hence $u \in L^q(\mathbb{R}^2)$. For $r > 0$ the mollification of u is defined by

$$u_r(s,t) := (\rho_r * u)(s,t) = \int_{\mathbb{R}^2} \rho_r(s-\sigma, t-\tau) u(\sigma,\tau)\, d\tau d\sigma.$$

Mollification is useful, because $u_r \in C_0^\infty(\mathbb{R}^2)$,

$$\|u_r - u\|_{L^q} \to 0 \quad \text{as } r \to 0,$$

and u_r converges to u pointwise almost everywhere; see e.g. [Jo98, app. A]. We denote $L^q(\mathbb{R}^2)$ by L^q. Moreover, note that by Young's convolution inequality the L^q norm of u_r is uniformly bounded, namely

$$\|u_r\|_{L^q} = \|\rho_r * u\|_{L^q} \le \|\rho_r\|_{L^1} \|u\|_{L^q} = \|u\|_{L^q}. \tag{21}$$

(3) We prove that each u_r is a temperature on the set Ω_r defined in step (1). Here assumption (19) enters. Let $\phi \in C_0^\infty(\Omega_r)$, then it follows that

$$\begin{aligned}
\int_\Omega \phi\,(\partial_s u_r - \partial_t\partial_t u_r)\, dt\, ds &= \int_\Omega (\rho_r * u)\,(-\partial_s\phi - \partial_t\partial_t\phi)\, dt\, ds \\
&= \int_\Omega u\,(\rho_r * (-\partial_s\phi - \partial_t\partial_t\phi))\, d\tau\, d\sigma \\
&= \int_\Omega u\,(-\partial_\sigma\phi_r - \partial_\tau\partial_\tau\phi_r)\, d\tau\, d\sigma \\
&= 0.
\end{aligned}$$

Here the first step is by integration by parts and the definition of u_r. The second step follows by commuting the integrals. To obtain the third step we first use integration by parts to throw the parabolic differential operator on the mollifier ρ_r. Then we observe that $(\partial_s - \partial_t \partial_t)\rho_r(s-\sigma, t-\tau) = (-\partial_\sigma - \partial_\tau \partial_\tau)\rho_r(s-\sigma, t-\tau)$ and pull out the latter differential operator of the integral (over the product of two smooth functions). To see the final step observe that $\phi_r = \rho_r * \phi \in C_0^\infty(\Omega)$, because $\phi \in C_0^\infty(\Omega_r)$, and therefore assumption (19) applies. Since the identity is true for all test functions $\phi \in C_0^\infty(\Omega_r)$ and u_r is smooth, it follows that $\partial_s u_r - \partial_t \partial_t u_r = 0$ pointwise in Ω_r.

(4) Fix a constant $R > 0$ and allow r to vary in the interval $(0, R/2)$. Then

$$\Omega_R \subset \Omega_{R/2} \subset \Omega_r \subset \Omega, \qquad \overline{E_{R/2}(s,t)} \subset \Omega_{R/2} \quad \forall (s,t) \in \Omega_R.$$

Hence by theorem 2.7 each temperature u_r satisfies the mean value equality on all heat balls with base point in Ω_R and radius less or equal to $R/2$.

(5) We prove that the family $\{u_r\}_{r \in (0,R/2)}$ is uniformly bounded on Ω_R. Fix a point $(s_0, t_0) \in \Omega_R$. Then by the mean value equality for the temperature u_r over the heat ball $E_{R/2}(s_0, t_0)$

$$\begin{aligned}
|u_r(s_0,t_0)| &\le \frac{1}{4\sqrt{\pi}R} \int_{E_{R/2}(s_0,t_0)} \frac{(t_0-\tau)^2}{(s_0-\sigma)^2} \left| u_r(\sigma,\tau) \right| \, d\tau d\sigma \\
&= \frac{1}{4\sqrt{\pi}R} \int_{E_{R/2}(0,0)} \frac{t^2}{s^2} \left| u_r(s+s_0, t+t_0) \right| \, dt ds \\
&\le \frac{1}{4\sqrt{\pi}R} \left\| t^2 s^{-2} \right\|_{L^p(E_{R/2})} \left\| u_r \right\|_{L^q} \\
&\le c_{q,R} \left\| u \right\|_{L^q}.
\end{aligned}$$

In the second step we introduced new variables $t = \tau - t_0$ and $s = \sigma - s_0$. In step three we use Hölder's inequality with $1/p + 1/q = 1$ and $p, q > 1$. Since the weight function $t^2 s^{-2}$ is not bounded on $E_{R/2}$ we can't get away with pulling out the sup norm. In the last step we applied (21). The constant $c_{q,R}$ is given by $\|t^2 s^{-2}\|_{L^p(E_{R/2})} / 4\sqrt{\pi}R$ with $p = \frac{q}{q-1}$. To see that it is finite observe that

$$\begin{aligned}
\left\| t^2 s^{-2} \right\|_{L^p(E_1)}^p &= \frac{2^{p+\frac{3}{2}}}{2p+1} \int_{-1}^{0} \frac{(s \ln(-s))^{p+\frac{1}{2}}}{(-s)^{2p}} ds \\
&= \frac{2^{p+\frac{3}{2}}}{2p+1} \int_0^\infty x^{p+\frac{1}{2}} e^{-x(\frac{3}{2}-p)} dx \\
&= \frac{2^{p+\frac{3}{2}}}{2p+1} \frac{\Gamma(p+\frac{3}{2})}{\left(\frac{3}{2}-p\right)^{p+\frac{3}{2}}}.
\end{aligned}$$

Here we used the change of variables $x = -\log(-s)$ in the second step, the last step is valid whenever $-\frac{3}{2} < p < \frac{3}{2}$, and Γ denotes the gamma function. The earlier use of Hölder's inequality further restricts p to the interval $(1, \frac{3}{2})$ and this is equivalent to $q = \frac{p}{p-1} > 3$. We still need to transform the unit heat ball E_1 to $E_{R/2}$. This leads to a finite constant which depends only on R and p.

(6) We prove that the family $\{u_r\}_{r \in (0,R/2)}$ is equicontinuous on Ω_R. Here we use uniform boundedness of the family which we proved in step (5). Given

two points (s_0,t_0) and (s_1,t_1) of Ω_R, the mean value equality for u_r over the heat balls $A=E_{R/2}(s_0,t_0)$ and $B=E_{R/2}(s_1,t_1)$, respectively, shows that

$$
\begin{aligned}
&4\sqrt{\pi}R\left|u_r(s_0,t_0)-u_r(s_1,t_1)\right|\\
&=\left|\int_A \frac{(t_0-\tau)^2}{(s_0-\sigma)^2}u_r(\sigma,\tau)\;d\tau d\sigma-\int_B \frac{(t_1-\tau)^2}{(s_1-\sigma)^2}u_r(\sigma,\tau)\;d\tau d\sigma\right|\\
&\le\int_{A\setminus B}\frac{(t_0-\tau)^2}{(s_0-\sigma)^2}\left|u_r(\sigma,\tau)\right|\;d\tau d\sigma+\int_{B\setminus A}\frac{(t_1-\tau)^2}{(s_1-\sigma)^2}\left|u_r(\sigma,\tau)\right|\;d\tau d\sigma\\
&\quad+\int_{A\cap B}\left|\frac{(t_0-\tau)^2}{(s_0-\sigma)^2}-\frac{(t_1-\tau)^2}{(s_1-\sigma)^2}\right|\left|u_r(\sigma,\tau)\right|\;d\tau d\sigma\\
&\le\sup_{\Omega_R}\left|u_r\right|(f+g+h)\,(s_0-s_1,t_0-t_1)\\
&\le c_{q,R}\left\|u\right\|_{L^q}C_R\left|(s_0,t_0)-(s_1,t_1)\right|
\end{aligned}
$$

where

$$
\begin{aligned}
f(s_0-s_1,t_0-t_1)&=\int_{E_{R/2}(s_0,t_0)\setminus E_{R/2}(s_1,t_1)}\frac{(t_0-\tau)^2}{(s_0-\sigma)^2}\;d\tau d\sigma\\
g(s_0-s_1,t_0-t_1)&=\int_{E_{R/2}(s_1,t_1)\setminus E_{R/2}(s_0,t_0)}\frac{(t_1-\tau)^2}{(s_1-\sigma)^2}\;d\tau d\sigma\\
h(s_0-s_1,t_0-t_1)&=\int_{E_{R/2}(s_0,t_0)\cap E_{R/2}(s_1,t_1)}\left|\frac{(t_0-\tau)^2}{(s_0-\sigma)^2}-\frac{(t_1-\tau)^2}{(s_1-\sigma)^2}\right|\;d\tau d\sigma.
\end{aligned}
$$

To see the final step in the estimate observe that

$$
f(s_0-s_1,t_0-t_1)=\int_{E_{R/2}\setminus E_{R/2}(s_1-s_0,t_1-t_0)}\frac{t^2}{s^2}\;dtds
$$

by change of variables. This shows that $f\ge 0$ depends continuously on the difference $(s_0,t_0)-(s_1,t_1)$ and vanishes precisely for $(s_0,t_0)=(s_1,t_1)$. If $s_0-s_1\ge -R^2/4$ or $t_0-t_1\ge\sqrt{2/e}R$, then f is constant and equal to $4\sqrt{\pi}R$. Denote this set by U_R. Its complement is compact. Hence f admits a uniform constant of continuity $C_R/3$. The same is true for g and h. Concerning h note that

$$
h(s_0-s_1,t_0-t_1)=\int_{E_{R/2}\cap E_{R/2}(s_1-s_0,t_1-t_0)}\left|\frac{t^2}{s^2}-\frac{(t_0-t_1-t)^2}{(s_0-s_1-s)^2}\right|\;dtds.
$$

Again $h\ge 0$ is continuous and constant on U_R, but this constant is zero because the integral is taken over the empty set.

(7) We conclude the proof in the case $u\in L^q(\Omega)$ with Ω bounded. By uniform boundedness and equicontinuity of the family $\{u_r:\Omega_R\to\mathbb{R}\}_{r\in(0,R/2)}$ the Arzela-Ascoli theorem asserts existence of a continuous function v on Ω_R and a sequence of positive reals $r_k\to 0$ such that $u_{r_k}\to v$ in $C^0(\Omega_R)$. On the other hand, the sequence u_{r_k} converges to u in L^q by step (2). Hence $u=v$ is continuous on Ω_R. Since the temperatures u_{r_k} satisfy the mean value equality

so does the uniform limit u. Hence $u : \Omega_R \to \mathbb{R}$ is a temperature by theorem 2.7. Since every point of Ω is contained in some Ω_R for $R > 0$ sufficiently small, it follows that u is a temperature on Ω.

(8) We prove the general case. Let $\Omega \subset \mathbb{R}^2$ be open and $u \in L^q_{loc}(\Omega)$. Given any point of Ω choose a sufficiently small bounded open neighborhood $\Omega' \subset \Omega$ on which u is L^q integrable. Then u is a temperature on Ω' by step (7). As the point was chosen arbitrarily, the function u is a temperature on Ω. □

2.2 Local regularity

The parabolic analogue of the Calderon-Zygmund inequality is the following fundamental L^p estimate. It is used to prove theorem 2.9 on local regularity and it implies the interior estimates of theorem 2.11 by induction.

Theorem 2.8 (Fundamental L^p estimate)**.** *For every $p > 1$, there is a constant $c = c(p)$ such that*

$$\|\partial_s v\|_p + \|\partial_t \partial_t v\|_p \le c \|\partial_s v - \partial_t \partial_t v\|_p$$

for every $v \in C_0^\infty(\mathbb{R}^2)$. The same statement is true for the domain $\mathbb{H}^-$.

Proof. A proof for $\mathbb{R}^2$ is given in [SW03, thm. C.2] by the Marcinkiewicz-Mihlin multiplier method. In the case of the lower half plane $\mathbb{H}^-$ choose a compactly supported smooth function v on $\mathbb{H}^-$ and constants $T > 0$ and $a < b$ such that $\operatorname{supp} u \subset (-T/2, 0] \times (a, b)$. Then [Li96, prop. 7.11] with $n = 1$, $A^{11} = 1$, $\lambda = \Lambda = 1$, the cube $K_0 = (-T/2, 0] \times (a, b)$ in $(-T, 0) \times \mathbb{R}$, and the function $f = \partial_s u - \partial_t \partial_t u$ proves the statement. Note that the case $\mathbb{H}^-$ implies the case $\mathbb{R}^2$ by translation. □

Theorem 2.9 (Local regularity)**.** *Fix a constant $1 < q < \infty$, an integer $k \ge 0$, and an open subset $\Omega \subset \mathbb{H}^-$. Then the following is true.*

a) *If $u \in L^1_{loc}(\Omega)$ and $f \in \mathcal{W}^{k,q}_{loc}(\Omega)$ satisfy*

$$\int_\Omega u \left(-\partial_s \phi - \partial_t \partial_t \phi\right) = \int_\Omega f \phi \tag{22}$$

for every $\phi \in C_0^\infty(\operatorname{int} \Omega)$, then $u \in \mathcal{W}^{k+1,q}_{loc}(\Omega)$.

b) *If $u \in L^1_{loc}(\Omega)$ and $f, h \in \mathcal{W}^{k,q}_{loc}(\Omega)$ satisfy*

$$\int_\Omega u \left(-\partial_s \phi - \partial_t \partial_t \phi\right) = \int_\Omega f \phi - \int_\Omega h \, \partial_t \phi \tag{23}$$

for every $\phi \in C_0^\infty(\operatorname{int} \Omega)$, then u and $\partial_t u$ are in $\mathcal{W}^{k,q}_{loc}(\Omega)$.

Here $\operatorname{int} \Omega$ denotes the interior of the set Ω. Part b) is used to prove theorem 3.1 on regularity for zeroes of the *linearized* heat equation. For convenience of the reader we recall Poincaré's inequality and its proof, since it is used in the proofs of theorem 2.9 and theorem 2.11.

Lemma 2.10 (Poincaré's inequality)**.** *Fix constants $q \geq 1$ and $r > 0$. Then*

$$\|\varphi\|_q \leq 2r \|\partial_t \varphi\|_q$$

for every $\varphi \in C_0^\infty((-r,r))$.

Proof. For such φ it holds that $\varphi(-r) = 0$ and hence

$$\varphi(t) = \int_{-r}^{t} \partial_t \varphi(\tau)\, d\tau$$

by the fundamental theorem of calculus. This implies that

$$|\varphi(t)| \leq \int_{-r}^{t} |\partial_t \varphi(\tau)|\, d\tau \leq \int_{-r}^{r} 1 \cdot |\partial_t \varphi(\tau)|\, d\tau \leq (2r)^{1/p} \|\partial_t \varphi\|_q$$

where the last step uses Hölder's inequality with $1/q + 1/p = 1$. Therefore

$$|\varphi(t)|^q \leq (2r)^{q-1} \|\partial_t \varphi\|_q^q$$

and integration over $t \in (-r,r)$ concludes the proof of the lemma. $\square$

Proof of theorem 2.9 . Since any given compact subset Q of Ω can be covered by finitely many parabolic rectangles whose closure is contained in Ω, we may assume without loss of generality that $\Omega = (-r^2, 0] \times (-r,r)$ for $r > 0$.
The proof of part a) consists of four steps I–IV and part b) requires another four steps V-VIII.

Step I: Fix two open subsets Ω' and U of $\Omega = (-r^2, 0] \times (-r,r)$ such that the closure of Ω' is contained in U and the closure of U is contained in Ω. Fix a smooth compactly supported cutoff function $\beta : \Omega \to [0,1]$ such that $\beta = 1$ on U. Then βf is compactly supported and $\mathcal{W}^{k,q}$ integrable over Ω. Now approximate βf in $\mathcal{W}^{k,q}(\Omega)$ through a sequence $(f_i) \subset C_0^\infty(\Omega)$, that is

$$\|f_i - \beta f\|_{\mathcal{W}^{k,q}(\Omega)} \longrightarrow 0, \qquad \text{as } i \to \infty.$$

Step II: Each smooth problem

$$(\partial_s - \partial_t \partial_t) u_i = f_i \tag{24}$$

with $f_i \in C_0^\infty(\Omega)$ admits a unique solution $u_i \in C_0^\infty(\Omega)$; see e.g. [Li96, thm. 5.6]. We prove below that the sequence of solutions u_i is a Cauchy sequence in $\mathcal{W}^{k+1,q}(\Omega)$. Therefore it admits a unique limit $\hat{u} \in \mathcal{W}^{k+1,q}(\Omega)$. Now the limit $\hat{u}$ solves the identity $(\partial_s - \partial_t\partial_t)\hat{u} = \beta f$ almost everywhere on Ω as can be seen as follows: The sequence $\partial_s u_i - \partial_t\partial_t u_i$ converges to $\partial_s \hat{u} - \partial_t\partial_t \hat{u}$ in L^q, since u_i is a Cauchy sequence in $\mathcal{W}^{k+1,q}(\Omega)$, and the sequence f_i converges to βf by step I. Uniqueness of the limit then proves equality in $L^q(\Omega)$.

It remains to prove that the sequence u_i is Cauchy. All norms are with respect to the domain Ω. Note that

$$\|u_i - u_j\|_q \leq 2r \|\partial_t (u_i - u_j)\|_q \leq (2r)^2 \|\partial_t \partial_t (u_i - u_j)\|_q .$$

The first inequality follows by integrating Poincaré's inequality (lemma 2.10) for $\varphi(t) = u_i(s,t) - u_j(s,t)$ over $s \in (-r^2, 0)$. The second inequality follows similarly. Now use equation (24) to obtain that

$$\left\|u_i - u_j\right\|_q \leq (2r)^2 \left(\left\|\partial_s(u_i - u_j)\right\|_q + \left\|f_i - f_j\right\|_q\right).$$

More generally, there is a constant $C = C(k,r)$ such that

$$\left\|u_i - u_j\right\|_{\mathcal{W}^{k+1,q}} \leq C\left(\left\|\partial_s^{k+1}(u_i - u_j)\right\|_q + \left\|f_i - f_j\right\|_{\mathcal{W}^{k,q}}\right)$$

for all i and j. This follows by inspecting the left hand side term by term replacing any two t-derivatives by one s-derivative and the error term f_i according to equation (24). If an odd number of t-derivatives appears then use lemma 2.10 to obtain an even number. Now the fundamental L^p estimate theorem 2.8 with constant $c = c(q)$ and function $v = \partial_s^k(u_i - u_j)$ asserts that

$$\begin{aligned}
\|\partial_s^{k+1}(u_i - u_j)\|_q &\leq c\|(\partial_s - \partial_t\partial_t)\partial_s^k(u_i - u_j)\|_q \\
&= c\|\partial_s^k(f_i - f_j)\|_q \\
&\leq c\|f_i - f_j\|_{\mathcal{W}^{k,q}}.
\end{aligned}$$

Here we used again equation (24). Next use the approximation of βf in step I to obtain that the sequence u_i in $\mathcal{W}^{k,q}(\Omega)$ is Cauchy, namely

$$\|f_i - f_j\|_{\mathcal{W}^{k,q}} \leq \|f_i - \beta f\|_{\mathcal{W}^{k,q}} + \|\beta f - f_j\|_{\mathcal{W}^{k,q}} \longrightarrow 0, \qquad \text{as } i,j \to \infty.$$

Step III: The restriction of $\hat{u} - u$ to the open subset $U \subset \Omega$ is a weak solution of the homogeneous problem. More precisely, it is true that

$$\begin{aligned}
\int_U (\hat{u} - u)(-\partial_s\phi - \partial_t\partial_t\phi) &= \int_U (\partial_s\hat{u} - \partial_t\partial_t\hat{u})\phi - \int_U u(-\partial_s\phi - \partial_t\partial_t\phi) \\
&= \int_U (\partial_s\hat{u} - \partial_t\partial_t\hat{u} - \beta f)\phi \\
&= 0
\end{aligned}$$

for every test function $\phi \in C_0^\infty(\operatorname{int} U)$. Here the first step is by integration by parts using step II and the second step is by assumption (22) and the fact that $f = \beta f$ on U. The last step uses the identity in step II.

Step IV: The difference $\hat{u} - u$ is in $L^1(U)$ by step II and assumption on u. Hence by the parabolic Weyl lemma 2.3 the function $F := \hat{u} - u$ is smooth on U. Together with the fact that $\hat{u} \in \mathcal{W}^{k+1,q}(\Omega)$ by step II this shows that $u = \hat{u} - F$ is of class $\mathcal{W}^{k+1,q}$ on each bounded open subset of U, hence on Ω'. This proves part a) of theorem 2.9. The proof of b) takes four further steps.

Step V: Let the sets Ω' and U, the cutoff function β, and the sequence $(f_i) \subset C_0^\infty(\Omega)$ be as in step I. Approximate the compactly supported function βh in

$\mathcal{W}^{k,q}(\Omega)$ through a sequence $(h_i) \subset C_0^\infty(\Omega)$. Now as in steps II and III each smooth problem

$$(\partial_s - \partial_t\partial_t)v_i = h_i \tag{25}$$

admits a unique solution $v_i \in C_0^\infty(\Omega)$ and the sequence (v_i) is Cauchy in $\mathcal{W}^{k+1,q}(\Omega)$ with unique limit $\hat{v}$ which solves the identity $(\partial_s - \partial_t\partial_t)\hat{v} = \beta h$ almost everywhere on Ω.

Step VI: Observe that the sequences

$$w_i := u_i + \partial_t v_i, \qquad \partial_t w_i = \partial_t u_i + \partial_t\partial_t v_i,$$

converge in $\mathcal{W}^{k,q}(\Omega)$ to the limits

$$\hat{w} = \hat{u} + \partial_t \hat{v}, \qquad \partial_t \hat{w} = \partial_t \hat{u} + \partial_t\partial_t \hat{v},$$

respectively. Moreover, each w_i satisfies the identity $(\partial_s - \partial_t\partial_t)w_i = f_i + \partial_t h_i$ on Ω. Integration by parts then shows that

$$\int_\Omega w_i \left(-\partial_s - \partial_t\partial_t\right)\phi = \int_\Omega f_i\phi - \int_\Omega h_i \partial_t\phi$$

for every $\phi \in C_0^\infty(\operatorname{int}\Omega)$. Taking the limit $i \to \infty$ we obtain that

$$\int_\Omega \hat{w} \left(-\partial_s - \partial_t\partial_t\right)\phi = \int_\Omega \beta f\phi - \int_\Omega \beta h\,\partial_t\phi \tag{26}$$

for every $\phi \in C_0^\infty(\operatorname{int}\Omega)$.

Step VII: The restriction of $\hat{w} - u$ to the open subset U of Ω is a weak solution of the homogeneous problem, meaning that

$$\begin{aligned}
\int_U (\hat{w} - u)(-\partial_s\phi - \partial_t\partial_t\phi) &= \int_U \hat{w}(-\partial_s - \partial_t\partial_t)\phi - \int_U u(-\partial_s\phi - \partial_t\partial_t\phi) \\
&= \int_U (\beta f\phi - \beta h\,\partial_t\phi) - \int_U (f\phi - h\,\partial_t\phi) \\
&= 0
\end{aligned}$$

for every test function $\phi \in C_0^\infty(\operatorname{int} U)$. Here step two uses the identity (26) for $\hat{w}$ and assumption (23) on u. Step three is true since $\beta = 1$ on U.

Step VIII: Note that the difference $\hat{w} - u$ is in $L^1(U)$ by step VI and assumption on u. Hence by the parabolic Weyl lemma 2.3 the function $G := \hat{w} - u$ is smooth on U. Since $\hat{w} \in \mathcal{W}^{k,q}(\Omega)$ by step VI, this shows that $u = \hat{w} - G$ is of class $\mathcal{W}^{k,q}$ on each bounded open subset of U. Since also $\partial_t\hat{w} \in \mathcal{W}^{k,q}(\Omega)$ by step VI, the function $\partial_t u = \partial_t \hat{w} - \partial_t G$ is of class $\mathcal{W}^{k,q}$ on each bounded open subset of U, in particular on Ω'. This concludes the proof of theorem 2.9. □

Interior estimates

Theorem 2.11 extends the fundamental L^p estimate theorem 2.8 to parabolic Sobolev spaces of higher order. The proof is by induction. Parabolic bootstrapping in section 2.4 relies on this extension. Also theorem 2.2 on interior regularity now follows readily. Then in proposition 2.13 we establish the linear version of the fundamental L^p estimate.

Theorem 2.11 (Interior estimates for parabolic rectangles)**.** *Fix an integer $k \geq 0$ and constants $1 < q < \infty$ and $0 < r < R$. Define $\Omega_r = (-r^2, 0] \times (-r, r)$. Then there is a constant $c = c(k, q, R - r)$ such that*

$$\|u\|_{\mathcal{W}^{k+1,q}(\Omega_r)} \leq c \left(\|\partial_s u - \partial_t \partial_t u\|_{\mathcal{W}^{k,q}(\Omega_R)} + \|u\|_{L^q(\Omega_R)} + \|\partial_t u\|_{L^q(\Omega_R)} \right) \quad (27)$$

for every $u \in C^\infty(\overline{\Omega_R})$.

Proof. The proof is by induction on k.
Step $k = 0$. Fix a smooth compactly supported cutoff function $\beta : \Omega_R \to [0, 1]$ such that $\beta = 1$ on Ω_r. Then

$$\begin{aligned}
&\|u\|_{\mathcal{W}^{1,q}(\Omega_r)} \\
&\leq \|\beta u\|_{L^q(\Omega_R)} + \|\partial_t (\beta u)\|_{L^q(\Omega_R)} + \|\partial_t \partial_t (\beta u)\|_{L^q(\Omega_R)} + \|\partial_s (\beta u)\|_{L^q(\Omega_R)} \\
&\leq 2R(1 + 2R) \|\partial_t \partial_t (\beta u)\|_{L^q(\Omega_R)} + \|\partial_s (\beta u)\|_{L^q(\Omega_R)} \\
&\leq c \|(\partial_s - \partial_t \partial_t) \beta u\|_{L^q(\Omega_R)} \\
&\leq c \|(\partial_s - \partial_t \partial_t) u\|_{L^q(\Omega_R)} + C \left(\|u\|_{L^q(\Omega_R)} + \|\partial_t u\|_{L^q(\Omega_R)} \right)
\end{aligned}$$

where $c = c_q (1 + 2R(1 + 2R))$ with c_q being the constant in theorem 2.8 and

$$C = \|\partial_s \beta\|_\infty + \|\partial_t \partial_t \beta\|_\infty + 2 \|\partial_t \beta\|_\infty .$$

The first step uses the fact that $\beta = 1$ on Ω_r, the definition of the $\mathcal{W}^{1,q}$ norm, and monotonicity of the integral. To obtain step two we fixed s and applied Poincaré's inequality lemma 2.10 to the functions $\beta u, \partial_t(\beta u) \in C_0^\infty(-R, R)$, then we integrated over $s \in (-R^2, 0]$. Step three is by theorem 2.8.
Induction step $k - 1 \Rightarrow k$. Fix $k \geq 1$. It suffices to estimate the $\mathcal{W}^{k+1,q}$ norms of u, $\partial_t u$, $\partial_t \partial_t u$, and $\partial_s u$ individually by the right hand side of (27). We provide details for the least trivial term and leave the others as an exercise. Fix constants $r < r_1 < r_2 < R$. Then by the induction hypothesis, that is case $k-1$ with pair of sets $\Omega_r \subset \Omega_{r_1}$ and function $v = \partial_s u$, we obtain that

$$\begin{aligned}
&\|\partial_s u\|_{\mathcal{W}^{k,q}(\Omega_r)} \\
&\leq c_1 \left(\|(\partial_s - \partial_t \partial_t) \partial_s u\|_{\mathcal{W}^{k-1,q}(\Omega_{r_1})} + \|\partial_s u\|_{L^q(\Omega_{r_1})} + \|\partial_t \partial_s u\|_{L^q(\Omega_{r_1})} \right) \\
&\leq c_1 \left(\|(\partial_s u - \partial_t \partial_t) u\|_{\mathcal{W}^{k,q}(\Omega_R)} + \|u\|_{\mathcal{W}^{1,q}(\Omega_{r_1})} + \|\partial_t u\|_{\mathcal{W}^{1,q}(\Omega_{r_1})} \right)
\end{aligned}$$

for some constant $c_1 = c_1(k-1, q, r_1 - r)$. To deal with the last term in the sum we apply the case $k = 0$ with pair of sets $\Omega_{r_1} \subset \Omega_{r_2}$ and function $v = \partial_t u$ to obtain that

$$\begin{aligned}\|\partial_t u\|_{\mathcal{W}^{1,q}(\Omega_{r_1})} &\le c_2 \left(\|(\partial_s - \partial_t\partial_t)\partial_t u\|_{L^q(\Omega_{r_2})} + \|\partial_t u\|_{L^q(\Omega_{r_2})} + \|\partial_t\partial_t u\|_{L^q(\Omega_{r_2})} \right) \\ &\le c_2 \left(\|(\partial_s u - \partial_t\partial_t)u\|_{\mathcal{W}^{k,q}(\Omega_R)} + \|\partial_t u\|_{L^q(\Omega_R)} + \|u\|_{\mathcal{W}^{1,q}(\Omega_{r_2})} \right)\end{aligned}$$

for some constant $c_2 = c_2(q, r_2 - r_1)$. It remains to estimate the last term in the sum. We apply again the case $k = 0$, but now for the pair of sets $\Omega_{r_2} \subset \Omega_R$ and the function u to obtain that

$$\|u\|_{\mathcal{W}^{1,q}(\Omega_{r_2})} \le c_3 \left(\|(\partial_s - \partial_t\partial_t)u\|_{L^q(\Omega_R)} + \|u\|_{L^q(\Omega_R)} + \|\partial_t u\|_{L^q(\Omega_R)} \right)$$

for some constant $c_3 = c_3(q, R - r_2)$. □

Proof of theorem 2.2 on interior regularity. a) Assume the parabolic rectangle $\Omega = (\sigma - r^2, \sigma] \times (\tau - r, \tau + r)$ is contained in the cylinder $Z_T = (-T, 0] \times S^1$. Then the assumptions of theorem 2.9 a) are satisfied for u and f restricted to Ω. Hence $u \in \mathcal{W}^{k+1,q}_{loc}(\Omega)$. Now u is locally $\mathcal{W}^{k+1,q}$ integrable on Z_T, because every compact subset of Z_T can be covered by finitely many parabolic rectangles. Part b) follows by induction over k based on theorem 2.11 and a covering argument by parabolic sets. □

Lemma 2.12 ([SW03, lemma D.4]). *Let $x \in C^\infty(S^1, M)$ and $p > 1$. Then*

$$\|\nabla_t \xi\|_p \le \kappa_p \left(\delta^{-1} \|\xi\|_p + \delta \|\nabla_t \nabla_t \xi\|_p \right)$$

for $\delta > 0$ and smooth vector fields ξ along x. Here κ_p equals $p/(p-1)$ for $p \le 2$ and it equals p for $p \ge 2$.

Proposition 2.13. *Assume $u : \mathbb{R} \times S^1 \to M$ is a smooth map such that $\|\partial_s u\|_\infty$, $\|\partial_t u\|_\infty$, and $\|\nabla_t \partial_t u\|_\infty$ are finite and $\lim_{s\to\pm\infty} u(s,t)$ exists, uniformly in t. Then, for every $p > 1$, there is a constant $c = c(p, u, M)$ such that*

$$\|\nabla_s \xi\|_p + \|\nabla_t \xi\|_p + \|\nabla_t \nabla_t \xi\|_p \le c \left(\|\nabla_s \xi - \nabla_t \nabla_t \xi\|_p + \|\xi\|_p \right) \tag{28}$$

for every smooth compactly supported vector field ξ along u. Estimate (28) remains valid for $-\nabla_s$ replacing ∇_s. Estimate (28) also remains valid if u is defined on the backward halfcylinder $(-\infty, 0] \times S^1$.

Proof. The proof of (28) for $\mathbb{R} \times S^1$ and $(-\infty, 0] \times S^1$ is based on theorem 2.8 for $\mathbb{R}^2$ and $\mathbb{H}^-$, respectively, using a covering argument. Full details in the case $\mathbb{R} \times S^1$ are provided by [SW03, prop. D.2]. Lemma 2.12 allows to add the term $\nabla_t \xi$ to the left hand side of (28). The underlying reason is periodicity in the t variable. The statement for $-\nabla_s$ follows by reflection $s \mapsto -s$. □

Applications of the proposition include closedness of the range of the linearized operator, proposition 3.18, estimate (77) in the proof of the exponential decay theorem 1.8, and step 2 in the proof of theorem 8.5.

2.3 A product estimate

The product estimate lemma 2.14 is the key tool to obtain the quadratic estimates of proposition 5.2. These in turn are used to prove the refined implicit function theorem 1.12. The Euclidean version corollary 2.16 of the product estimate is crucial in section 2.4 on parabolic bootstrapping. Namely, it allows to estimate the quadratic first order term $\Gamma(u)\,(\partial_t u, \partial_t u)$ of the heat equation (30) in the L^p norm as opposed to the $L^{p/2}$ norm which one expects at first sight.

Lemma 2.14. *Let N be a Riemannian manifold with Levi-Civita connection ∇ and Riemannian curvature tensor R. Fix constants $2 \le p < \infty$ and $c_0 > 0$. Then there is a constant $C = C(p, c_0, \|R\|_\infty)$ such that the following holds. If $u : (a, b] \times S^1 \to N$ is a smooth map such that*

$$\|\partial_s u\|_\infty + \|\partial_t u\|_\infty \le c_0,$$

then

$$\left(\int_a^b \int_0^1 (|\nabla_t \xi|\,|\nabla_t X|)^p \;dtds\right)^{1/p} \le C\,\|\xi\|_{\mathcal{W}^{1,p}} \left(\|\nabla_t X\|_p + \|\nabla_t \nabla_t X\|_p\right)$$

for all smooth compactly supported vector fields ξ and X along u.

Remark 2.15. Lemma 2.14 continues to hold for smooth maps u that are defined on the whole cylinder $\mathbb{R} \times S^1$. In this case the (compact) supports of ξ and X are contained in an interval of the form $(a, b]$.

Corollary 2.16. *Fix $2 \le p < \infty$. Then there is a constant $C = C(p)$ such that*

$$\left(\int_{-T}^0 \int_0^1 (|\partial_t v|\,|\partial_t w|)^p \;dtds\right)^{1/p} \le C\,\|v\|_{\mathcal{W}^{1,p}} \left(\|\partial_t w\|_p + \|\partial_t \partial_t w\|_p\right)$$

for all compactly supported smooth maps $v, w : (-T, 0] \times S^1 \to \mathbb{R}^k$.

Proof. Lemma 2.14 with $N = \mathbb{R}^k$, $u \equiv$ *const*, $\xi = v$, and $X = w$. ∎

Proof of lemma 2.14. The proof has three steps. Step 2 requires $p \ge 2$. Abbreviate $I = (a, b]$ and for $q, r \in [1, \infty]$ consider the norm

$$\|\xi\|_{q;r} := \|\xi\|_{L^q(I, L^r(S^1))}\,.$$

STEP 1. *Fix reals $\alpha \ge 1$ and $q, r, q', r' \in [\alpha, \infty]$ such that $\frac{1}{q} + \frac{1}{r} = \frac{1}{\alpha}$ and $\frac{1}{q'} + \frac{1}{r'} = \frac{1}{\alpha}$. Then $\|fg\|_\alpha \le \|f\|_{q';q}\,\|g\|_{r';r}$ for all functions $f, g \in C^\infty(I \times S^1)$.*

Let $f_s(t) := f(s, t)$. Apply Hölder's inequality twice to obtain

$$\begin{aligned}
\|fg\|^\alpha_{L^\alpha(I\times S^1)} &= \int_a^b \|f_s g_s\|^\alpha_{L^\alpha(S^1)}\; ds \\
&\le \int_a^b \left(\|f_s\|_{L^q(S^1)}\,\|g_s\|_{L^r(S^1)}\right)^\alpha ds \\
&= \|uv\|^\alpha_{L^\alpha(I)} \\
&\le \left(\|u\|_{L^{q'}(I)}\,\|v\|_{L^{r'}(I)}\right)^\alpha
\end{aligned}$$

where $u(s) := \|f_s\|_{L^q(S^1)}$ and $v(s) := \|g_s\|_{L^r(S^1)}$. This proves step 1.

STEP 2. *Given p, c_0, and u as in the hypothesis of the lemma, then there is a constant $c = c(p, c_0)$ such that*

$$\|\nabla_t \xi\|_{\infty;p} \le c \|\xi\|_{\mathcal{W}^{1,p}}$$

for every smooth compactly supported vector field ξ along $u : I \times S^1 \to N$.

The proof uses the generalized Young inequality: Given reals $a, b, c \ge 0$ and $1 < \alpha, \beta, \gamma < \infty$ such that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$, then

$$abc \le \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta} + \frac{c^\gamma}{\gamma}. \tag{29}$$

To prove this inequality one applies twice the standard Young inequality. The first application uses the exponents α and $\alpha/(\alpha-1)$ and the second application $\ell := \beta(\alpha-1)/\alpha$ and $m := \gamma(\alpha-1)/\alpha$. The sum of the inverted exponents equals one in both cases. It follows that

$$\begin{aligned} abc &\le \frac{a^\alpha}{\alpha} + \frac{\alpha-1}{\alpha} b^{\alpha/(\alpha-1)} c^{\alpha/(\alpha-1)} \\ &\le \frac{a^\alpha}{\alpha} + \frac{\alpha-1}{\alpha} \left(\frac{b^{\alpha\ell/(\alpha-1)}}{\ell} + \frac{c^{\alpha m/(\alpha-1)}}{m} \right) \\ &= \frac{a^\alpha}{\alpha} + \frac{\alpha-1}{\alpha\ell} b^{\alpha\ell/(\alpha-1)} + \frac{\alpha-1}{\alpha m} c^{\alpha m/(\alpha-1)} \end{aligned}$$

and this proves (29). Next straightforward calculation using integration by parts and abbreviating $\xi(s,t)$ by ξ shows that

$$\begin{aligned} &\frac{d}{ds} \int_0^1 |\nabla_t \xi(s,t)|^p \, dt \\ &= p \int_0^1 |\nabla_t \xi|^{p-2} \langle \nabla_t \xi, \nabla_t \nabla_s \xi + [\nabla_s, \nabla_t] \xi \rangle \, dt \\ &= -p \int_0^1 \left(\frac{d}{dt} |\nabla_t \xi|^{p-2} \right) \langle \nabla_t \xi, \nabla_s \xi \rangle \, dt - p \int_0^1 |\nabla_t \xi|^{p-2} \langle \nabla_t \nabla_t \xi, \nabla_s \xi \rangle \, dt \\ &\quad + p \int_0^1 |\nabla_t \xi|^{p-2} \langle \nabla_t \xi, R(\partial_s u, \partial_t u) \xi \rangle \, dt \\ &= -p(p-2) \int_0^1 |\nabla_t \xi|^{p-4} \langle \nabla_t \xi, \nabla_t \nabla_t \xi \rangle \langle \nabla_t \xi, \nabla_s \xi \rangle \, dt \\ &\quad - p \int_0^1 |\nabla_t \xi|^{p-2} \left(\langle \nabla_t \nabla_t \xi, \nabla_s \xi \rangle - \langle \nabla_t \xi, R(\partial_s u, \partial_t u) \xi \rangle \right) \, dt. \end{aligned}$$

Take the absolute value of the right hand side, apply the generalized Young inequality (29) in the case[2] $p > 2$ with $\alpha = p/(p-2)$, $\beta = p$, $\gamma = p$, and the

[2]The case $p = 2$ is taken care of by the standard Young inequality.

standard Young inequality with $\alpha = p/(p-1)$, $\beta = p$ to obtain the inequality

$$\begin{aligned}
&\frac{d}{ds}\int_0^1 |\nabla_t \xi(s,t)|^p \, dt \\
&\le p(p-1)\int_0^1 |\nabla_t \xi|^{p-2} |\nabla_t\nabla_t\xi| \cdot |\nabla_s \xi| \, dt + pc_0^2 \|R\|_\infty \int_0^1 |\nabla_t\xi|^{p-1} |\xi| \, dt \\
&\le p(p-1)\int_0^1 \left(\frac{p-2}{p} |\nabla_t\xi|^p + \frac{1}{p} |\nabla_t\nabla_t\xi|^p + \frac{1}{p}|\nabla_s\xi|^p \right) dt \\
&\quad + pc_0^2 \|R\|_\infty \int_0^1 \left(\frac{p-1}{p} |\nabla_t\xi|^p + \frac{1}{p}|\xi|^p \right) dt \\
&\le C_1 \left(\|\xi_s\|^p_{L^p(S^1)} + \|\nabla_s\xi_s\|^p_{L^p(S^1)} + \|\nabla_t\nabla_t\xi_s\|^p_{L^p(S^1)} \right).
\end{aligned}$$

Here $C_1 > 0$ is a constant depending only on p, c_0, and $\|R\|_\infty$ and $\xi_s(t) := \xi(s,t)$. Note that we used lemma 2.12 to estimate the terms involving $\nabla_t\xi_s$. Now fix $\sigma \in (a,b]$ and integrate this inequality over $s \in (a,\sigma]$ to obtain the estimate

$$\|\nabla_t\xi_\sigma\|^p_{L^p(S^1)} \le c\,\|\xi\|^p_{\mathcal{W}^{1,p}((a,b]\times S^1)}.$$

Here we used compactness of the support of ξ and monotonicity of the integral. Since the right hand side is independent of σ the proof of step 2 is complete.

Step 3. *We prove the lemma.*

Define

$$f(s,t) := |\nabla_t\xi(s,t)|, \qquad g(s,t) := |\nabla_t X(s,t)|.$$

By step 1 with $\alpha = q = r'$ equal to p and with $r = q' = \infty$ we have that

$$\begin{aligned}
\int_a^b\int_0^1 \left(|\nabla_t\xi(s,t)|\,|\nabla_t X(s,t)|\right)^p \, dtds &= \|fg\|_p^p \\
&\le \|\nabla_t\xi\|^p_{\infty;p}\,\|\nabla_t X\|^p_{p;\infty}.
\end{aligned}$$

Now apply step 2 to the first factor. For the second one we exploit the fact that, since the slices $s\times S^1$ of our domain are compact, there is the Sobolev embedding

$$W^{1,p}(S^1) \hookrightarrow L^\infty(S^1)$$

with constant $\mu = \mu(p) > 0$. It follows that

$$\begin{aligned}
\int_a^b \|\nabla_t X_s\|^p_{L^\infty(S^1)}\, ds &\le \int_a^b \mu^p\, \|\nabla_t X_s\|^p_{W^{1,p}(S^1)}\, ds \\
&= \mu^p \int_a^b \|\nabla_t X_s\|^p_{L^p(S^1)} + \|\nabla_t\nabla_t X_s\|^p_{L^p(S^1)}\, ds.
\end{aligned}$$

This concludes the proof of lemma 2.14. $\square$

2.4 Parabolic bootstrapping

In this section we establish by parabolic bootstrapping uniform Sobolev bounds for strong solutions u of the perturbed heat equation (30). Proposition 2.18 is a refined version of theorem 2.1 providing Sobolev bounds for $\partial_t u$, $\partial_s u$, and $\partial_t\partial_t u$. In order to deal with the quadratic lower order nonlinearity Γ of the heat equation we shall first prove apriori continuity of $\partial_t u$ in lemma 2.17. This provides a C^0 bound for $\partial_t u$ and we can think of the quadratic nonlinearity becoming linear. This is crucial in the first step $\ell = 1$ of the parabolic bootstrap in the proof of proposition 2.18.

Throughout this section we fix a closed smooth submanifold $M \hookrightarrow \mathbb{R}^N$ and a smooth family of vector-valued symmetric bilinear forms $\Gamma : M \to \mathbb{R}^{N\times N\times N}$. Abbreviate $\mathcal{W}^{k,p}(Z) = \mathcal{W}^{k,p}(Z,\mathbb{R}^N)$. Moreover, for $T > T' > 0$ we abbreviate

$$Z = Z_T = (-T,0]\times S^1, \qquad Z' = Z_{T'} = (-T',0]\times S^1.$$

Lemma 2.17 (Apriori continuity of $\partial_t u$)**.** *Fix constants $p > 2$, $\mu_0 > 1$, and $T > 0$. Fix a map $F : Z \to \mathbb{R}^N$ such that F and $\partial_t F$ are of class L^p. Assume that $u : Z \to \mathbb{R}^N$ is a $\mathcal{W}^{1,p}$ map taking values in M with $\|u\|_{\mathcal{W}^{1,p}} \le \mu_0$ and such that the perturbed heat equation*

$$\partial_s u - \partial_t\partial_t u = \Gamma(u)\,(\partial_t u, \partial_t u) + F \tag{30}$$

is satisfied almost everywhere. Then $\partial_t u$ is continuous. More precisely, for every $T' \in (0,T)$ there is a constant $c = c(p,\mu_0,T,T',\|\Gamma\|_{C^1})$ such that

$$\|\partial_t u\|_{C^0(Z')} \le c\left(1 + \|\partial_t F\|_{L^p(Z)}\right).$$

Note that by the Sobolev embedding theorem the assumption $p > 2$ guarantees that every $\mathcal{W}^{1,p}$ map u is continuous. Hence it makes sense to specify that u takes values in the submanifold M of $\mathbb{R}^N$.

Proposition 2.18. *Under the assumptions of lemma 2.17 the following is true for every integer $k \ge 1$ such that $F, \partial_t F \in \mathcal{W}^{k-1,p}(Z)$ and every $T' \in (0,T)$.*

(i) *There is a constant a_k depending on p, μ_0, T, T', $\|\Gamma\|_{C^{2k+2}}$, and the $\mathcal{W}^{k-1,p}(Z)$ norms of F and $\partial_t F$ such that*

$$\|\partial_t u\|_{\mathcal{W}^{k,p}(Z')} \le a_k.$$

(ii) *If $\partial_s F \in \mathcal{W}^{k-1,p}(Z)$ then there is a constant b_k depending on p, μ_0, T, T', $\|\Gamma\|_{C^{2k+2}}$, and the $\mathcal{W}^{k-1,p}(Z)$ norms of F, $\partial_t F$, and $\partial_s F$ such that*

$$\|\partial_s u\|_{\mathcal{W}^{k,p}(Z')} \le b_k.$$

(iii) *If $\partial_t\partial_t F \in \mathcal{W}^{k-1,p}(Z)$ then there is a constant c_k depending on p, μ_0, T, T', $\|\Gamma\|_{C^{2k+2}}$, and the $\mathcal{W}^{k-1,p}(Z)$ norms of F, $\partial_t F$, and $\partial_t\partial_t F$ such that*

$$\|\partial_t\partial_t u\|_{\mathcal{W}^{k,p}(Z')} \le c_k.$$

Remark 2.19. Since the proof of lemma 2.17 relies heavily on the product estimate corollary 2.16 it seems unlikely that the assumption $u \in \mathcal{W}^{1,p}$ can be weakened to $u \in W^{1,p}$ – unless we also replace the assumption $p > 2$ by $p > 3$.

Remark 2.20. The assumption $u \in \mathcal{W}^{1,p}$ in lemma 2.17 and proposition 2.18 can likewise be replaced by $u, \partial_t\partial_t u \in L^p \cap C^0$. To see this observe that the new assumption implies firstly that $\partial_t u \in L^p$, see e.g. [SW03, lemma D.4], and secondly that $\partial_s u$ is in L^p, though on a smaller domain. This follows similarly to the argument in the proof of lemma 2.17 leading to an L^p bound for $\partial_t\partial_t u$.

Notation. In the proofs of lemma 2.17 and proposition 2.18 we use the following notation. Given two constants $T > T' > 0$ consider the sequence given by

$$T_k := T' + \frac{T - T'}{k}, \quad k \in \mathbb{N}. \tag{31}$$

Note that $T_1 = T$. The definition also makes sense if we replace k by a real number $r \geq 1$. Now consider cylinders $Z_r = (-T_r, 0] \times S^1$. By $\operatorname{int} Z_r$ we denote the interior $(-T_r, 0) \times S^1$ of Z_r. It is useful to memorize that $Z_{r+1} \subset Z_r$. For each positive integer k fix a smooth compactly supported cutoff function

$$\rho_k : (-T_k, 0] \to [0, 1] \tag{32}$$

such that $\rho_k = 1$ on Z_{k+1} and $\|\partial_s\rho\|_\infty \geq 1$. Recall that $\mathcal{C}^k$ is defined by (14).

Proof of lemma 2.17. Denote the nonlinear part of the heat equation (30) by

$$h = h(u) = \Gamma(u)\,(\partial_t u, \partial_t u) + F$$

and the first cutoff function fixed in (32) by $\rho = \rho_1$. Then $h \in L^p(Z_2)$, namely

$$\begin{aligned} \|h\|_{L^p(Z_2)} &\leq \left\|\rho^2 h\right\|_{L^p(Z_1)} \\ &\leq \|\Gamma\|_\infty \left\||\partial_t(\rho u)| \cdot |\partial_t(\rho u)|\right\|_{L^p(Z_1)} + \left\|\rho^2 F\right\|_{L^p(Z_1)} \\ &\leq C_p \|\Gamma\|_\infty \|\partial_s\rho\|_\infty^2 \|u\|^2_{\mathcal{W}^{1,p}(Z_T)} + \|F\|_{L^p(Z_T)} \end{aligned}$$

where in step one and two we used that $\rho^2 = 1$ on Z_2 and independence of ρ on the t variable, respectively. The last step is by the product estimate corollary 2.16 with constant $C_p > 0$ applied to the compactly supported $\mathcal{W}^{1,p}$ map $\rho u : Z_T \to \mathbb{R}^N$ using a density argument. Compactness of M implies that $\|\Gamma\|_\infty < \infty$. Next observe that

$$\partial_t h = d\Gamma(u)\,(\partial_t u, \partial_t u, \partial_t u) + 2\Gamma(u)\,(\partial_t\partial_t u, \partial_t u) + \partial_t F. \tag{33}$$

Now we indicate the main idea of the proof. Suppose we knew that $\partial_t h \in L^\chi(Z_{k+1})$ for some $\chi > 1$ and some $k \in \mathbb{N}$, then

$$\begin{aligned} \int_{Z_{k+1}} \partial_t u\,(-\partial_s\phi - \partial_t\partial_t\phi) &= -\int_{Z_{k+1}} \partial_s u\,\partial_t\phi + \int_{Z_{k+1}} \partial_t\partial_t u\,\partial_t\phi \\ &= -\int_{Z_{k+1}} h\,\partial_t\phi \\ &= \int_{Z_{k+1}} \partial_t h\,\phi \end{aligned} \tag{34}$$

for every $\phi \in C_0^\infty(\operatorname{int} Z_{k+1})$. Here all steps use integration by parts. Step two is by definition of h and the assumption that u satisfies the heat equation (30) almost everywhere. Now theorem 2.2 on interior regularity asserts that $\partial_t u \in \mathcal{W}^{1,\chi}(Z_{k+2})$. Hence we have improved the regularity of $\partial_t u$ which in turn improves the of regularity $\partial_t h$ as given by (33). Now start over again. We prove below that under this iteration χ eventually converges to p. But $p > 2$, hence continuity of $\partial_t u$ follows by the Sobolev embedding $\mathrm{W}^{1,\chi} \hookrightarrow C^0$.

To get the iteration started at $k = 1$ we first need to prove that $\partial_t h \in L^\chi(Z_2)$ for some $\chi > 1$. As a first try recall that by assumption $u \in \mathcal{W}^{1,p}(Z_1)$. Therefore the first term in (33) is in $L^{p/3}$ only whereas the second term is in $L^{p/2}$. Hence $\partial_t h \in L^{p/3}$, but $p/3$ is not necessarily larger than 1. Fortunately, using the product estimate corollary 2.16 we can do better. By assumption $p > 2$ is given and fixed. Consider the function

$$\chi = \chi_p(q) = \frac{pq}{p+q}$$

and observe that $1/p + 1/q = 1/\chi$. Apply Hölder's inequality to obtain that

$$\begin{aligned}
\|\partial_t h\|_{L^\chi(Z_{k+1})} &\le \left\|{\rho_k}^2 \partial_t h\right\|_{L^\chi(Z_k)} \\
&\le \|d\Gamma\|_\infty \||\partial_t(\rho_k u)| \cdot |\partial_t(\rho_k u)|\|_{L^p(Z_k)} \|\partial_t u\|_{L^q(Z_k)} \\
&\quad + 2\|\Gamma\|_\infty \|\partial_t \partial_t u\|_{L^p(Z_k)} \|\partial_t u\|_{L^q(Z_k)} + \|\partial_t F\|_{L^\chi(Z_k)} \\
&\le C_p \|d\Gamma\|_\infty \|\partial_s \rho_k\|_\infty^2 \|u\|_{\mathcal{W}^{1,p}(Z_T)}^2 \|\partial_t u\|_{L^q(Z_k)} \\
&\quad + 2\|\Gamma\|_\infty \|\partial_t \partial_t u\|_{L^p(Z_T)} \|\partial_t u\|_{L^q(Z_k)} + \|\partial_t F\|_{L^p(Z_k)} \\
&\le \alpha \|\partial_t u\|_{L^q(Z_k)} + \|\partial_t F\|_{L^p(Z_T)} .
\end{aligned} \tag{35}$$

Here the third step is by the product estimate corollary 2.16 with constant C_p and the constant α in the last line depends on p, μ_0, $\|\Gamma\|_{C^1}$, and ρ_k. We used again one of the cutoff functions in (32) to produce a compactly supported function as required by the product estimate. Consequently the domain shrinks.

Now we start the iteration with initial value $q_1 = p$. Then $\chi(q_1) = p/2 > 1$. Hence $\partial_t h \in L^{p/2}(Z_2)$ by estimate (35) for $k = 1$. Therefore by (34) theorem 2.2 applies to the functions $\partial_t u$ and $f = \partial_t h$ and proves that $\partial_t u \in \mathcal{W}^{1,p/2}_{loc}(Z_2)$ and

$$\begin{aligned}
\|\partial_t u\|_{\mathcal{W}^{1,p/2}(Z_3)} &\le c_2 \left(\|\partial_t h\|_{L^{p/2}(Z_2)} + \mu_0\right) \\
&\le c_2 \left(\alpha\mu_0 + \|\partial_t F\|_{L^p(Z_T)} + \mu_0\right)
\end{aligned} \tag{36}$$

for some constant $c_2 = c_2(p, T_2 - T_3)$. Step two uses (35) for $k = 1$ and $q = p/2$, the fact that $\|\partial_t u\|_{p/2} \le \|\partial_t u\|_p$, and the assumption $\|\partial_t u\|_p \le \mu_0$.
Now there are three cases: If $p > 4$ then we are done by the Sobolev embedding $W^{1,p/2} \hookrightarrow C^0$ on the domain Z_3; see e.g. [MS04, app. B.1] for relevant embedding theorems. If $p < 4$ then the value of $\chi = \chi_p(q_1) = p/2$ is in the interval $(1, 2)$. In this case there is the Sobolev embedding

$$\mathcal{W}^{1,\chi}(Z_3) \subset W^{1,\chi}(Z_3) \hookrightarrow L^{2\chi/(2-\chi)}(Z_3) = L^{q_2}(Z_3)$$

with constant $C_2 = C_2(p, T_3) > 0$. Here we abbreviated

$$q_2 := \frac{2\chi}{2-\chi} = \frac{2pq_1}{2p + 2q_1 - pq_1} = \frac{2p}{4-p}.$$

Hence $\partial_t u \in L^{2p/(4-p)}(Z_3)$. Since $2p/(4-p) > p$ is equivalent to $2 < p < 4$, this means that the regularity of $\partial_t u$ has been improved – on the expense of a smaller domain though. The case $p = 4$ means that $u : Z_T \to \mathbb{R}^N$ is a $\mathcal{W}^{1,4}$ map to start with. But then it is also a $\mathcal{W}^{1,3}$ map and we are in the former case.

Repeating the same argument with new initial value q_2 proves that $\partial_t u \in \mathcal{W}^{1,\chi_p(q_2)}(Z_5)$. Again this space embedds either in $C^0(Z_5)$ and we are done or it embedds in $L^{q_3}(Z_5)$ where $q_3 = 2pq_2/(2p + 2q_2 - pq_2) > q_2$. It is crucial that in (35) the value of p is fixed. Firstly, because the product estimate corollary 2.16 requires $p \geq 2$ and, secondly, because we only know that $\partial_t \partial_t u \in L^p$. Proceeding this way we obtain the sequence q_k determined by

$$q_{k+1} = \frac{2pq_k}{2p + 2q_k - pq_k}, \qquad q_1 = p. \tag{37}$$

Observe again that the condition $p > 2$ implies that $q_{k+1} > q_k$. Hence the sequence is strictly monotone increasing. Next we prove that $q_k \to \infty$ as $k \to \infty$. Assume by contradiction that this is not true. Then by strict monotonicity the sequence is bounded and admits a unique limit, say q. By (37) this limit satisfies $q = 2pq/(2p + 2q - pq)$. But this is equivalent to $p = 2$ contradicting $p > 2$. It follows that $\chi_p(q_k)$ converges to p as $k \to \infty$. But $p > 2$, hence whenever k is sufficiently large there is the Sobolev embedding

$$\mathcal{W}^{1,\chi_p(q_k)}(Z_{2k+1}) \hookrightarrow C^0(Z_{2k+1}) \subset C^0(Z_{T'})$$

and this implies the estimate in lemma 2.17. Clearly $\partial_t u$ is continuous on the whole cylinder Z_T since every point is contained in some subcylinder $Z_{T'}$. □

Proof of proposition 2.18. We prove the following claim by induction on ℓ. Recall from (31) the definition of the real T_ℓ and the cylinder Z_ℓ. The claim with $\ell = k$ proves proposition 2.18.

Claim. *Given $0 < T' < T$ and an integer $k \geq 1$ such that F and $\partial_t F$ are in $\mathcal{W}^{k-1,p}$, then the following is true for every $\ell \in \{1, \ldots, k\}$.*

(a) $\partial_t u \in \mathcal{W}^{\ell,p}_{loc}(Z_{3\ell-1})$ *and there exists a constant A_ℓ depending on p, μ_0, $\|\Gamma\|_{C^{2\ell+2}}$, $\|F\|_{\mathcal{W}^{\ell-1,p}}$, and $\|\partial_t F\|_{\mathcal{W}^{\ell-1,p}}$ such that*

$$\|\partial_t u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell})} \leq A_\ell.$$

(b) *If $\partial_s F \in \mathcal{W}^{k-1,p}(Z_T)$ then $\partial_s u \in \mathcal{W}^{\ell,p}_{loc}(Z_{3\ell})$ and there exists a constant B_ℓ depending on p, μ_0, $\|\Gamma\|_{C^{2\ell+2}}$, $\|F\|_{\mathcal{W}^{\ell-1,p}}$, $\|\partial_t F\|_{\mathcal{W}^{\ell-1,p}}$, and $\|\partial_s F\|_{\mathcal{W}^{\ell-1,p}}$ such that*

$$\|\partial_s u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+1})} \leq B_\ell.$$

(c) *If* $\partial_t\partial_t F \in \mathcal{W}^{k-1,p}(Z_T)$ *then* $\partial_t\partial_t u \in \mathcal{W}^{\ell,p}_{loc}(Z_{3\ell+1})$ *and there exists a constant* C_ℓ *depending on* p, μ_0, $\|\Gamma\|_{C^{2\ell+2}}$, $\|F\|_{\mathcal{W}^{\ell-1,p}}$, $\|\partial_t F\|_{\mathcal{W}^{\ell-1,p}}$, *and* $\|\partial_t\partial_t F\|_{\mathcal{W}^{\ell-1,p}}$ *such that*

$$\|\partial_t\partial_t u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})} \leq C_\ell.$$

Here and throughout the domain of all norms is Z_T, unless specified otherwise. An exception are the various norms of Γ for which the domain is the compact manifold M. We abbreviate $h = \Gamma(u)(\partial_t u, \partial_t u) + F$.

Step $\ell = 1$. By lemma 2.17 with $T' = T_2$ there is a constant C_0 depending on p, μ_0, T, T_2, and $\|\Gamma\|_{C^1}$, such that

$$\|\partial_t u\|_{C^0(Z_2)} \leq C_0 \left(1 + \|\partial_t F\|_p\right). \tag{38}$$

(a) Recall that $\partial_t h$ is given by (33). Straightforward calculation shows that

$$\begin{aligned}\|\partial_t h\|_{L^p(Z_2)} &\leq \|d\Gamma\|_\infty \|\partial_t u\|^2_{C^0(Z_2)} \|\partial_t u\|_{L^p(Z_2)} + \|\partial_t F\|_{L^p(Z_2)} \\ &\quad + 2\|\Gamma\|_\infty \|\partial_t u\|_{C^0(Z_2)} \|\partial_t\partial_t u\|_{L^p(Z_2)} \\ &\leq \alpha \left(1 + \|\partial_t F\|_p^2\right)\end{aligned}$$

for some constant $\alpha = \alpha(p, \mu_0, T, T_2, \|\Gamma\|_{C^1})$. We used (38) and the assumption $\|u\|_{\mathcal{W}^{1,p}} \leq \mu_0$. Recall from (34) that $\partial_t u$ satisfies

$$\int_{Z_2} \partial_t u\,(-\partial_s\phi - \partial_t\partial_t\phi) = \int_{Z_2} \partial_t h\,\phi$$

for every $\phi \in C_0^\infty(\operatorname{int} Z_2)$. Hence theorem 2.2 on interior regularity for $q = p$, $T = T_2$, $T' = T_3$, $k = 0$, and the functions $f = \partial_t h$ and $\partial_t u$ in $L^p(Z_2)$ proves that $\partial_t u \in \mathcal{W}^{1,p}_{loc}(Z_2)$ and

$$\|\partial_t u\|_{\mathcal{W}^{1,p}(Z_3)} \leq \mu \left(\|\partial_t h\|_{L^p(Z_2)} + \|\partial_t u\|_{L^p(Z_2)}\right)$$

for some constant $\mu = \mu(p, T_2, T_3)$. Now use the estimate for $\partial_t h$ to see that

$$\|\partial_t u\|_{\mathcal{W}^{1,p}(Z_3)} \leq A \left(1 + \|\partial_t F\|_p^2\right)$$

for some constant $A = A(p, \mu_0, T, T_2, T_3, \|\Gamma\|_{C^1})$.

(b) Straightforward calculation shows that

$$\begin{aligned}\|\partial_s h\|_{L^p(Z_3)} &\leq \|d\Gamma\|_\infty \|\partial_t u\|^2_{C^0(Z_3)} \|\partial_s u\|_{L^p(Z_3)} + \|\partial_s F\|_{L^p(Z_3)} \\ &\quad + 2\|\Gamma\|_\infty \|\partial_t u\|_{C^0(Z_3)} \|\partial_s\partial_t u\|_{L^p(Z_3)} \\ &\leq \beta \left(1 + \|\partial_t F\|_p^3\right) + \|\partial_s F\|_p\end{aligned}$$

for some constant $\beta = \beta(p, \mu_0, T, T_2, T_3, \|\Gamma\|_{C^1}) > 1$. Here we estimated the L^p norm of $\partial_s\partial_t u$ by the $\mathcal{W}^{1,p}$ estimate for $\partial_t u$ just proved in (a). We also used the C^0 estimate (38). Next observe that

$$\begin{aligned}\int_{Z_3} \partial_s u \left(-\partial_s\phi - \partial_t\partial_t\phi\right) &= -\int_{Z_3} \left(\partial_s u - \partial_t\partial_t u\right)\partial_s\phi \\ &= -\int_{Z_3} \left(\Gamma(u)\left(\partial_t u, \partial_t u\right) + F(u)\right)\partial_s\phi \\ &= \int_{Z_3} \partial_s h\,\phi\end{aligned} \tag{39}$$

for every $\phi \in C_0^\infty(\operatorname{int} Z_3)$. Here steps one and three are by integration by parts. Step two uses the assumption that u satisfies the heat equation (30) almost everywhere. Now theorem 2.2 proves that $\partial_s u \in \mathcal{W}^{1,p}_{loc}(Z_3)$ and

$$\|\partial_s u\|_{\mathcal{W}^{1,p}(Z_4)} \le \mu\left(\|\partial_s h\|_{L^p(Z_3)} + \|\partial_s u\|_{L^p(Z_3)}\right)$$

for some constant $\mu = \mu(p, T_3, T_4)$. Now use the estimate for $\partial_s h$ to see that

$$\|\partial_s u\|_{\mathcal{W}^{1,p}(Z_4)} \le B\left(1 + \|\partial_t F\|_p^3 + \|\partial_s F\|_p\right)$$

for some constant $B = B(p, \mu_0, T, T_2, T_3, T_4, \|\Gamma\|_{C^1})$.
(c) Straighforward calculation shows that

$$\begin{aligned}\|\partial_t\partial_t h\|_{L^p(Z_4)} \le{}& \left\|d^2\Gamma\right\|_\infty \|\partial_t u\|^3_{C^0(Z_4)} \|\partial_t u\|_{L^p(Z_4)} + \|\partial_t\partial_t F\|_{L^p(Z_4)} \\ &+ 4\,\|d\Gamma\|_\infty \|\partial_t u\|^2_{C^0(Z_4)} \|\partial_t\partial_t u\|_{L^p(Z_4)} \\ &+ 2\,\|\Gamma\|_\infty \|\partial_t u\|_{C^0(Z_4)} \|\partial_t\partial_t\partial_t u\|_{L^p(Z_4)} \\ &+ 2\,\|\Gamma\|_\infty \|\partial_t\partial_t u\|_{C^0(Z_4)} \|\partial_t\partial_t u\|_{Lp(Z_4)} \\ \le{}& \gamma\left(1 + \|\partial_t F\|_p^4\right) + \|\partial_t\partial_t F\|_p\end{aligned}$$

for some constant $\gamma = \gamma(p, \mu_0, T, T_2, T_3, T_4, \|\Gamma\|_{C^2})$. In the final inequality we used the C^0 estimate (38) for $\partial_t u$ and the $\mathcal{W}^{1,p}$ estimate for $\partial_t u$ proved above in (a). This takes care of all terms but one, namely the C^0 norm of $\partial_t\partial_t u$. Here we use that $\partial_t\partial_t\partial_t u$ and $\partial_s\partial_t\partial_t u = \partial_t\partial_t\partial_s u$ are in $L^p(Z_4)$ by (a) and (b), respectively. Hence $\partial_t\partial_t u \in C^0$ by the Sobolev embedding $W^{1,p} \hookrightarrow C^0$. Similarly to the calculation in (34) it follows that

$$\int_{Z_4} \partial_t\partial_t u \left(-\partial_s\phi - \partial_t\partial_t\phi\right) = \int_{Z_4} \partial_t\partial_t h\,\phi$$

for every $\phi \in C_0^\infty(\operatorname{int} Z_4)$. Theorem 2.2 then proves that $\partial_t\partial_t u \in \mathcal{W}^{1,p}_{loc}(Z_4)$ and

$$\|\partial_t\partial_t u\|_{\mathcal{W}^{1,p}(Z_5)} \le \mu\left(\|\partial_t\partial_t h\|_{L^p(Z_4)} + \|\partial_t\partial_t u\|_{L^p(Z_4)}\right)$$

for some constant $\mu = \mu(p, T_4, T_5)$. Now use the estimate for $\partial_t\partial_t h$ to see that

$$\|\partial_t\partial_t u\|_{\mathcal{W}^{1,p}(Z_5)} \le C\left(1 + \|\partial_t F\|_p^4 + \|\partial_t\partial_t F\|_p\right)$$

for some constant $C = C(p, \mu_0, T, T_2, T_3, T_4, T_5, \|\Gamma\|_{C^2})$.

Induction step $\ell \Rightarrow \ell+1$. Fix an integer $\ell \in \{1, \ldots, k-1\}$ and assume that (a–c) are true for this choice of ℓ. We indicate this by the notation $(\text{a–c})_\ell$. The task at hand is to prove $(\text{a–c})_{\ell+1}$. Recall the parabolic $\mathcal{C}^\ell$ norm (14). An immediate consequence of the induction hypothesis $(\text{a–c})_\ell$ is that

$$\|u\|_{\mathcal{W}^{\ell+1,p}(Z_{3\ell+2})} \le D'_{\ell+1}$$

for some constant $D'_{\ell+1} = D'_{\ell+1}(p, \mu_0, \|\Gamma\|_{C^{2\ell+2}}, \|F\|_{\mathcal{W}^{\ell,p}})$. Hence

$$\|u\|_{\mathcal{C}^\ell(Z_{3\ell+2})} \le D_{\ell+1} \tag{40}$$

for some constant $D_{\ell+1} = D_{\ell+1}(p, \mu_0, \|\Gamma\|_{C^{2\ell+2}}, \|F\|_{\mathcal{W}^{\ell,p}})$. To see this observe that up to a constant the $\mathcal{C}^\ell$ norm can be estimated by the $\mathcal{W}^{\ell+1,p}$ norm. (This boils down to the Sobolev embedding $W^{1,p} \hookrightarrow C^0$ for each individual derivative of u showing up in $\mathcal{C}^\ell$.)

$(\text{a})_{\ell+1}$ Straightforward calculation shows that

$$\begin{aligned}
&\|\partial_t h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})} \\
&\le \|d\Gamma\|_{C^{2\ell}}\, d_\ell\, \|u\|^2_{\mathcal{C}^\ell(Z_{3\ell+2})} \|\partial_t u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})} + \|\partial_t F\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})} \\
&\quad + 2\,\|\Gamma\|_{C^{2\ell}}\, d_\ell\, \|u\|_{\mathcal{C}^\ell(Z_{3\ell+2})} \left(\|\partial_t u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})} + \|\partial_t\partial_t u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})}\right) \\
&\le \alpha_{\ell+1} + \|\partial_t F\|_{\mathcal{W}^{\ell,p}}
\end{aligned}$$

for some constant $\alpha_{\ell+1} = \alpha_{\ell+1}(p, \mu_0, \|\Gamma\|_{C^{2\ell+2}}, \|F\|_{\mathcal{W}^{\ell,p}})$. The first inequality follows from the identity (33) and the last two estimates of corollary 2.22 with constant d_ℓ. Notice the difference between the standard C^ℓ and the parabolic $\mathcal{C}^\ell$ norms. To obtain the second inequality we applied (40) and the induction hypotheses $(\text{a})_\ell$ and $(\text{c})_\ell$ to estimate the $\mathcal{W}^{\ell,p}$ norms of $\partial_t u$ and $\partial_t\partial_t u$, respectively. Next observe that theorem 2.2 applies by (34) and shows that $\partial_t u \in \mathcal{W}^{\ell+1,p}_{loc}(Z_{3\ell+2})$ and

$$\|\partial_t u\|_{\mathcal{W}^{\ell+1,p}(Z_{3\ell+3})} \le \mu\left(\|\partial_t h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+2})} + \|\partial_t u\|_{L^p(Z_{3\ell+2})}\right)$$

for some constant $\mu = \mu(p, Z_{3\ell+2}, Z_{3\ell+3})$. Now the assumption $\|u\|_{\mathcal{W}^{1,p}} \le \mu_0$ and the estimate for $\partial_t h$ conclude the proof of $(\text{a})_{\ell+1}$. For latter reference we remark that $(\text{a})_{\ell+1}$ implies – similarly to (40) – the estimate

$$\|\partial_t u\|_{\mathcal{C}^\ell(Z_{3\ell+3})} \le E_\ell \tag{41}$$

for some constant $E_\ell = E_\ell(p, \mu_0, \|\Gamma\|_{C^{2\ell+2}}, \|F\|_{\mathcal{W}^{\ell,p}}, \|\partial_t F\|_{\mathcal{W}^{\ell,p}})$.

$(\mathrm{b})_{\ell+1}$ Straightforward calculation using the $\mathcal{W}^{\ell+1,p}$ estimate for $\partial_t u$ just proved and the induction hypotheses $(\mathrm{a}\text{–}\mathrm{c})_\ell$ implies that

$$\begin{aligned}\|\partial_s h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} &\le \|d\Gamma\|_{C^{2\ell}} \|\partial_t u\|^2_{\mathcal{C}^\ell(Z_{3\ell+3})} \|\partial_s u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} + \|\partial_s F\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} \\ &\quad + 2\|\Gamma\|_{C^{2\ell}} \|\partial_t u\|_{\mathcal{C}^\ell(Z_{3\ell+3})} \|\partial_s\partial_t u\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} \\ &\le \beta_{\ell+1} + \|\partial_s F\|_{\mathcal{W}^{\ell,p}}\end{aligned}$$

for some constant $\beta_{\ell+1} = \beta_{\ell+1}(p, \mu_0, \|\Gamma\|_{C^{2\ell+2}}, \|F\|_{\mathcal{W}^{\ell,p}}, \|\partial_t F\|_{\mathcal{W}^{\ell,p}})$. To obtain the first inequality we simply pulled out the $\mathcal{C}^\ell$ norms. In the second inequality we used (41), the induction hypothesis $(\mathrm{b})_\ell$ to estimate the $\mathcal{W}^{\ell,p}$ norm of $\partial_s u$, and the induction hypothesis $(\mathrm{a})_{\ell+1}$ just proved to estimate the $\mathcal{W}^{\ell,p}$ norm of $\partial_s\partial_t u$. Next observe that theorem 2.2 applies by the identity (39) with Z_3 replaced by $Z_{3\ell+3}$ and shows that $\partial_s u \in \mathcal{W}^{\ell+1,p}_{loc}(Z_{3\ell+3})$ and

$$\|\partial_s u\|_{\mathcal{W}^{\ell+1,p}(Z_{3\ell+4})} \le \mu\left(\|\partial_s h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+4})} + \|\partial_t u\|_{L^p(Z_{3\ell+4})}\right)$$

for some constant $\mu = \mu(p, Z_{3\ell+3}, Z_{3\ell+4})$. Now use the estimate for $\partial_s h$.

$(\mathrm{c})_{\ell+1}$ Straighforward calculation shows that

$$\begin{aligned}\|\partial_t\partial_t h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} &\le \left\|d^2\Gamma\right\|_{C^{2\ell}} \|\partial_t u\|^3_{\mathcal{C}^\ell} \|\partial_t u\|_{\mathcal{W}^{\ell,p}} \\ &\quad + 5\|d\Gamma\|_{C^{2\ell}} \|\partial_t u\|^2_{\mathcal{C}^\ell} \|\partial_t\partial_t u\|_{\mathcal{W}^{\ell,p}} \\ &\quad + 2\|\Gamma\|_{C^{2\ell}} \|\partial_t u\|_{\mathcal{C}^\ell} \|\partial_t\partial_t\partial_t u\|_{\mathcal{W}^{\ell,p}} + \|\partial_t\partial_t F\|_{\mathcal{W}^{\ell,p}} \\ &\quad + 2\|\Gamma\|_{C^{2\ell}} C'_k \|\partial_t u\|_{\mathcal{C}^\ell} \|\partial_t\partial_t u\|_{\mathcal{W}^{\ell,p}}.\end{aligned}$$

Here all norms are taken on the domain $Z_{3\ell+3}$ except those involving Γ which are taken over M. Notice that in the first three terms of the sum we simply pulled out the $\mathcal{C}^\ell$ norms. However, in the last term there appears originally the product $\partial_t\partial_t u$ times $\partial_t\partial_t u$. To deal with this product we applied the first estimate of corollary 2.22 (where in both factors u is replaced by $\partial_t u$).
Now the $\mathcal{C}^\ell$ estimate (41) for $\partial_t u$ and the $\mathcal{W}^{\ell+1,p}$ estimate for $\partial_t u$ established in $(\mathrm{a})_{\ell+1}$ above prove that

$$\|\partial_t\partial_t h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} \le \gamma_{\ell+1} + \|\partial_t\partial_t F\|_{\mathcal{W}^{\ell,p}}$$

for some constant $\gamma_{\ell+1} = \gamma_{\ell+1}(\ell, p, \mu_0, \|\Gamma\|_{C^{2\ell+2}}, \|F\|_{\mathcal{W}^{\ell,p}}, \|\partial_t F\|_{\mathcal{W}^{\ell,p}})$. Apply again theorem 2.2 to see that $\partial_t\partial_t u \in \mathcal{W}^{\ell+1,p}_{loc}(Z_{3\ell+3})$ and

$$\|\partial_t\partial_t u\|_{\mathcal{W}^{\ell+1,p}(Z_{3\ell+4}))} \le \mu\left(\|\partial_t\partial_t h\|_{\mathcal{W}^{\ell,p}(Z_{3\ell+3})} + \|\partial_t\partial_t u\|_{L^p(Z_{3\ell+3})}\right)$$

for some constant $\mu = \mu(p, Z_{3\ell+3}, Z_{3\ell+4})$. The estimate for $\partial_t\partial_t h$ implies $(\mathrm{c})_{\ell+1}$. This proves the induction step, hence the claim. □

Lemma 2.21. *Fix a constant $p > 2$ and a bounded open subset $\Omega \subset \mathbb{R}^2$ with area $|\Omega|$. Then for every integer $k \ge 1$ there is a constant $c = c(k, |\Omega|)$ such that*

$$\|\partial_t u \cdot v\|_{\mathcal{W}^{k,p}} \le c\left(\|\partial_t u\|_{\mathcal{W}^{k,p}} \|v\|_\infty + \|u\|_{\mathcal{C}^k} \|v\|_{\mathcal{W}^{k,p}}\right)$$

for all functions $u, v \in C^\infty(\overline{\Omega})$.

Proof. The proof is by induction on k. By definition of the $\mathcal{W}^{\ell,p}$ norm

$$\begin{aligned}\|\partial_t u \cdot v\|_{\mathcal{W}^{\ell+1,p}} \leq \|\partial_t u \cdot v\|_{\mathcal{W}^{\ell,p}} &+ \|\partial_t\partial_t u \cdot v + \partial_t u \cdot \partial_t v\|_{\mathcal{W}^{\ell,p}} \\ &+ \|\partial_t\partial_t\partial_t u \cdot v + 2\partial_t\partial_t u \cdot \partial_t v + \partial_t u \cdot \partial_t\partial_t v\|_{\mathcal{W}^{\ell,p}} \\ &+ \|\partial_s\partial_t u \cdot v + \partial_t u \cdot \partial_s v\|_{\mathcal{W}^{\ell,p}} .\end{aligned} \tag{42}$$

Step $k = 1$. Estimate (42) for $\ell = 0$ shows that

$$\begin{aligned}\|\partial_t u \cdot v\|_{\mathcal{W}^{1,p}} \leq &\Big(\|\partial_t u\|_p + \|\partial_t\partial_t u\|_p + \|\partial_t\partial_t\partial_t u\|_p + \|\partial_s\partial_t u\|_p\Big) \|v\|_\infty \\ &+ \left(\|\partial_t u\|_\infty + 2\,\|\partial_t\partial_t u\|_\infty\right) \|\partial_t v\|_p \\ &+ \|\partial_t u\|_\infty \Big(\|\partial_t\partial_t v\|_p + \|\partial_s v\|_p\Big)\end{aligned}$$

and this proves the lemma for $k = 1$.
Induction step $k \Rightarrow k+1$. Consider estimate (42) for $\ell = k$, then inspect the right hand side term by term using the induction hypothesis to conclude the proof. To illustrate this we give full details for the last term in (42), namely

$$\begin{aligned}\|\partial_t u \cdot \partial_s v\|_{\mathcal{W}^{k,p}} &\leq c\left(\|\partial_t u\|_{\mathcal{W}^{k,p}} \|\partial_s v\|_\infty + \|u\|_{\mathcal{C}^k} \|\partial_s v\|_{\mathcal{W}^{k,p}}\right) \\ &\leq c\left(c' \,|\Omega|\, \|\partial_t u\|_{\mathcal{C}^k} \|\partial_s v\|_{\mathcal{W}^{1,p}} + \|u\|_{\mathcal{C}^k} \|v\|_{\mathcal{W}^{k+1,p}}\right) \\ &\leq c\left(c' \,|\Omega|\, \|u\|_{\mathcal{C}^{k+1}} \|v\|_{\mathcal{W}^{2,p}} + \|u\|_{\mathcal{C}^k} \|v\|_{\mathcal{W}^{k+1,p}}\right).\end{aligned}$$

Step one is by the induction hypothesis. In step two we pulled out the L^∞ norms of all derivatives of $\partial_t u$ and for the term $\partial_s v$ we used the Sobolev embedding $\mathcal{W}^{1,p} \subset W^{1,p} \hookrightarrow C^0$ with constant c'. Here we use the assumptions $p > 2$ and Ω bounded. Step three is obvious. Now $\mathcal{W}^{k+1,p} \hookrightarrow \mathcal{W}^{2,p}$ since $k \geq 1$. □

Corollary 2.22. *Fix a constant $p > 2$ and a bounded open subset $\Omega \subset \mathbb{R}^2$. Then for every integer $k \geq 1$ there is a constant $d = d(k, |\Omega|)$ such that*

$$\begin{aligned}\|\partial_t u \cdot \partial_t u\|_{\mathcal{W}^{k,p}} &\leq d_k \|u\|_{\mathcal{C}^k} \|\partial_t u\|_{\mathcal{W}^{k,p}} \\ \|\partial_t u \cdot \partial_t\partial_t u\|_{\mathcal{W}^{k,p}} &\leq d_k \|u\|_{\mathcal{C}^k} \left(\|\partial_t u\|_{\mathcal{W}^{k,p}} + \|\partial_t\partial_t u\|_{\mathcal{W}^{k,p}}\right) \\ \|\partial_t u \cdot \partial_t u \cdot \partial_t u\|_{\mathcal{W}^{k,p}} &\leq d_k \|u\|_{\mathcal{C}^k}^2 \|\partial_t u\|_{\mathcal{W}^{k,p}}\end{aligned}$$

for every function $u \in C^\infty(\overline{\Omega})$.

Proof. All three estimates follow from lemma 2.21. To obtain the first and the second estimate set $v = \partial_t u$ and $v = \partial_t\partial_t u$, respectively, and use that

$$\|\partial_t u\|_\infty \leq \|u\|_{\mathcal{C}^k}, \qquad \|\partial_t\partial_t u\|_\infty \leq \|u\|_{\mathcal{C}^k} .$$

To obtain estimate three set $v = \partial_t u \cdot \partial_t u$ and use estimate one. □

Proof of theorem 2.1. The $\mathcal{W}^{k+1,p}$ norm of u is equivalent to the sum of the $\mathcal{W}^{k,p}$ norms of u, $\partial_t u$, $\partial_s u$, and $\partial_t\partial_t u$. Apply proposition 2.18 (i–iii). □

3 The linearized heat equation

Fix a smooth function $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3) and a smooth map $u : \mathbb{R} \times S^1 \to M$. In this chapter we study the linear parabolic PDE

$$\nabla_s \xi - \nabla_t \nabla_t \xi - R(\xi, \partial_t u)\partial_t u - \mathcal{H}_{\mathcal{V}}(u)\xi = 0 \tag{43}$$

for vector fields ξ along u. Throughout R denotes the Riemannian curvature tensor associated to the closed Riemannian manifold M and the covariant Hessian $\mathcal{H}_{\mathcal{V}}$ of $\mathcal{V}$ at a loop $u(s, \cdot)$ is defined by (4).

In section 3.1 we show that *strong solutions*, that is solutions of class $\mathcal{W}_u^{1,p}$, are automatically smooth. More generally, for $\xi \in \mathcal{L}_u^p$ we define the notion of *weak solution* and show that even weak solutions are smooth. In section 3.2 we derive pointwise estimates of ξ and certain partial derivatives in terms of the L^2 norm of ξ over small backward cylinders. In section 3.3 we establish asymptotic exponential decay of the slicewise L^2 norm $\|\xi_s\|_{L^2(S^1)}$ of a solution ξ whenever the covariant Hessian A_{u_s} given by (10) is asymptotically injective. Still assuming asymptotic injectivity we prove in section 3.4 that the linear operator

$$\mathcal{D}_u : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p$$

given by the left hand side of (43) is Fredholm.

Observe that if u solves the (nonlinear) heat equation (6) then $\xi := \partial_s u$ solves the linear equation (43). Hence the results of this chapter will be useful in chapter 4 on solutions of the nonlinear heat equation.

3.1 Regularity

Define the operator $\mathcal{D}_u^*$ by the left hand side of (43) with ∇_s replaced by $-\nabla_s$.

Theorem 3.1 (Local regularity of weak solutions)**.** *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V3) *and constants $q > 1$ and $a < b$. Let $u : (a, b] \times S^1 \to M$ be a smooth map with bounded derivatives of all orders. Then the following is true. If η is a vector field along u of class L^q_{loc} such that*

$$\langle \eta, \mathcal{D}_u^* \xi \rangle = 0$$

for every smooth vector field ξ along u of compact support in $(a, b) \times S^1$, then η is smooth. Here $\langle \cdot, \cdot \rangle$ denotes integration over the pointwise inner products.

Remark 3.2. Theorem 3.1 remains true if we replace $\mathcal{D}_u^*$ by $\mathcal{D}_u$ and define u on $[a, b) \times S^1$. This follows by the variable substitution $s \mapsto -s$.

Proof. It suffices to prove the conclusion in a neighborhood of any point $z \in (a, b] \times S^1$. Shifting the s and t variables, if necessary, we may assume that $z \in \Omega_r = (-r^2, 0] \times (-r, r)$ for some sufficiently small $r > 0$. Now choose local coordinates on the manifold M around the point $u(z)$ and fix $r > 0$ sufficiently small such that $u(\overline{\Omega_r})$ is contained in the local coordinate patch. In these local coordinates the vector field η is represented by the map $(\eta^1, \ldots, \eta^n) : \Omega_r \to \mathbb{R}^n$ of class L^q_{loc} and the Riemannian metric g by the matrix with components g_{ij}. Throughout we use Einstein's sum convention. By induction we will prove that

$$v_\mu := g_{\mu j}\eta^j \in \bigcap_{m=1}^{\infty} \mathcal{W}^{m,q}_{loc}(\Omega_r), \qquad \mu = 1, \ldots, n.$$

Note that the intersection of spaces equals $C^\infty(\Omega_r)$; see e.g. [MS04, app. B.1]. Now apply the inverse metric matrix to obtain that $\eta^j = g^{j\mu} v_\mu \in C^\infty(\Omega_r)$ and this proves the theorem.

Step $m = 1$. Fix $\mu \in \{1, \ldots, n\}$ and consider vector fields of the form

$$\xi^{(\mu,\phi)} = (0, \ldots, 0, \phi, 0, \ldots, 0) : \Omega_r \to \mathbb{R}^n$$

where a function $\phi \in C^\infty_0(\operatorname{int} \Omega_r)$ occupies slot μ. Via extension by zero we view $\xi^{(\mu,\phi)}$ as a compactly supported smooth vector field along u. Now our assumption implies that $\langle \eta, \mathcal{D}^*_u \xi^{(\mu,\phi)} \rangle = 0$ for every $\phi \in C^\infty_0(\operatorname{int} \Omega_r)$. By straightforward calculation this is equivalent to

$$\int_{\Omega_r} v_\mu \left(-\partial_s \phi - \partial_t \partial_t \phi\right) = \int_{\Omega_r} f_\mu \phi - \int_{\Omega_r} h_\mu \, \partial_t \phi$$

for every $\phi \in C^\infty_0(\operatorname{int} \Omega_r)$, where $h_\mu = -2 v_k \Gamma^k_{i\mu} \, \partial_t u^i$ and

$$\begin{aligned} f_\mu = v_k \Big(&\Gamma^k_{i\mu} \, \partial_s u^i + \frac{\partial \Gamma^k_{i\mu}}{\partial u^r} \, \partial_t u^r \, \partial_t u^i + \Gamma^k_{i\mu} \, \partial_t \partial_t u^i \\ &+ \Gamma^k_{ij} \, \partial_t u^i \Gamma^j_{r\mu} \, \partial_t u^r + R^k_{\mu ij} \, \partial_t u^i \, \partial_t u^j + H^k_\mu \Big). \end{aligned}$$

Here $R^k_{\ell ij}$ represents the Riemann curvature operator and H^k_ℓ the Hessian $\mathcal{H}_{\mathcal{V}}(u)$ in local coordinates. The Christoffel symbols associated to the Levi Civita connection ∇ are denoted by Γ^k_{ij}.

From now on the domain of all spaces will be Ω_r, unless specified differently. Observe that $v_\mu \in L^q_{loc} \subset L^1_{loc}$ by smoothness of the metric, compactness of M, and the fact that $\eta^\ell \in L^q_{loc}$ by assumption. It follows that h_μ and f_μ are in L^q_{loc}. Here we used in addition boundedness of the derivatives of u and axiom (V1). Hence $\partial_t v_\mu \in L^q_{loc}$ by theorem 2.9 b) and this implies that $\partial_t h_\mu \in L^q_{loc}$. Now integration by parts shows that

$$\int_{\Omega_r} v_\mu \left(-\partial_s \phi - \partial_t \partial_t \phi\right) = \int_{\Omega_r} \left(f_\mu + \partial_t h_\mu\right) \phi$$

for every $\phi \in C^\infty_0(\operatorname{int} \Omega_r)$ and therefore $v_\mu \in \mathcal{W}^{1,q}_{loc}$ by theorem 2.9 a).

Induction step $m \Rightarrow m+1$. Assume that $v_\mu \in \mathcal{W}^{m,q}_{loc}$. Then $f_\mu, h_\mu \in \mathcal{W}^{m,q}_{loc}$ by compactness of M, boundedness of the derivatives of u, and axiom (V3). Hence $\partial_t v_\mu \in \mathcal{W}^{m,q}_{loc}$ by theorem 2.9 b). But this implies that $\partial_t h_\mu$ is in $\mathcal{W}^{m,q}_{loc}$ and so is $f_\mu + \partial_t h_\mu$. Therefore $v_\mu \in \mathcal{W}^{m+1,q}_{loc}$ by theorem 2.9 a). □

3.2 Apriori estimates

Theorem 3.3. *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V2) *and a constant $C_0 > 0$. Then there is a constant $C = C(C_0, \mathcal{V}) > 0$ such that the following is true. Assume $u : \mathbb{R} \times S^1 \to M$ is a smooth map with $\|\partial_t u\|_\infty \le C_0$ and ξ is a smooth vector field along u satisfying the linear heat equation (43). Then*

$$|\xi(s,t)| \le C \, \|\xi\|_{L^2([s-\frac{1}{2},s]\times S^1)}$$

for every $(s,t) \in \mathbb{R} \times S^1$. If in addition $\|\partial_s u\|_\infty + \|\nabla_t \partial_t u\|_\infty \le C_0$, then

$$|\nabla_t \xi(s,t)| \le C \, \|\xi\|_{L^2([s-1,s]\times S^1)}$$

for every $(s,t) \in \mathbb{R} \times S^1$.

Theorem 3.4. *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V2) *and a constant $C_0 > 0$. Then there is a constant $C = C(C_0, \mathcal{V}) > 0$ such that the following is true. Assume $u : \mathbb{R} \times S^1 \to M$ is a smooth map with*

$$\|\partial_t u\|_\infty + \|\partial_s u\|_\infty + \|\nabla_t \partial_t u\|_\infty + \|\nabla_t \partial_s u\|_\infty + \|\nabla_t \nabla_t \partial_t u\|_\infty \le C_0$$

and ξ is a smooth vector field along u satisfying the linear heat equation (43). Then

$$|\nabla_t \nabla_t \xi(s,t)| + |\nabla_s \xi(s,t)| \le C \, \|\xi\|_{L^2([s-2,s]\times S^1)}$$

for every $(s,t) \in \mathbb{R} \times S^1$.

Remark 3.5. If in theorem 3.3 or theorem 3.4 the vector field ξ solves $\mathcal{D}_u^* \xi = 0$, then $\eta(s,t) := \xi(-s,t)$ solves (43). The apriori estimates for η then translate into apriori estimates for ξ. For example, it follows that

$$|\xi(s,t)| \le C \, \|\xi\|_{L^2([s,s+\frac{1}{2}]\times S^1)}$$

for every $(s,t) \in \mathbb{R} \times S^1$ and similarly for the higher order derivatives.

The proof of theorem 3.3 and theorem 3.4 is based on the following mean value inequalities. Consider the **parabolic domain** defined for $r > 0$ by

$$P_r := (-r^2, 0) \times (-r, r).$$

Lemma 3.6 ([SW03, lemma B.1])**.** *There is a constant $c_1 > 0$ such that the following holds for all $r \in (0,1]$ and $a \ge 0$. If $w : P_r \to \mathbb{R}$, $(s,t) \mapsto w(s,t)$, is C^1 in the s-variable and C^2 in the t-variable such that*

$$(\partial_t \partial_t - \partial_s) w \ge -aw, \qquad w \ge 0,$$

then

$$w(0) \le \frac{c_1 e^{ar^2}}{r^3} \int_{P_r} w.$$

Corollary 3.7. *Let c_1 be the constant of lemma 3.6 and fix two constants $r \in (0, 1]$ and $\mu \geq 0$. Then the following is true. If $F : [-r^2, 0] \to \mathbb{R}$ is a C^1 function such that*

$$-F' + \mu F \geq 0, \qquad F \geq 0,$$

then

$$F(0) \leq \frac{2c_1 e^{\mu r^2}}{r^2} \int_{-r^2}^{0} F(s)\, ds.$$

Proof. Lemma 3.6 with $w(s,t) := F(s)$. □

Lemma 3.8 ([SW03, lemma B.4])**.** *Let $R, r > 0$ and $u : P_{R+r} \to \mathbb{R}$, $(s,t) \mapsto u(s,t)$, be C^1 in the s-variable and C^2 in the t-variable and $f, g : P_{R+r} \to \mathbb{R}$ be continuous functions such that*

$$(\partial_t \partial_t - \partial_s)\, u \geq g - f, \qquad u \geq 0, \qquad f \geq 0, \qquad g \geq 0.$$

Then

$$\int_{P_R} g \leq \int_{P_{R+r}} f + \left(\frac{4}{r^2} + \frac{1}{Rr} \right) \int_{P_{R+r} \setminus P_R} u.$$

Corollary 3.9. *Fix two positive constants r, R and three functions $U, F, G : [-(R+r)^2, 0] \to \mathbb{R}$ such that U is C^1 and F, G are continuous. If*

$$-U' \geq G - F, \qquad U \geq 0, \qquad F \geq 0, \qquad G \geq 0,$$

then

$$\int_{-R^2}^{0} G(s)\, ds \leq \frac{R+r}{R} \left(\int_{-(R+r)^2}^{0} F(s)\, ds + \left(\frac{4}{r^2} + \frac{1}{Rr} \right) \int_{-(R+r)^2}^{0} U(s)\, ds \right).$$

Proof. Lemma 3.8 with $u(s,t) = U(s)$, $f(s,t) = F(s)$, and $g(s,t) = G(s)$. □

Proof of theorem 3.3. We prove the theorem in three steps. The idea is to prove in step 1 the desired pointwise estimate in its integrated form (slicewise estimate). In steps 2 and 3 this is then used to prove the pointwise estimates. Note that in step 3 we provide an estimate which is not used in the current proof, but later on in the proof of theorem 3.4. Occasionaly we denote $\xi(s,t)$ by $\xi_s(t)$ and in this case $\|\xi_s\|$ abbreviates $\|\xi_s\|_{L^2(S^1)}$.

Step 1. *There is a constant $C_1 = C_1(C_0, \mathcal{V}) > 0$ such that*

$$\int_0^1 |\xi(s,t)|^2 \, dt + \int_{s-\frac{1}{16}}^{s} \int_0^1 |\nabla_t \xi(s,t)|^2 \, dt ds \leq C_1 \|\xi\|^2_{L^2([s-\frac{1}{4}, s] \times S^1)}$$

for every $s \in \mathbb{R}$.

Define the functions $f, g : \mathbb{R} \times S^1 \to \mathbb{R}$ and $F, G : \mathbb{R} \to \mathbb{R}$ by

$$2f := |\xi|^2, \quad 2g := |\nabla_t \xi|^2, \quad F(s) := \int_0^1 f(s,t)\, dt, \quad G(s) := \int_0^1 g(s,t)\, dt,$$

and abbreviate

$$L := \partial_t\partial_t - \partial_s, \qquad \mathcal{L} := \nabla_t\nabla_t - \nabla_s.$$

Then

$$Lf = 2g + U, \qquad U := \langle \xi, \mathcal{L}\xi\rangle. \tag{44}$$

Assume that U satisfies the pointwise inequality

$$|U| \le \mu f + \frac{1}{2}\|\xi_s\|^2 \tag{45}$$

for a suitable constant $\mu = \mu(C_0, \mathcal{V}) > 0$. Hence $Lf + \mu f + F \ge 2g$ by (44) and integration over the interval $0 \le t \le 1$ shows that

$$-F' + (\mu+1)F \ge 2G.$$

Step 1 follows by Corollary 3.7 with $r = \frac{1}{2}$ and corollary 3.9 with $R = r = \frac{1}{4}$.

It remains to prove (45). Since ξ solves the linear heat equation (43), it follows that

$$\begin{aligned}
|U| &= |\langle \xi, \nabla_t\nabla_t\xi - \nabla_s\xi\rangle| \\
&= |\langle \xi, R(\xi, \partial_t u)\partial_t u + \mathcal{H}_{\mathcal{V}}(u)\xi\rangle| \\
&\le \|R\|_\infty \|\partial_t u\|_\infty^2 |\xi|^2 + c_1 |\xi| \left(|\xi| + \|\xi_s\|_{L^1(S^1)}\right) \\
&\le \left(2C_0^2 \|R\|_\infty + 2c_1 + {c_1}^2\right) \frac{1}{2} |\xi|^2 + \frac{1}{2} \|\xi_s\|^2 .
\end{aligned}$$

Here we used the assumption on $\partial_t u$, axiom (V1) with constant c_1, and the fact that $\|\cdot\|_{L^1(S^1)} \le \|\cdot\|_{L^2(S^1)}$ by Hölder's inequality. This proves (45).

Step 2. *We prove the estimate for $|\xi|$ in theorem 3.3.*

Note that $Lf \ge -|U|$ by (45). Hence the estimate (45) for $|U|$ and the slicewise estimate for ξ_s provided by step 1 prove the pointwise inequality

$$Lf \ge -\mu f - 2C_1 \|\xi\|^2_{L^2([s-\frac{1}{4},s]\times S^1)}$$

for all s and t. Fix (s_0, t_0) and set $a = a(s_0) := \frac{2C_1}{\mu} \|\xi\|^2_{L^2([s_0-\frac{1}{2},s_0]\times S^1)}$. Then

$$L\left(f + a\right) \ge -\mu\left(f + a\right)$$

for all t and $s \in [s_0 - \frac{1}{4}, s_0]$. Hence lemma 3.6 with $r = \frac{1}{2}$ applies to the function $w(s,t) := f(s_0 + s, t_0 + t) + a$ and we obtain that

$$\begin{aligned}
f(s_0, t_0) &\le 8c_1 e^{\mu/4} \int_{-\frac{1}{4}}^{0} \int_0^1 \left(f(s_0 + s, t_0 + t) + a\right)\, dtds \\
&\le 8c_1 e^{\mu/4} \left(\frac{1}{2} + \frac{C_1}{2\mu}\right) \|\xi\|^2_{L^2([s_0-\frac{1}{2},s_0]\times S^1)} .
\end{aligned}$$

Since $s_0 \in \mathbb{R}$ and $t_0 \in S^1$ were chosen arbitrarily, this proves step 2.

STEP 3. *There is a constant $C_3 = C_3(C_0, \mathcal{V}) > 0$ such that*

$$\int_{s-\frac{1}{4}}^{s} \int_0^1 |\nabla_t \nabla_t \xi(s,t)|^2 \; dtds \le C_3 \|\xi\|^2_{L^2([s-\frac{5}{4},s]\times S^1)}$$

for every $s \in \mathbb{R}$. Moreover, the estimate for $|\nabla_t \xi|$ in theorem 3.3 holds true.
Define the functions $f_1, g_1 : \mathbb{R} \times S^1 \to \mathbb{R}$ by

$$2f_1 := |\nabla_t \xi|^2, \quad 2g_1 := |\nabla_t \nabla_t \xi|^2$$

and the functions $F_1, G_1 : \mathbb{R} \to \mathbb{R}$ by

$$F_1(s) := \int_0^1 f_1(s,t)\, dt, \quad G_1(s) := \int_0^1 g_1(s,t)\, dt.$$

Then

$$Lf_1 = 2g_1 + U_t, \qquad U_t := \langle \nabla_t \xi, \mathcal{L} \nabla_t \xi \rangle. \tag{46}$$

Since ξ solves the linear heat equation (43), it follows that

$$\begin{aligned}
\mathcal{L}\nabla_t \xi &= \nabla_t \left(\nabla_t \nabla_t \xi - \nabla_s \xi \right) - [\nabla_s, \nabla_t]\xi \\
&= \nabla_t \left(-R(\xi, \partial_t u)\partial_t u - \mathcal{H}_{\mathcal{V}}(u)\xi \right) - R(\partial_s u, \partial_t u)\xi \\
&= -\left(\nabla_t R \right)(\xi, \partial_t u)\partial_t u - R(\nabla_t \xi, \partial_t u)\partial_t u - R(\xi, \nabla_t \partial_t u)\partial_t u \\
&\quad - R(\xi, \partial_t u)\nabla_t \partial_t u - \nabla_t \mathcal{H}_{\mathcal{V}}(u)\xi - R(\partial_s u, \partial_t u)\xi.
\end{aligned}$$

Now take the pointwise inner product of this identity and $\nabla_t \xi$ and estimate the resulting six terms separately using the L^∞ boundedness assumption of the various derivatives of u. For instance, term five satisfies the estimate

$$|\langle \nabla_t \xi, \nabla_t \mathcal{H}_{\mathcal{V}}(u)\xi \rangle| \le c_2 |\nabla_t \xi| \left(|\nabla_t \xi| + (1 + |\partial_t u|) \left(|\xi| + \|\xi_s\|_{L^1(S^1)} \right) \right)$$

by the second inequality of axiom (V2) with constant c_2. It follows that U_t satisfies the pointwise inequality

$$|U_t| \le \mu f_1 + \mu |\xi|^2 + \mu \|\xi_s\|^2_{L^2(S^1)}$$

for a suitable constant $\mu = \mu(C_0, \mathcal{V}) > 0$. Hence

$$Lf_1 \ge 2g_1 - \mu f_1 - \mu |\xi|^2 - \mu \|\xi_s\|^2_{L^2(S^1)} \tag{47}$$

pointwise for all s and t. Integrate this inequality over $t \in [0,1]$ to obtain that

$$-F_1' \ge 2G_1 - \mu F_1 - 2\mu F$$

pointwise for every $s \in \mathbb{R}$. Then corollary 3.9 with $R = r = \frac{1}{2}$ shows that

$$\int_{s_0-\frac{1}{4}}^{s_0} \|\nabla_t \nabla_t \xi_s\|^2 ds \le (\mu + 20) \int_{s_0-1}^{s_0} \|\nabla_t \xi_s\|^2 ds + 2\mu \int_{s_0-1}^{s_0} \|\xi_s\|^2 ds$$

for every $s_0 \in \mathbb{R}$. Now

$$\int_{s_0-1}^{s_0} \|\nabla_t \xi_s\|^2 \, ds \le 16C_1 \int_{s_0-\frac{5}{4}}^{s_0} \|\xi_s\|^2 \, ds$$

by step 1 and this proves the first assertion of step 3. (We need this result only in the proof of theorem 3.4 below.)

To prove the second assertion of step 3, that is the estimate for $|\nabla_t \xi|$, note that estimate (47), step 1, and step 2 imply the pointwise estimate

$$Lf_1 \ge -\mu f_1 - \mu \|\xi\|^2_{L^2([s-\frac{1}{2},s]\times S^1)}$$

for all s and t. Here we have chosen a larger value for the constant μ. Fix $(s_0, t_0) \in \mathbb{R} \times S^1$ and set $a = a(s_0) := \|\xi\|^2_{L^2([s_0-1,s_0]\times S^1)}$. Then

$$L\left(f_1 + a\right) \ge -\mu \left(f_1 + a\right)$$

for all t and $s \in [s_0 - \frac{1}{2}, s_0]$. Hence lemma 3.6 with $r = \frac{1}{2}$ applies to the function $w(s,t) := f_1(s_0 + s, t_0 + t) + a$ and proves the desired estimate, namely

$$\begin{aligned}
f_1(s_0,t_0) &\le 8c_1 e^{\mu/4} \int_{-\frac{1}{4}}^{0} \int_0^1 \left(f_1(s_0+s, t_0+t) + a\right) \, dtds \\
&= 8c_1 e^{\mu/4} \left(\frac{1}{2} \int_{s_0-\frac{1}{4}}^{s_0} \int_0^1 |\nabla_t \xi(s,t)|^2 \, dtds + \frac{a}{4} \right) \\
&\le 8c_1 e^{\mu/4} \left(2\|\xi\|^2_{L^2([s_0-\frac{1}{2},s_0]\times S^1)} + \frac{1}{4}\|\xi\|^2_{L^2([s_0-1,s_0]\times S^1)} \right)
\end{aligned}$$

for all $s_0 \in \mathbb{R}$ and $t_0 \in S^1$. The final inequality uses the estimate of step 1. This concludes the proof of step 3 and theorem 3.3. □

Proof of theorem 3.4. Occasionaly we denote $\xi(s,t)$ by $\xi_s(t)$. Define the functions $f_2, g_2 : \mathbb{R} \times S^1 \to \mathbb{R}$ by

$$f_2 := \frac{1}{2}|\nabla_t \nabla_t \xi|^2, \qquad g_2 := \frac{1}{2}|\nabla_t \nabla_t \nabla_t \xi|^2$$

and abbreviate $L := \partial_t \partial_t - \partial_s$ and $\mathcal{L} := \nabla_t \nabla_t - \nabla_s$. Then

$$Lf_2 = 2g_2 + U_{tt}, \qquad U_{tt} := \langle \nabla_t \nabla_t \xi, \mathcal{L} \nabla_t \nabla_t \xi \rangle. \tag{48}$$

We estimate $|U_{tt}|$. Since ξ solves the linear heat equation (43), it follows that

$$\begin{aligned}
\mathcal{L}\nabla_t\nabla_t\xi &= \nabla_t\nabla_t\left(\nabla_t\nabla_t\xi - \nabla_s\xi\right) + [\nabla_t\nabla_t, \nabla_s]\xi \\
&= \nabla_t\nabla_t\left(-R(\xi,\partial_t u)\partial_t u - \mathcal{H}_{\mathcal{V}}(u)\xi\right) + \nabla_t[\nabla_t,\nabla_s]\xi + [\nabla_t,\nabla_s]\nabla_t\xi \\
&= \nabla_t\Big(- \left(\nabla_t R\right)(\xi,\partial_t u)\partial_t u - R(\nabla_t\xi,\partial_t u)\partial_t u - R(\xi,\nabla_t\partial_t u)\partial_t u \\
&\quad - R(\xi,\partial_t u)\nabla_t\partial_t u\Big) - \nabla_t\nabla_t\mathcal{H}_{\mathcal{V}}(u)\xi + \left(\nabla_t R\right)(\partial_t u, \partial_s u)\xi \\
&\quad + R(\nabla_t\partial_t u, \partial_s u)\xi + R(\partial_t u, \nabla_t\partial_s u)\xi + 2R(\partial_t u,\partial_s u)\nabla_t\xi.
\end{aligned}$$

Now take the pointwise inner product of this identity and $\nabla_t\nabla_t\xi$. Estimate the resulting sum term by term and use the assumption that various derivatives of u are bounded in L^∞. It follows that

$$|U_{tt}| \le \mu_1 |\nabla_t\nabla_t\xi| (|\xi| + |\nabla_t\xi| + |\nabla_t\nabla_t\xi|) + |\nabla_t\nabla_t\xi| \cdot |\nabla_t\nabla_t\mathcal{H}_{\mathcal{V}}(u)\xi|$$

for some positive constant μ_1 which depends only on the L^∞ bound C_0. Note that by axiom (V3) there is a positive constant $c_3 = c_3(\mathcal{V})$ such that

$$\begin{aligned}|\nabla_t\nabla_t\mathcal{H}_{\mathcal{V}}(u)\xi| &\le c_3 |\nabla_t\nabla_t\xi| + c_3 (1 + |\partial_t u|) |\nabla_t\xi| \\ &\quad + c_3 \left(1 + |\partial_t u|^2 + |\nabla_t\partial_t u|\right) \left(|\xi| + \|\xi_s\|_{L^1(S^1)}\right).\end{aligned}$$

Hence there is a positive constant $\mu_2 = \mu_2(C_0, \mathcal{V})$ such that

$$|U_{tt}| \le \mu_2 \left(f_2 + |\nabla_t\xi|^2 + |\xi|^2 + \|\xi_s\|_{L^2(S^1)}^2\right).$$

Theorem 3.3 applied to the last three terms of this sum implies that

$$|U_{tt}| \le \mu f_2 + \mu \|\xi\|_{L^2([s-1,s]\times S^1)}^2$$

pointwise for all s and t and with a suitable constant $\mu = \mu(C_0, \mathcal{V}) > 0$. Now $Lf_2 \ge -|U_{tt}|$ by (48) and therefore

$$Lf_2 \ge -\mu f_2 - \mu \|\xi\|_{L^2([s-1,s]\times S^1)}^2$$

pointwise for all s and t. Fix $s_0 \in \mathbb{R}$ and set $a := \|\xi\|_{L^2([s_0-2,s_0]\times S^1)}^2$, then

$$L(f_2 + a) \ge -\mu (f_2 + a)$$

for all $t \in S^1$ and $s \in [s_0 - 1, s_0]$. Fix $t_0 \in S^1$ and apply lemma 3.6 with $r = 1$ to the function $w(s,t) := f_2(s_0 + s, t_0 + t) + a$ to obtain that

$$\begin{aligned}f_2(s_0, t_0) &\le c_1 e^\mu \int_{-1}^0 \int_{-1}^{+1} (f_2(s_0 + s, t_0 + t) + a)\, dtds \\ &= c_1 e^\mu \left(\int_{s_0-1}^{s_0} \int_0^1 |\nabla_t\nabla_t\xi(s,t)|^2\, dtds + 2a\right) \\ &\le c_1 e^\mu (4C_3 + 2) \|\xi\|_{L^2([s_0-2,s_0]\times S^1)}^2.\end{aligned}$$

Here the last inequality follows by the estimate of step 3 in the proof of theorem 3.3 with constant $C_3 = C_3(C_0, \mathcal{V}) > 0$. Since $s_0 \in \mathbb{R}$ and $t_0 \in S^1$ were chosen arbitrarily, the proof of the first estimate of theorem 3.4 is complete.

The second estimate, that is the one for $|\nabla_s\xi|$, follows easily from the fact that ξ solves the linear heat equation (43), the estimate for $|\nabla_t\nabla_t\xi|$ which we just proved, the estimate for $|\xi|$ of theorem 3.3, and the estimate for $|\mathcal{H}_{\mathcal{V}}(u)\xi|$ provided by axiom (V1). This concludes the proof of theorem 3.4. □

3.3 Exponential decay

Given a smooth loop $x : S^1 \to M$ consider the linear operator defined by

$$A_x\xi = -\nabla_t\nabla_t\xi - R(\xi, \partial_t x)\partial_t x - \mathcal{H}_\mathcal{V}(x)\xi \tag{49}$$

on $L^2(S^1, x^*TM)$ with dense domain $W^{2,2}(S^1, x^*TM)$. With respect to the L^2 inner product $\langle\cdot,\cdot\rangle$ this operator is self-adjoint; see e.g. [W02] for the case of geometric perturbations V_t and use lemma 3.14 in the general case.

Theorem 3.10 (Backward exponential decay)**.** *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V2) *and a constant $c_0 > 0$. Then there exist positive constants δ, ρ, C such that the following holds. Let $x : S^1 \to M$ be a smooth loop such that A_x given by (49) is injective and $\|\partial_t x\|_2 + \|\nabla_t\partial_t x\|_2 \le c_0$. Assume $u : (-\infty, 0] \times S^1 \to M$ is a smooth map and $T_0 > 0$ is a constant such that*

$$u_s = \exp_x \eta_s, \quad \|\eta_s\|_{W^{2,2}} \le \delta, \quad \|\partial_s u_s\|_2 + \|\nabla_s\partial_t u_s\|_2 \le \delta,$$

whenever $s \le -T_0$. Assume further that ξ is a smooth vector field along u such that the function $s \mapsto \|\xi_s\|_2$ is bounded by a constant $c = c(\xi)$ and ξ solves one of two equations

$$\pm\nabla_s\xi - \nabla_t\nabla_t\xi - R(\xi, \partial_t u)\partial_t u - \mathcal{H}_\mathcal{V}(u)\xi = 0. \tag{50}$$

Then

$$\|\xi_s\|_2^2 \le e^{\rho(s+T_0)} \|\xi_{-T_0}\|_2^2 \le c^2 e^{\rho(s+T_0)}$$

and

$$\|\xi\|^2_{L^2((-\infty,s]\times S^1)} \le \tfrac{C^2}{\rho} e^{\rho(s+T_0)} \|\xi\|^2_{L^2([-T_0-1,-T_0]\times S^1)}$$

for every $s \le -T_0$.

Note the weak assumption (L^2 versus L^∞) on the s-derivatives of $\partial_t u_s$ and its base component u_s. To prove theorem 3.10 we need two lemmas.

Remark 3.11 (Forward exponential decay)**.** If the domain of u is the forward half cylinder $[0, \infty) \times S^1$ and the vector field ξ along u solves $\pm$(50), then theorem 3.10 applies to $v(\sigma, t) := u(-\sigma, t)$ and $\eta(\sigma, t) := \xi(-\sigma, t)$, since η solves $\mp$(50). The estimates obtained for η provide estimates for ξ, for instance

$$\|\xi\|^2_{L^2([\sigma,\infty)\times S^1)} \le \tfrac{C^2}{\rho} e^{\rho(-\sigma+T_0)} \|\xi\|^2_{L^2([T_0,T_0+1]\times S^1)}$$

for every $\sigma \ge T_0$.

Lemma 3.12 (Stability of injectivity)**.** *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V2) *and a constant $c_0 > 1$. Then there are constants $\mu, \delta_0 > 0$ such that the following holds. If x and γ are smooth loops in M such that the operator A_x is injective and*

$$\gamma = \exp_x(\eta), \qquad \|\eta\|_{W^{2,2}} \le \delta_0, \qquad \|\partial_t x\|_2 + \|\nabla_t\partial_t x\|_2 \le c_0,$$

then

$$\|\xi\|_2 + \|\nabla_t\xi\|_2 + \|\nabla_t\nabla_t\xi\|_2 \le \mu \|A_\gamma\xi\|_2$$

for every $\xi \in \Omega^0(S^1, \gamma^*TM)$.

Proof. By self-adjointness and injectivity the operator A_x is bijective. Hence it admits a bounded inverse by the open mapping theorem. This proves the estimate in the case $\gamma = x$ for some positive constant, say $\mu_0 = \mu_0(\mathcal{V}, c_0) > 1$. Since bijectivity is preserved under small perturbations (with respect to the operator norm), the result for general x follows from continuous dependence of the operator family on η with respect to the $W^{2,2}$ topology. More precisely, given a smooth vector field ξ along γ, define $X = \Phi^{-1}\xi$ where $\Phi = \Phi(x, \eta)$ denotes parallel transport along the geodesic $[0,1] \ni \tau \mapsto \exp_x(\tau\eta)$. Recall that Φ is pointwise an isometry, then straightforward calculation shows that

$$\|\xi\|_2 + \|\nabla_t \xi\|_2 + \|\nabla_t \nabla_t \xi\|_2 \leq c c_0^2 \mu_0 \left\|\Phi A_x \Phi^{-1} \xi\right\|_2$$

where the constant $c > 1$ depends only on the closed Riemannian manifold M and the constant c_1 associated to the Sobolev embedding $W^{1,2} \hookrightarrow C^0$. Now

$$\left\|\Phi A_x \Phi^{-1}\xi - A_\gamma \xi\right\|_2 \leq C \|\eta\|_{W^{2,2}} \|\xi\|_{W^{1,2}} \leq \delta_0 C \|\xi\|_{W^{1,2}}$$

by straightforward calculation, where the constant $C > 1$ depends on $\|R\|_\infty$, c_0, c_1, δ_0, and the constant in axiom (V2) and where we estimated the term quadratic in $\nabla_t \eta$ by $\|\nabla_t \eta\|_\infty^2 \leq c_1^2 \|\eta\|_{W^{2,2}}^2$. The second inequality uses the assumption on η. Now combine both estimates and choose $\delta_0 > 0$ sufficiently small to obtain the assertion of the lemma with $\mu = 2cc_0^2\mu_0$. □

Lemma 3.13. *Let $f \geq 0$ be a C^2 function on the interval $(-\infty, -T_0]$. If f is bounded by a constant c and satisfies the differential inequality $f'' \geq \rho^2 f$ for some constant $\rho \geq 0$, then*

$$f(s) \leq e^{\rho(s+T_0)} f(-T_0)$$

for every $s \leq -T_0$.

Proof. Although the argument is standard, see e.g. [DS94], we provide the details for the sake of completeness. The main point is to observe that $f'(s) - \rho f(s) \geq 0$ for every $s \leq -T_0$. To see this assume by contradiction that $f'(s_0) - \rho f(s_0) < 0$ for some time $s_0 \leq -T_0$. Note that the function $g(s) = e^{\rho s}\left(f'(s) - \rho f(s)\right)$ satisfies $g' \geq 0$ on $(-\infty, -T_0]$. Hence $g(s) \leq g(s_0)$, or equivalently

$$f'(s) \leq e^{\rho(s_0 - s)}\left(f'(s_0) - \rho f(s_0)\right) + \rho c$$

for every $s \leq s_0$. It follows that $f'(s) \to -\infty$ as $s \to -\infty$ and therefore

$$\int_s^{s_0} f'(\sigma)\, d\sigma \to -\infty, \quad \text{as } s \to -\infty.$$

But this contradicts the fact that by boundedness of f

$$\int_s^{s_0} f'(\sigma)\, d\sigma = f(s_0) - f(s) \geq -c$$

for every $s \leq s_0$. To conclude the proof consider the function $h(s) = e^{-\rho s} f(s)$ on the interval $(-\infty, -T_0]$. It follows from the observation above that $h' \geq 0$. Hence $h(s) \leq h(-T_0)$ for every $s \leq -T_0$ and this proves the lemma. □

To prove theorem 3.10 it is useful to denote $\exp_u(\xi)$ by $E(u, \xi)$ and define linear maps

$$E_i(u, \xi) : T_u M \to T_{exp_u \xi} M, \qquad E_{ij}(u, \xi) : T_u M \times T_u M \to T_{exp_u \xi} M$$

for $\xi \in T_x M$ and $i, j \in \{1, 2\}$. If $u : \mathbb{R} \to M$ is a smooth curve and ξ, η are smooth vector fields along u, then the maps E_i and E_{ij} are characterized by the identities

$$\begin{aligned} \frac{d}{ds} \exp_u(\xi) &= E_1(u, \xi)\partial_s u + E_2(u, \xi)\nabla_s \xi \\ \nabla_s \left(E_1(u, \xi)\eta\right) &= E_{11}(u, \xi)\left(\eta, \partial_s u\right) + E_{12}(u, \xi)\left(\eta, \nabla_s \xi\right) + E_1(u, \xi)\nabla_s \eta \\ \nabla_s \left(E_2(u, \xi)\eta\right) &= E_{21}(u, \xi)\left(\eta, \partial_s u\right) + E_{22}(u, \xi)\left(\eta, \nabla_s \xi\right) + E_2(u, \xi)\nabla_s \eta. \end{aligned} \tag{51}$$

These maps satisfy the symmetry properties

$$E_{12}(u, \xi)\left(\eta, \eta'\right) = E_{21}(u, \xi)\left(\eta', \eta\right), \quad E_{22}(u, \xi)\left(\eta, \eta'\right) = E_{22}(u, \xi)\left(\eta', \eta\right), \tag{52}$$

and the identities

$$E_{11}(u, 0) = E_{12}(u, 0) = E_{22}(u, 0) = 0, \qquad E_1(u, 0) = E_2(u, 0) = \mathbb{1}. \tag{53}$$

Alternatively E_2 can be defined by

$$E_2(u, \xi)\eta := \left.\frac{d}{d\tau}\right|_{\tau=0} \exp_u(\xi + \tau\eta)$$

for $\xi, \eta \in T_u M$ and $\tau \in \mathbb{R}$. An explicit definition of E_1 and the maps E_{ij} can be given in local coordinates.

Proof of theorem 3.10. Fix c_0 and $\mathcal{V}$ and let μ and δ_0 be the constants of lemma 3.12 and C be the constant of theorem 3.3 with this choice. Set $\delta := \delta_0$ and suppose u, x, T_0, ξ satisfy the assumptions of the theorem. Then lemma 3.12 for $\gamma = u_s$ and vector fields $\eta = \eta_s$ and $\xi = \xi_s$ asserts that

$$\left\|\xi_s\right\|_2^2 + \left\|\nabla_t \xi_s\right\|_2^2 + \left\|\nabla_t \nabla_t \xi_s\right\|_2^2 \leq \mu^2 \left\|A_{u_s} \xi_s\right\|_2^2 = \mu^2 \left\|\nabla_s \xi_s\right\|_2^2 \tag{54}$$

whenever $s \leq -T_0$. The last step uses the consequence $\nabla_s \xi_s = \mp A_{u_s} \xi_s$ of (49) and (50). From now on we assume that $s \leq -T_0$. Observe that

$$\begin{aligned} \partial_t u_s &= E_1(x, \eta_s)\partial_t x + E_2(x, \eta_s)\nabla_t \eta_s \\ \nabla_t \partial_t u_s &= E_{11}(x, \eta_s)\left(\partial_t x, \partial_t x\right) + 2E_{12}(x, \eta_s)\left(\partial_t x, \nabla_t \eta_s\right) + E_1(x, \eta_s)\nabla_t \partial_t x \\ &\quad + E_{22}(x, \eta_s)\left(\nabla_t \eta_s, \nabla_t \eta_s\right) + E_2(x, \eta_s)\nabla_t \nabla_t \eta_s. \end{aligned}$$

By the identities (53) we can choose $\delta > 0$ smaller, if necessary, such that

$$\left\|\partial_t u_s\right\|_2 \leq \left\|E_1(x, \eta_s)\right\|_\infty \left\|\partial_t x\right\|_2 + \left\|E_2(x, \eta_s)\right\|_\infty \left\|\nabla_t \eta_s\right\|_2 \leq 2c_0.$$

and, similarly, that $\left\|\nabla_t \partial_t u_s\right\|_2 \leq 2c_0$.

Claim. *Consider the function*

$$F(s) := \frac{1}{2}\left\|\xi_s\right\|_2^2 = \frac{1}{2}\int_0^1 |\xi(s,t)|^2 \, dt.$$

Then there is a sufficiently small constant $\delta > 0$ *such that*

$$F''(s) \geq \frac{1}{\mu^2} F(s)$$

whenever $s \leq -T_0$.

Before proving the claim we show how it implies the conclusions of theorem 3.10. Set $\rho = \rho(c_0, \mathcal{V}) := 1/\mu$, then $F'' \geq \rho^2 F$ on $(-\infty, T_0]$. Hence lemma 3.13 proves the first conclusion of theorem 3.10. Use this conclusion, the fact that $\|\cdot\|_2 \leq \|\cdot\|_\infty$ on the domain S^1, and theorem 3.3 with constant $C = C(c_0, \mathcal{V})$ to obtain that

$$\left\|\xi_s\right\|_2^2 \leq e^{\rho(s+T_0)} \left\|\xi_{-T_0}\right\|_\infty^2 \leq C^2 e^{\rho(s+T_0)} \left\|\xi\right\|^2_{L^2([-T_0-1,-T_0]\times S^1)}$$

whenever $s \leq -T_0$. Fix $\sigma \leq -T_0$ and integrate this estimate over $s \in (-\infty, \sigma]$. This proves the final conclusion of theorem 3.10.

It remains to prove the claim. In the following calculation we drop the subindex s for simplicity and denote the $L^2(S^1)$ inner product by $\langle \cdot, \cdot \rangle$. By straightforward computation it follows that

$$F''(s) = \left\|\nabla_s \xi_s\right\|_2^2 + \left\langle \xi, \nabla_s \nabla_s \xi \right\rangle$$

and

$$\begin{aligned}
\left\langle \xi, \nabla_s \nabla_s \xi \right\rangle &= \pm\left\langle \xi, \nabla_s \left(\nabla_t \nabla_t \xi + R(\xi, \partial_t u)\partial_t u + \mathcal{H}_{\mathcal{V}}(u)\xi\right)\right\rangle \\
&= \pm\left\langle \xi, [\nabla_s, \nabla_t \nabla_t]\xi + \nabla_t \nabla_t \nabla_s \xi + \nabla_s \left(R(\xi, \partial_t u)\partial_t u + \mathcal{H}_{\mathcal{V}}(u)\xi\right)\right\rangle \\
&= \pm\left\langle \xi, \nabla_t [\nabla_s, \nabla_t]\xi + [\nabla_s, \nabla_t]\nabla_t \xi + \nabla_s \left(R(\xi, \partial_t u)\partial_t u + \mathcal{H}_{\mathcal{V}}(u)\xi\right)\right\rangle \\
&\quad \pm \left\langle \nabla_t \nabla_t \xi, \nabla_s \xi \right\rangle \\
&= \pm\left\langle \pm\nabla_s \xi - R(\xi, \partial_t u)\partial_t u - \mathcal{H}_{\mathcal{V}}(u)\xi, \nabla_s \xi \right\rangle \\
&\quad \pm \Big\langle \xi, \big(\nabla_t R\big)(\partial_s u, \partial_t u)\xi + R(\nabla_t \partial_s u, \partial_t u)\xi + R(\partial_s u, \nabla_t \partial_t u)\xi \\
&\qquad + 2R(\partial_s u, \partial_t u)\nabla_t \xi + \big(\nabla_s R\big)(\xi, \partial_t u)\partial_t u + R(\nabla_s \xi, \partial_t u)\partial_t u \\
&\qquad + R(\xi, \nabla_s \partial_t u)\partial_t u + R(\xi, \partial_t u)\nabla_s \partial_t u + \nabla_s \mathcal{H}_{\mathcal{V}}(u)\xi \Big\rangle \\
&= \left\|\nabla_s \xi\right\|_2^2 \pm \left\langle \xi, \nabla_s \mathcal{H}_{\mathcal{V}}(u)\xi - \mathcal{H}_{\mathcal{V}}(u)\nabla_s \xi \right\rangle \\
&\quad \pm \Big\langle \xi, \big(\nabla_t R\big)(\partial_s u, \partial_t u)\xi + 2R(\xi, \partial_t u)\nabla_t \partial_s u + R(\partial_s u, \nabla_t \partial_t u)\xi \\
&\qquad + 2R(\partial_s u, \partial_t u)\nabla_t \xi + \big(\nabla_s R\big)(\xi, \partial_t u)\partial_t u \Big\rangle.
\end{aligned}$$

To obtain the first and the fourth step we replaced ξ according to (50). The third step is by integration by parts. In the final step we used twice the first Bianchi identity and lemma 3.14 on symmetry of the Hessian. Note that the term $\nabla_t \partial_s u$ forces us to assume $W^{1,2}$ and not only L^∞ smallness of $\partial_s u_s$.

Abbreviate $\|\cdot\|_{1,2} := \|\cdot\|_{W^{1,2}(S^1)}$ and assume from now on that $s \le -T_0$. Recall that $\|\partial_t u_s\|_\infty \le c_1 \|\partial_t u_s\|_{1,2} \le 4c_0 c_1$ where c_1 is the Sobolev constant of the embedding $W^{1,2}(S^1) \hookrightarrow C^0(S^1)$. Then the former two identities imply that

$$\begin{aligned} F''(s) &\ge 2\left\|\nabla_s \xi_s\right\|_2^2 - C_1 \left(\left\|\partial_s u_s\right\|_\infty + \left\|\nabla_t \partial_s u_s\right\|_2\right)\left(\left\|\xi_s\right\|_\infty^2 + \left\|\xi_s\right\|_\infty \left\|\nabla_t \xi\right\|_2\right) \\ &\ge 2\left\|\nabla_s \xi_s\right\|_2^2 - C_2 \left\|\partial_s u_s\right\|_{1,2} \left\|\xi_s\right\|_{1,2}^2 \end{aligned}$$

for positive constants $C_1 = C_1(c_0, c_1, \mathcal{V}, \|R\|_{C^2})$ and $C_2 = C_2(c_1, C_1)$. Choose $\delta > 0$ again smaller, if necessary, namely such that $\delta < 1/(2\mu^2 C_2)$. Hence

$$\left\|\partial_s u_s\right\|_{1,2} \le \delta < \frac{1}{2\mu^2 C_2}$$

where the first inequality is by assumption. Therefore

$$F''(s) \ge 2\left\|\nabla_s \xi_s\right\|_2^2 - \frac{1}{2\mu^2}\left\|\xi_s\right\|_{1,2}^2 \ge \left\|\nabla_s \xi_s\right\|_2^2$$

where the second inequality is by (54). But

$$\left\|\nabla_s \xi_s\right\|_2^2 \ge \frac{1}{\mu^2}\left\|\xi_s\right\|_2^2 = \frac{2}{\mu^2} F(s)$$

again by (54) and definition of F. This proves the claim and theorem 3.10. □

Lemma 3.14 (Symmetry of the Hessian)**.** *Fix a smooth map $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ and let $x : S^1 \to M$ be a smooth loop. Then*

$$\langle \mathcal{H}_\mathcal{V}(x)\xi, \eta \rangle = \langle \xi, \mathcal{H}_\mathcal{V}(x)\eta \rangle$$

for all smooth vector fields ξ and η along x.

Proof. Let $h : \mathbb{R}^2 \to \mathcal{L}M$, $(\sigma, \tau) \mapsto h(\sigma, \tau)$ be a smooth map such that

$$h(0,0) = x, \qquad \left.\frac{\partial}{\partial\sigma}\right|_0 h(\sigma, 0) = \xi, \qquad \left.\frac{\partial}{\partial\tau}\right|_0 h(0, \tau) = \eta.$$

Observe that

$$\begin{aligned} &\left.\frac{\partial^2}{\partial\tau\partial\sigma}\right|_{(0,0)} \mathcal{V}(h(\sigma,\tau)) \\ &= \left.\frac{d}{d\tau}\right|_0 d\mathcal{V}\,|_{h(0,\tau)} \left(\left.\frac{\partial}{\partial\sigma}\right|_0 h(\sigma,\tau)\right) \\ &= \left.\frac{d}{d\tau}\right|_0 \left\langle \mathrm{grad}\mathcal{V}\,|_{h(0,\tau)}, \left.\frac{\partial}{\partial\sigma}\right|_0 h(\sigma,\tau)\right\rangle \\ &= \left\langle \left.\frac{D}{d\tau}\right|_0 \mathrm{grad}\mathcal{V}\,|_{h(0,\tau)}, \left.\frac{\partial}{\partial\sigma}\right|_0 h(\sigma,0)\right\rangle + \left\langle \mathrm{grad}\mathcal{V}(x), \left.\frac{D}{d\tau}\right|_0 \left.\frac{\partial}{\partial\sigma}\right|_0 h(\sigma,\tau)\right\rangle \\ &= \langle \mathcal{H}_\mathcal{V}(x)\eta, \xi \rangle + \left\langle \mathrm{grad}\mathcal{V}(x), \left.\frac{D}{d\tau}\right|_0 \left.\frac{\partial}{\partial\sigma}\right|_0 h(\sigma,\tau)\right\rangle. \end{aligned}$$

Now interchange the order of partial differentiation and use the fact that this is still valid for two-parameter maps. □

3.4 The Fredholm operator

Hypothesis 3.15. Throughout this section we fix a perturbation $\mathcal{V}$ that satisfies (V0)–(V3) and two nondegenerate critical points $x^\pm$ of $\mathcal{S}_\mathcal{V}$. Fix a smooth map $u : \mathbb{R} \times S^1 \to M$ such that u_s converges to $x^\pm$ in $W^{2,2}(S^1)$ and $\partial_s u_s$ converges to zero in $W^{1,2}(S^1)$, as $s \to \pm\infty$. Moreover, assume that $\|\nabla_t \nabla_t \partial_s u_s\|_2$ is bounded, uniformly in $s \in \mathbb{R}$; see footnote below. Set $x = x^-$ and $y = x^+$.

Note that by theorem 1.8, proved in section 4.4 below, these assumptions are satisfied if $\mathcal{S}_\mathcal{V}$ is Morse and u is a finite energy solution of the heat equation (6). On the other hand, the hypothesis guarantees that the assumptions of the exponential decay theorem 3.10 and the local regularity theorem 3.1 – only here (V3) is needed – are satisfied. More precisely, set $a = \max\{\mathcal{S}_\mathcal{V}(x), \mathcal{S}_\mathcal{V}(y)\}$. Then (5) and (7) imply that

$$\|\partial_t x\|_2^2 = 2a + 2\mathcal{V}(x) \le 2(a + C_0), \qquad \|\nabla_t \partial_t x\|_2 = \|\mathrm{grad}\, \mathcal{V}(x)\|_2 \le C_0.$$

Here $C_0 > 0$ is the constant in axiom (V0). Similar estimates hold true for y. Precisely as in the proof of theorem 3.10 it follows that $T = T(u) > 0$ can be chosen sufficiently large such that

$$\|\partial_t u_s\|_2^2 \le 2c_0, \qquad \|\nabla_t \partial_t u_s\|_2 = \|\mathrm{grad}\, \mathcal{V}(x)\|_2 \le 2c_0$$

whenever $|s| \ge T_0$ and where $c_0 = 2(|a| + C_0)$. Hence by smoothness of u and compactness of the remaining domain $[-T, T] \times S^1$ we conclude that

$$\|\partial_t u_s\|_\infty \le c_1 \|\partial_t u_s\|_{W^{1,2}} \le c_2 \tag{55}$$

for every $s \in \mathbb{R}$ and where $c_2 = c_2(x, y, u, \mathcal{V})$. Similarly it follows that

$$\|\partial_s u_s\|_\infty \le c_1 \|\partial_s u_s\|_{W^{1,2}} \le c_3 \tag{56}$$

for every $s \in \mathbb{R}$ and some constant $c_3 = c_3(x, y, u, \mathcal{V})$.

Now consider the linear operator $\mathcal{D}_u$ given by

$$\mathcal{D}_u \xi = \nabla_s \xi - \nabla_t \nabla_t \xi - R(\xi, \partial_t u)\partial_t u - \mathcal{H}_\mathcal{V}(u)\xi \tag{57}$$

for smooth vector fields ξ along u. Recall that R denotes the Riemannian curvature tensor on M. The operator $\mathcal{D}_u$ arises, for instance, by linearizing the heat equation (6) at a solution u; see [W99, app. A.2]. Recall the definition of the Banach spaces $\mathcal{L}^p_u$ and $\mathcal{W}^{1,p}_u$ and their norms in (12). The goal of this section is to prove that $\mathcal{D}_u : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u$ is a **Fredholm operator** whenever $p > 1$ and u satisfies nondegenerate asymptotic boundary conditions as in hypothesis 3.15. By definition this means that $\mathcal{D}_u$ is a bounded linear operator with closed range and finite dimensional kernel and cokernel. The difference of these dimensions is called the **Fredholm index** of $\mathcal{D}_u$ and denoted by $\mathrm{index}\, \mathcal{D}_u$. The **formal adjoint operator** $\mathcal{D}_u^* : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u$ with respect to the L^2-inner product has the form

$$\mathcal{D}_u^* \xi = -\nabla_s \xi - \nabla_t \nabla_t \xi - R(\xi, \partial_t u)\partial_t u - \mathcal{H}_\mathcal{V}(u)\xi. \tag{58}$$

We proceed as follows. In the case $p = 2$ we show that our situation suits the assumptions of [RS95] where the Fredholm property is proved. Then we reduce the case $p > 1$ to the case $p = 2$ by proving that the kernel and the cokernel do actually not depend on p. The argument is based on exponential decay and local regularity, theorem 3.10 and theorem 3.1, respectively.

Fredholm property and index for $p = 2$

To prove that $\mathcal{D}_u$ is Fredholm it is useful to choose a representation with respect to an orthonormal frame along u. However, since M is not necessarily orientable, a frame which is periodic in the t-variable might not exist. Hence, given a smooth map $u : \mathbb{R} \times S^1 \to M$, we define

$$\sigma = \sigma(u) := \begin{cases} +1, & \text{if } u^*TM \to \mathbb{R} \times S^1 \text{ is trivial} \\ -1, & \text{else} \end{cases}$$

and $E_\sigma := \operatorname{diag}(\sigma, 1, \ldots, 1) \in \mathbb{R}^{n \times n}$. The orthogonal group $\mathrm{O}(n)$ has two connected components, one contains $E_1 = \mathbb{1}$ and the other one E_{-1}. Hence there exists an orthonormal frame $\phi = \phi_\sigma : \mathbb{R} \times [0,1] \to u^*TM$ such that $\phi(s,1) = \phi(s,0)E_\sigma$ for all $s \in \mathbb{R}$. The vector space of smooth sections of u^*TM is isomorphic to the space C^∞_σ of all maps $X \in C^\infty(\mathbb{R} \times [0,1], \mathbb{R}^n)$ such that $X(s,1) = E_\sigma X(s,0)$, for every $s \in \mathbb{R}$, and such that this condition also holds for all derivatives of X with respect to the t-variable.

Denote by W the closure of C^∞_σ with respect to the Sobolev $W^{2,2}$ norm and by H its closure with respect to the L^2 norm. Then $\mathcal{D}_u : \mathcal{W}^{1,2}_u \to \mathcal{L}^2_u$ given by (57) is represented by the Atiyah-Patodi-Singer type operator

$$D_{A+C} := \phi^{-1}\mathcal{D}_u\phi = \frac{d}{ds} + A(s) + C(s) \tag{59}$$

from $\mathcal{W}^{1,2} := L^2(\mathbb{R}, W) \cap W^{1,2}(\mathbb{R}, H)$ to $L^2(\mathbb{R}, H)$. Here $A(s)$ is the family of symmetric second order operators on H with dense domain W given by

$$A(s) = -\frac{d^2}{dt^2} - B(s,t) - Q(s,t)$$

where

$$Q = \phi^{-1}R(\phi, \partial_t u)\partial_t u + \phi^{-1}\mathcal{H}_\mathcal{V}(u)\phi$$

and $B = (\partial_t P) + 2P\partial_t + P^2$. The families of skew-symmetric matrices $P(s,t)$ and $C(s,t)$ are determined by the identities

$$\phi^{-1}\nabla_t\phi = \partial_t + P, \qquad \phi^{-1}\nabla_s\phi = \partial_s + C.$$

Hypothesis 3.15 implies that $\partial_s u_s$ converges to zero in $C^0(S^1)$, as $s \to \pm\infty$, and therefore $\lim_{s\to\pm\infty} C(s,t) = 0$, uniformly in t. It follows that the family $C(s)$ of bounded operators on H – defined pointwise by matrix multiplication with $C(s,t)$ – converges to zero in the norm topology as $s \to \pm\infty$. Hence the linear operator $C : \mathcal{W}^{1,2} \to L^2$ is a compact perturbation of D_A by [RS95, lem. 3.18]. Since the Fredholm property and the Fredholm index are invariant under compact perturbations, it suffices to prove that D_A is Fredholm and compute its index. By [RS95, thm. A] it remains to verify the following properties.

(i) The inclusion of Hilbert spaces $W \hookrightarrow H$ is compact with dense image.

(ii) The operator $A(s) : H \to H$ with dense domain W is unbounded and self-adjoint for every s.

(iii) The norm of W is equivalent to the graph norm of $A(s)$ for every s.

(iv) The map $\mathbb{R} \to \mathcal{L}(W, H) : s \mapsto A(s)$ is continuously differentiable with respect to the weak operator topology.

(v) There exist invertible operators $A^\pm \in \mathcal{L}(W, H)$ which are the limits of $A(s)$ in the norm topology, as s tends to $\pm\infty$.

Statements (i) and (ii) follow by the Sobolev embedding theorem, the well known fact that the 1-dimensional Laplacian $-d^2/dt^2$ on $[0, 1]$ with periodic boundary conditions is self-adjoint, and by the Kato-Rellich Theorem since the perturbation $B+Q$ is of relative bound zero; see [ReS75]. To prove (iii) one has to establish that the W norm is bounded above by a constant times the graph norm and vice versa. The first inequality uses the elliptic estimate for the operator $A(s)$ and the second one follows since $\|\partial_t u_s\|_\infty$ and $\|\nabla_t \partial_t u_s\|_2$ are bounded by (55) and the Hessian $\mathcal{H}_\mathcal{V}(u_s)$ is a bounded linear operator on $L^2(S^1, {u_s}^*TM)$ by axiom (V1). To prove (iv) we need to show that, given any $\xi \in W$ and $\eta \in H$, the map $s \mapsto \langle \eta, A(s)\xi \rangle$ is in $C^1(\mathbb{R}, \mathbb{R})$. This follows by the bounds in (55) and (56), by the final estimate in axiom (V2), and the apparently unnatural[3] assumption in hypothesis 3.15 that $\nabla_t \nabla_t \partial_s u_s$ be uniformly L^2 bounded. Statement (v) is true, since the critical points $x^\pm$ are nondegenerate and u_s and $\partial_t u_s$ converge in C^0 to $x^\pm$ and $\partial_t x^\pm$, respectively, and $\nabla_t \partial_t u_s$ converges in L^2 to $\nabla_t \partial_t x^\pm$, all as $s \to \pm\infty$.

The properties (i–v) are precisely the assumptions of theorem A in [RS95] which asserts that the operator $D_A : \mathcal{W}^{1,2} \to L^2$ is Fredholm and its index is given by the spectral flow of the operator family $A(s)$. The spectral flow represents the net change in the number of negative eigenvalues of $A(s)$ as s runs from $-\infty$ to ∞. It is equal to $\mathrm{ind}(A^-) - \mathrm{ind}(A^+)$ where $\mathrm{ind}(A^\pm)$ denotes the Morse index, i.e. the number of negative eigenvalues of the self-adjoint operator $A^\pm$. To see this observe that $\mathrm{ind}(A^+)$ equals $\mathrm{ind}(A^-)$ plus the number of eigenvalues changing from positive to negative minus the number of those changing sign in the opposite direction. Finally, the Fredholm indices of D_A and D_{A+C} are equal, since $\{D_{A+\tau C}\}_{\tau \in [0,1]}$ is an interpolating family of Fredholm operators. This proves theorem 1.9 in the case $p = 2$.

[3]If in [RS95, thm.A], hence in (iv), *continuously differentiable* could be replaced by *continuous*, then the assumption on $\|\nabla_t \nabla_t \partial_s u_s\|_2$ can be dropped in hypothesis 3.15 and theorem 1.9.

Remark 3.16 (The formal adjoint)**.** If $\mathcal{D}_u : \mathcal{W}^{1,2}_u \to \mathcal{L}^2_u$ is represented with respect to an orthonormal frame by the operator D_{A+C} in (59), then $\mathcal{D}_u^*$ is represented by $-D_{-A-C}$. Above we proved that A satisfies (i-v), hence so does $-A$. Thus D_{-A} is a Fredholm operator again by [RS95, thm. A] and its index is given by minus the spectral flow of the operator family $A = A(s)$. But if D_{-A} is Fredholm, so is its negative $-D_{-A}$ and both Fredholm indices are equal, since both kernels and both cokernels coincide. Now $-D_{-A}$ and $-D_{-A-C}$ are homotopic through the family $\{-D_{-A-\tau C}\}_{\tau\in[0,1]}$ of Fredholm operators. This proves that the formal adjoint operator $\mathcal{D}_u^* : \mathcal{W}^{1,2}_u \to \mathcal{L}^2_u$ is Fredholm and $\mathrm{index}\mathcal{D}_u^* = -\mathrm{index}\mathcal{D}_u$.

Fredholm property and index for $p > 1$

Still assuming hypothesis 3.15 consider the vector space given by

$$X_0 := \Big\{\xi \in C^\infty(\mathbb{R}\times S^1, u^*TM) \mid \mathcal{D}_u\xi = 0,\, \exists c, \delta > 0\ \forall s \in \mathbb{R} : \\ \|\xi_s\|_\infty + \|\nabla_t\xi_s\|_\infty + \|\nabla_t\nabla_t\xi_s\|_\infty + \|\nabla_s\xi_s\|_\infty \le ce^{\delta|s|}\Big\}.$$

Define X_0^* by using $\mathcal{D}_u^*$ in the definition. Note that p does not enter.

Proposition 3.17. *Let $p > 1$, then*

$$\ker\left[\mathcal{D}_u : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u\right] = X_0, \qquad \ker\left[\mathcal{D}_u^* : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u\right] = X_0^*.$$

Proof. The inclusion $\supset$ is trivial. To prove the inclusion $\subset$ assume that $\xi \in \mathcal{W}^{1,p}$ solves $\mathcal{D}_u\xi = 0$ almost everywhere. Being a local property smoothness of ξ follows from theorem 3.1 using integration by parts. Exponential L^∞ decay follows by combining the apriori estimates theorem 3.3 and theorem 3.4 with the L^2 exponential decay results theorem 3.10 and remark 3.11. The last two results require nondegeneracy of the critical points $x^\pm$ and boundedness of the map

$$s \mapsto \|\xi_s\|_2.$$

To see the latter use the Sobolev embedding theorem together with the fact that the vector field ξ along the cylinder u is of class $\mathcal{W}^{1,p}_u$ and satisfies $\mathcal{D}_u\xi = 0$. This proves that X_0 is the kernel of $\mathcal{D}_u$. The result for $\mathcal{D}_u^*$ follows by reflection $s \mapsto -s$. □

Proposition 3.18. *The range of $\mathcal{D}_u, \mathcal{D}_u^* : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u$ is closed whenever $p > 1$.*

Proof. The structure of proof is standard; see e.g. [S99, sec. 2]. We sketch the two key steps for $\mathcal{D}_u$. Step one is the linear estimate

$$\|\xi\|_{\mathcal{W}^{1,p}} \le c_p\left(\|\mathcal{D}_u\xi\|_p + \|\xi\|_p\right)$$

for compactly supported vector fields ξ along u. This follows immediately from proposition 2.13, lemma 2.12, the L^∞ bound for $\partial_t u$ in (55) and axiom (V1).

Step two is to prove bijectivity of $\mathcal{D}_u$ in the case of the constant cylinder $u(s,t) = x(t)$, whenever x is a nondegenerate critical point of $\mathcal{S}_\mathcal{V}$. We give a proof for $p \geq 2$ in the related situation of half cylinders in theorem 8.5 below. The case $1 < p \leq 2$ follows by duality; see [S99, exc. 2.5]. Both steps are then combined by a cutoff function argument; see [S99, thm 2.2]. $\square$

Proposition 3.18 enables us to define the cokernels of $\mathcal{D}_u : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p$ and $\mathcal{D}_u^* : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p$ as Banach space quotients, namely for $p > 1$ set

$$\operatorname{coker} \mathcal{D}_u := \frac{\mathcal{L}_u^p}{\operatorname{im} \mathcal{D}_u}, \qquad \operatorname{coker} \mathcal{D}_u^* := \frac{\mathcal{L}_u^p}{\operatorname{im} \mathcal{D}_u^*}.$$

The next result shows that these spaces are again independent of p.

Proposition 3.19. *Let $p > 1$, then*

$$\operatorname{coker} \left[\mathcal{D}_u : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p \right] = X_0^*, \qquad \operatorname{coker} \left[\mathcal{D}_u^* : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p \right] = X_0.$$

Proof. We prove the second identity. The other one follows by reflection $s \mapsto -s$. We identify the image of $\mathcal{D}_u^*$ in $\mathcal{L}_u^p$ with its annihilator $(\operatorname{im} \mathcal{D}_u^*)^\perp$ in $\mathcal{L}_u^q = (\mathcal{L}_u^p)^*$ where $\frac{1}{q} + \frac{1}{p} = 1$, that is we identify

$$\operatorname{coker} \mathcal{D}_u^* \simeq (\operatorname{im} \mathcal{D}_u^*)^\perp .$$

The inclusion $\supset$ is trivial. To prove the inclusion $\subset$ assume that $\xi \in (\operatorname{im} \mathcal{D}_u^*)^\perp$. This means that $\xi \in \mathcal{L}_u^p$ and that $\langle \xi, \mathcal{D}_u^* \eta \rangle = 0$ for all $\eta \in C_0^\infty(\mathbb{R} \times S^1)$. Hence ξ is smooth by theorem 3.1. Integration by parts then shows that $\mathcal{D}_u \xi = 0$. Exponential decay follows by combining theorem 3.3 and theorem 3.4 with theorem 3.10 and remark 3.11 as explained in the proof of proposition 3.17. $\square$

Remark 3.20. It is an easy but important consequence of proposition 3.19 that if $\mathcal{D}_u : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p$ is surjective for some $p > 1$, then it is surjective for all $p > 1$. This justifies the phrase "$\mathcal{D}_u$ is surjective" encountered occasionally.

Proof of theorem 1.9. The range of $\mathcal{D}_u : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p$ is closed by proposition 3.18. Moreover, by proposition 3.17 and proposition 3.19 the kernel and the cokernel of $\mathcal{D}_u : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p$ are given by X_0 and X_0^*, respectively. Now these vector spaces do not depend on $p > 1$. But for $p = 2$ we proved in the previous subsection that they are finite dimensional and the difference of their dimensions equals $\operatorname{ind}_\mathcal{V}(x^-) - \operatorname{ind}_\mathcal{V}(x^+)$. The claim for $\mathcal{D}_u^*$ follows similarly. $\square$

4 Solutions of the nonlinear heat equation

4.1 Regularity and compactness

Throughout this subsection we embed the compact Riemannian manifold M isometrically into some Euclidean space $\mathbb{R}^N$ and view any continuous map $u : Z = (-T, 0] \times S^1 \to M$ as a map into $\mathbb{R}^N$ taking values in the embedded manifold. We indicate this by the notation $u : Z \to M \hookrightarrow \mathbb{R}^N$. Then the heat equation (6) is of the form

$$\partial_s u - \partial_t \partial_t u = \Gamma(u) \left(\partial_t u, \partial_t u\right) + F. \tag{60}$$

Here and throughout this section Γ denotes the second fundamental form associated to the embedding $M \hookrightarrow \mathbb{R}^N$ and the map $F : Z \to \mathbb{R}^N$ is given by

$$F(s,t) := (\mathrm{grad}\mathcal{V}(u_s))(t). \tag{61}$$

Recall the definition of the $\mathcal{W}^{k,p}$ and the $\mathcal{C}^k$ norm in (13) and (14), respectively.

Proposition 4.1. *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V3)*, constants $p > 2$ and $\mu_0 > 0$, and cylinders*

$$Z = (-T, 0] \times S^1, \qquad Z' = (-T', 0] \times S^1, \qquad T > T' > 0.$$

Then for every integer $k \geq 1$ there is a constant $c_k = c_k(p, \mu_0, T, T', \mathcal{V})$ such that the following is true. If $u : Z \to M \hookrightarrow \mathbb{R}^N$ is a $\mathcal{W}^{1,p}$ map such that

$$\left\|u\right\|_p + \left\|\partial_s u\right\|_p + \left\|\partial_t u\right\|_p + \left\|\partial_t \partial_t u\right\|_p \leq \mu_0 \tag{62}$$

and which satisfies the heat equation (60) almost everywhere, then

$$\left\|u\right\|_{\mathcal{W}^{k,p}(Z',\mathbb{R}^N)} \leq c_k.$$

Proposition 4.1 follows by induction from the bootstrap proposition 2.18 using all axioms (V0)–(V3) and a product estimate, lemma 4.4 below. By standard arguments proposition 4.1 immediately implies theorem 4.2 on regularity and theorem 4.3 on compactness.

Theorem 4.2 (Regularity)**.** *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V3) *and constants $p > 2$ and $a < b$. Let u be a map $(a, b] \times S^1 \to M \hookrightarrow \mathbb{R}^N$ which is of Sobolev class $\mathcal{W}^{1,p}$ and solves the heat equation (60) almost everywhere. Then u is smooth.*

Theorem 4.3 (Compactness)**.** *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V3) *and constants $p > 2$ and $a < b$. Let $u^\nu : (a, b] \times S^1 \to M \hookrightarrow \mathbb{R}^N$ be a sequence of smooth solutions of the heat equation (60) such that*

$$\sup_\nu \left\|\partial_t u^\nu\right\|_\infty + \sup_\nu \left\|\partial_s u^\nu\right\|_p < \infty.$$

Then there is a smooth solution $u : (a, b] \times S^1 \to M$ of (60) and a subsequence, still denoted by u^ν, such that u^ν converges to u, uniformly with all derivatives on every compact subset of $(a, b] \times S^1$.

Lemma 4.4. *Fix a constant $p > 2$ and a bounded open subset $\Omega \subset \mathbb{R}^2$ with area $|\Omega|$. Then for every integer $k \geq 1$ there is a constant $c = c(k, |\Omega|)$ such that*

$$\|\partial_s u \cdot v\|_{\mathcal{W}^{k,p}} \leq c \, \|\partial_s u\|_{\mathcal{W}^{k,p}} \|v\|_\infty + c \left(\|u\|_{\mathcal{C}^k} + \|\partial_t u\|_{\mathcal{C}^k} \right) \|v\|_{\mathcal{W}^{k,p}}$$

for all functions $u, v \in C^\infty(\overline{\Omega})$.

Proof. The proof is by induction on k. By definition of the $\mathcal{W}^{\ell,p}$ norm

$$\begin{aligned} \|\partial_s u \cdot v\|_{\mathcal{W}^{\ell+1,p}} \leq & \|\partial_s u \cdot v\|_{\mathcal{W}^{\ell,p}} + \|\partial_t \partial_s u \cdot v + \partial_s u \cdot \partial_t v\|_{\mathcal{W}^{\ell,p}} \\ & + \|\partial_t \partial_t \partial_s u \cdot v + 2 \partial_t \partial_s u \cdot \partial_t v + \partial_s u \cdot \partial_t \partial_t v\|_{\mathcal{W}^{\ell,p}} \\ & + \|\partial_s \partial_s u \cdot v + \partial_s u \cdot \partial_s v\|_{\mathcal{W}^{\ell,p}} . \end{aligned} \tag{63}$$

Step $k = 1$. Estimate (63) for $\ell = 0$ shows that

$$\begin{aligned} \|\partial_s u \cdot v\|_{\mathcal{W}^{1,p}} \leq & \Big(\|\partial_s u\|_p + \|\partial_t \partial_s u\|_p + \|\partial_t \partial_t \partial_s u\|_p + \|\partial_s \partial_s u\|_p \Big) \|v\|_\infty \\ & + 2 \, \|\partial_t \partial_s u\|_\infty \|\partial_t v\|_p \\ & + \|\partial_s u\|_\infty \Big(\|\partial_t v\|_p + \|\partial_t \partial_t v\|_p + \|\partial_s v\|_p \Big) . \end{aligned}$$

Since $\partial_t \partial_s u = \partial_s \partial_t u$ this proves the lemma for $k = 1$.
Induction step $k \Rightarrow k + 1$. Consider estimate (63) for $\ell = k$, then inspect the right hand side term by term using the induction hypothesis for the appropriate functions to conclude the proof. To illustrate this we give full details for the last term in (63), namely

$$\begin{aligned} \|\partial_s u \cdot \partial_s v\|_{\mathcal{W}^{k,p}} & \leq c \, \|\partial_s u\|_{\mathcal{W}^{k,p}} \|\partial_s v\|_\infty + c \left(\|u\|_{\mathcal{C}^k} + \|\partial_t u\|_{\mathcal{C}^k} \right) \|\partial_s v\|_{\mathcal{W}^{k,p}} \\ & \leq c c_1 \, |\Omega| \, \|\partial_s u\|_{\mathcal{C}^k} \|\partial_s v\|_{\mathcal{W}^{1,p}} + c \left(\|u\|_{\mathcal{C}^k} + \|\partial_t u\|_{\mathcal{C}^k} \right) \|v\|_{\mathcal{W}^{k+1,p}} \\ & \leq c c_1 \, |\Omega| \, \|u\|_{\mathcal{C}^{k+1}} \|v\|_{\mathcal{W}^{2,p}} + c \left(\|u\|_{\mathcal{C}^k} + \|\partial_t u\|_{\mathcal{C}^k} \right) \|v\|_{\mathcal{W}^{k+1,p}} . \end{aligned}$$

The first step is by the induction hypothesis for the function $\partial_s v$. In the second step we pulled out the L^∞ norms of all derivatives of $\partial_s u$ and for the term $\partial_s v$ we applied the Sobolev embedding $\mathcal{W}^{1,p} \subset W^{1,p} \hookrightarrow C^0$ with constant c_1. Here our assumptions $p > 2$ and Ω bounded enter. Step three is obvious. Note that $k \geq 1$ implies that $\mathcal{W}^{k+1,p} \hookrightarrow \mathcal{W}^{2,p}$. □

Proof of proposition 4.1. Consider the family

$$T_r := T' + \frac{T - T'}{r}, \qquad r \in [1, \infty),$$

and the corresponding nested sequence of cylinders $Z_r := (-T_r, 0] \times S^1$ with

$$Z = Z_1 \supset Z_2 \supset Z_3 \supset \ldots \supset Z'.$$

Denote by C_0 the constant in (V0). More generally, for $\ell \geq 1$ choose C_ℓ larger than $C_{\ell-1}$ and larger than all constants $C(k', \ell', \mathcal{V})$ in (V3) for which $2k' + \ell' \leq \ell$.

CLAIM. *The map F given by (61) is in $\mathcal{W}^{\ell,p}(Z_{\ell+1})$ for every integer $\ell \geq 1$.*

Proposition 4.1 immediately follows: Given any integer $k \geq 1$, then $F \in \mathcal{W}^{k,p}(Z_{k+1})$ by the claim. Furthermore, by inclusion $Z_{k+1} \subset Z$ and (62)

$$\|u\|_{\mathcal{W}^{1,p}(Z_{k+1})} \leq \|u\|_{\mathcal{W}^{1,p}(Z)} \leq \mu_0.$$

Hence by theorem 2.1 for the pair $Z_{k+2} \subset Z_{k+1}$ there is a constant c_{k+1} depending on p, μ_0, Z_{k+2}, Z_{k+1}, $\|\Gamma\|_{C^{2k+2}}$, and $\|F\|_{\mathcal{W}^{k,p}(Z_{k+1})}$ such that

$$\|u\|_{\mathcal{W}^{k+1,p}(Z')} \leq \|u\|_{\mathcal{W}^{k+1,p}(Z_{k+2})} \leq c_{k+1}.$$

It remains to prove the claim. The proof is by induction.

Step $\ell = 1$. We need to prove that F, $\partial_t F$, $\partial_s F$, and $\partial_t \partial_t F$ are in $L^p(Z_2)$. The domain of all norms of Γ and its derivatives is the compact manifold M. The domain of all other norms is the cylinder Z unless indicated differently. By axiom (V0) with constant C_0 it follows (even on the larger domain Z) that

$$\|F\|_\infty = \sup_{s \in (-T,0]} \|\mathrm{grad}\mathcal{V}(u_s)\|_{L^\infty(S^1)} \leq C_0 \tag{64}$$

and therefore

$$\|F\|_p \leq \|F\|_\infty \left(\mathrm{Vol}\, Z\right)^{1/p} \leq C_0 T^{1/p}.$$

Next we use axiom (V1) with constant $C_1 \geq C_0$ to obtain that

$$\begin{aligned}
\|\partial_t F\|_p &\leq \|\nabla_t \mathrm{grad}\mathcal{V}(u)\|_p + \|\Gamma(u)\left(\partial_t u, \mathrm{grad}\mathcal{V}(u)\right)\|_p \\
&\leq C_1 \left(1 + \|\partial_t u\|_p\right) + \|\Gamma\|_\infty \|\partial_t u\|_p \|F\|_\infty \\
&\leq C_1(1+\mu_0) + \|\Gamma\|_\infty \mu_0 C_0.
\end{aligned}$$

Here we used the assumption (62) in the last step. Now by the bootstrap proposition 2.18 (i) for $k = 1$ and the pair $Z_{4/3} \subset Z$ there is a constant a_1 depending on p, μ_0, $Z_{4/3}$, Z, $\|\Gamma\|_{C^4}$, and the $L^p(Z)$ norms of F and $\partial_t F$ such that $\|\partial_t u\|_{\mathcal{W}^{1,p}(Z_{4/3})} \leq a_1$. Then by the Sobolev embedding $W^{1,p} \hookrightarrow C^0$ with constant $c' = c'(p, Z_{5/3})$ it follows that $\partial_t u$ is continuous on $Z_{4/3}$ and

$$\|\partial_t u\|_{C^0(Z_{5/3})} \leq c' \|\partial_t u\|_{\mathcal{W}^{1,p}(Z_{5/3})} \leq a_1 c'. \tag{65}$$

Again using axiom (V1) we obtain similarly that

$$\begin{aligned}
\|\partial_s F\|_p &\leq \|\nabla_s \mathrm{grad}\mathcal{V}(u)\|_p + \|\Gamma(u)\left(\partial_s u, \mathrm{grad}\mathcal{V}(u)\right)\|_p \\
&\leq 2C_1 \|\partial_s u\|_p + \|\Gamma\|_\infty \|\partial_s u\|_p \|F\|_\infty \\
&\leq \mu_0 \left(2C_1 + \|\Gamma\|_\infty C_0\right).
\end{aligned}$$

In order to estimate $\partial_t \partial_t F$ observe first that

$$\begin{aligned}
\|\nabla_t \partial_t u\|_{L^p(Z_{5/3})} &\leq \|\partial_t \partial_t u\|_{L^p(Z_{5/3})} + \|\Gamma\|_\infty \||\partial_t u| \cdot |\partial_t u|\|_{L^p(Z_{5/3})} \\
&\leq \mu_0 + \|\Gamma\|_\infty \|\partial_t u\|_{C^0(Z_{5/3})} \|\partial_t u\|_{L^p(Z_{5/3})} \\
&\leq \mu_0 + \|\Gamma\|_\infty a_1 c' \mu_0.
\end{aligned}$$

Here the last step uses assumption (62) and the C^0 estimate (65) for $\partial_t u$ which requires shrinking of the domain. Now by axiom (V3) for $k=0$ and $\ell=2$ there is a constant still denoted by $C_1 = C_1(\mathcal{V})$ such that

$$|\nabla_t \nabla_t F| \le C_1 \Big(1 + |\partial_t u| + |\nabla_t \partial_t u| \Big) \tag{66}$$

pointwise for every (s,t). Integrating this inequality to the power p implies that

$$\begin{aligned} \|\nabla_t \nabla_t F\|_{L^p(Z_{5/3})} &\le C_1 \left(1 + \|\partial_t u\|_{L^p(Z_{5/3})} + \|\nabla_t \partial_t u\|_{L^p(Z_{5/3})} \right) \\ &\le C_1 \left(1 + 2\mu_0 + \|\Gamma\|_\infty a_1 c' \mu_0 \right). \end{aligned}$$

Straightforward calculation shows that

$$\begin{aligned} \|\partial_t \partial_t F\|_{L^p(Z_{5/3})} \le& \|\nabla_t \nabla_t F\|_{L^p} + \|d\Gamma\|_\infty \|\partial_t u\|_{C^0} \|\partial_t u\|_{L^p} \|F\|_{C^0} \\ &+ \|\Gamma\|_\infty \|\partial_t \partial_t u\|_{L^p} \|F\|_{C^0} + 2 \|\Gamma\|_\infty \|\partial_t u\|_{C^0} \|\partial_t F\|_{L^p} \\ &+ \|\Gamma\|_\infty^2 \|\partial_t u\|_{C^0} \|\partial_t u\|_{L^p} \|F\|_{C^0} \end{aligned}$$

is bounded by a constant $c = c(p, \mu_0, c', C_1, \|\Gamma\|_{C^1})$. Here all C^0 and L^p norms are on the domain $Z_{5/3}$. We used again assumption (62), the estimates for F and its derivatives obtained earlier, and (65).

Induction step $\ell \Rightarrow \ell+1$. Let $\ell \ge 1$ and assume that the claim is true for ℓ. This means that F is in $\mathcal{W}^{\ell,p}(Z_{\ell+1}$, hence

$$\alpha_\ell := \|F\|_{\mathcal{W}^{\ell,p}(Z_{\ell+1})} < \infty.$$

Therefore by theorem 2.1 for the integer ℓ and the pair of sets $Z_{\ell+1} \supset Z_{\ell+3/2}$ there is a constant $c_\ell = c_\ell(p, \mu_0, T_{\ell+1}, T_{\ell+3/2}, \|\Gamma\|_{C^{2\ell+2}}, \alpha_\ell)$ such that

$$\|u\|_{\mathcal{W}^{\ell+1,p}(Z_{\ell+3/2})} \le c_\ell, \qquad \|u\|_{\mathcal{C}^\ell(Z_{\ell+3/2})} \le c_\ell. \tag{67}$$

The second inequality follows from the first by the Sobolev embedding $W^{1,p} \hookrightarrow C^0$ applied to each term in the $\mathcal{C}^\ell$ norm. Then choose c_ℓ larger, if necessary. It remains to prove that the $\mathcal{W}^{\ell,p}(Z_{\ell+2})$ norms of $\partial_t F$, $\partial_s F$, and $\partial_t \partial_t F$ are finite. Similarly as in step $\ell = 1$ we obtain that

$$\begin{aligned} \|\partial_t F\|_{\mathcal{W}^{\ell,p}(Z_{\ell+3/2})} &\le \|\nabla_t F\|_{\mathcal{W}^{\ell,p}} + \|\Gamma(u)\,(\partial_t u, F)\|_{\mathcal{W}^{\ell,p}} \\ &\le C_1 \left(\|1\|_{\mathcal{W}^{\ell,p}} + \|\partial_t u\|_{\mathcal{W}^{\ell,p}} \right) \\ &\quad + \tilde{c} \|\Gamma\|_{\mathcal{C}^\ell} \left(\|\partial_t u\|_{\mathcal{W}^{\ell,p}} \|F\|_\infty + \|u\|_{\mathcal{C}^\ell} \|F\|_{\mathcal{W}^{\ell,p}} \right) \\ &\le C_1 \left(T^{1/p} + c_\ell \right) + \tilde{c} \|\Gamma\|_{\mathcal{C}^\ell} \left(c_\ell C_0 + c_\ell \alpha_\ell \right). \end{aligned}$$

Here the domain of all norms, except the one of Γ, is $Z_{\ell+3/2}$. The first step is by definition of the covariant derivative and the triangle inequality. Step two uses axiom (V1) and lemma 2.21 with constant $\tilde{c}$. The last step uses the estimates (64), (67), and the definition of α_ℓ in the induction hypothesis. Now by the refined bootstrap proposition 2.18 there is a constant $a_{\ell+1}$ such that

$$\|\partial_t u\|_{\mathcal{W}^{\ell+1,p}(Z_{\ell+2})} \le a_{\ell+1}, \qquad \|\partial_t u\|_{\mathcal{C}^\ell(Z_{\ell+2})} \le a_{\ell+1}. \tag{68}$$

Next observe that

$$
\begin{aligned}
&\|\partial_s F\|_{\mathcal{W}^{\ell,p}(Z_{\ell+2})} \\
&\le \|\nabla_s F\|_{\mathcal{W}^{\ell,p}} + \|\Gamma(u)\,(\partial_s u, F)\|_{\mathcal{W}^{\ell,p}} \\
&\le 2C_1 \|\partial_s u\|_{\mathcal{W}^{\ell,p}} + C' \|\Gamma\|_{\mathcal{C}^\ell} \left(\|\partial_s u\|_{\mathcal{W}^{\ell,p}} \|F\|_\infty + (\|u\|_{\mathcal{C}^\ell} + \|\partial_t u\|_{\mathcal{C}^\ell}) \|F\|_{\mathcal{W}^{\ell,p}}\right) \\
&\le 2C_1 c_\ell + C' \|\Gamma\|_{\mathcal{C}^\ell} \left(c_\ell C_0 + (c_\ell + a_{\ell+1})\alpha_\ell\right).
\end{aligned}
$$

Here the domain of all norms, except the one of Γ, is $Z_{\ell+2}$. Again the first step is by definition of the covariant derivative and the triangle inequality. Step two uses axiom (V1) and lemma 4.4 with constant C'. The last step uses the estimates (64), (67), (68), and the definition of α_ℓ in the induction hypothesis. Similarly as in step $\ell = 1$ we obtain that

$$
\begin{aligned}
&\|\partial_t\partial_t F\|_{\mathcal{W}^{\ell,p}(Z_{\ell+2})} \\
&\le \|\nabla_t\nabla_t F\|_{\mathcal{W}^{\ell,p}} + \|d\Gamma(u)\,(\partial_t u, \partial_t u, F)\|_{\mathcal{W}^{\ell,p}} \\
&\quad + \|\Gamma(u)\,(\partial_t\partial_t u, F)\|_{\mathcal{W}^{\ell,p}} + 2\,\|\Gamma(u)\,(\partial_t u, \partial_t F)\|_{\mathcal{W}^{\ell,p}} \\
&\quad + \|\Gamma(u)\,(\partial_t u, \Gamma(u)\,(\partial_t u, F))\|_{\mathcal{W}^{\ell,p}} \\
&\le C_1 \Big(T^{1/p} + \|\partial_t u\|_{\mathcal{W}^{\ell,p}} + \|\partial_t\partial_t u\|_{\mathcal{W}^{\ell,p}} + \|\Gamma\|_{\mathcal{C}^\ell} \|\partial_t u\|_{\mathcal{C}^\ell} \|\partial_t u\|_{\mathcal{W}^{\ell,p}}\Big) \\
&\quad + \|d\Gamma\|_{\mathcal{C}^\ell} \|\partial_t u\|_{\mathcal{C}^\ell}^2 \|F\|_{\mathcal{W}^{\ell,p}} \\
&\quad + \tilde{c}\,\|\Gamma\|_{\mathcal{C}^\ell} \left(\|\partial_t\partial_t u\|_{\mathcal{W}^{\ell,p}} \|F\|_\infty + \|\partial_t u\|_{\mathcal{C}^\ell} \|F\|_{\mathcal{W}^{\ell,p}}\right) \\
&\quad + 2\,\|\Gamma\|_{\mathcal{C}^\ell} \|\partial_t u\|_{\mathcal{C}^\ell} \|\partial_t F\|_{\mathcal{W}^{\ell,p}} \\
&\quad + \|\Gamma\|_{\mathcal{C}^\ell}^2 \|\partial_t u\|_{\mathcal{C}^\ell}^2 \|F\|_{\mathcal{W}^{\ell,p}}.
\end{aligned}
$$

Here the domain of all norms, except the one of Γ, is $Z_{\ell+2}$. In the second step we used axiom (V2) with constant C_1 to estimate the term $\nabla_t\nabla_t F$ and we spelled out the covariant derivative arising in $\nabla_t\partial_t u$. Moreover, crudely pulling out $\mathcal{C}^\ell$ norms worked for all terms but the third one, the one involving $\partial_t\partial_t u$, here we used lemma 4.4 with constant $\tilde{c}$ for the functions $\partial_t\partial_t u$ and F. Now all terms appearing on the right hand side have been estimated earlier. This proves the induction step and therefore the claim and proposition 4.1. □

Proof of theorem 4.2. Fix any point $z \in Z = (a,b] \times S^1$ and a subcylinder $Z' = (a',b] \times S^1$ that contains z and where $a' \in (a,b)$. Set $\mu_0 = \|u\|_{\mathcal{W}^{1,p}(Z,\mathbb{R}^N)}$, then proposition 4.1 for the function $\tilde{u}(s,t) := u(s+b,t)$ and the constants $T = b-a$ and $T' = b-a'$ implies that

$$
u \in \bigcap_{k\ge 0} \mathcal{W}^{k,p}(Z',\mathbb{R}^N) = \bigcap_{k\ge 0} W^{k,p}(Z',\mathbb{R}^N) = C^\infty(\overline{Z'},\mathbb{R}^N).
$$

See [MS04, app. B.1] for the last step. Hence u is locally smooth. □

Proof of theorem 1.5. Theorem 4.2. □

Proof of theorem 4.3. Shifting the s variable by b and setting $T = b - a$, if necessary, we may assume without loss of generality that the maps u^ν are defined on $(-T, 0]$ and, furthermore, by composition with the isometric embedding $M \hookrightarrow \mathbb{R}^N$ that they take values in $\mathbb{R}^N$. All norms are taken on the domain $(-T, 0] \times S^1$, unless indicated otherwise. To apply proposition 4.1 we need to verify that the maps $u^\nu : (-T, 0] \times S^1 \to \mathbb{R}^N$ satisfy the four apriori estimates in (62) for some constant μ_0 independent of ν. To see this observe that

$$\|u^\nu\|_p \leq \|u^\nu\|_\infty \operatorname{Vol}\left((-T, 0] \times S^1\right) \leq c_1 T^{1/p}$$

for some constant c_1 depending only on the isometric embedding $M \hookrightarrow \mathbb{R}^N$ and the diameter of the compact manifold M. By assumption there is a constant c_2 independent of ν such that

$$\|\partial_t u^\nu\|_p \leq \|\partial_t u^\nu\|_\infty T^{1/p} \leq c_2 T^{1/p}$$

and

$$\|\partial_s u^\nu\|_p \leq c_2.$$

Then it follows by the heat equation (60) that

$$\|\nabla_t \partial_t u^\nu\|_p \leq \|\partial_s u^\nu\|_p + \|\operatorname{grad}\mathcal{V}(u^\nu)\|_p \leq c_2 + C_0 T^{1/p}.$$

In the second step we used (V0) to estimate $\operatorname{grad}\mathcal{V}(u^\nu)$ in L^∞ from above by a constant $C_0 = C_0(\mathcal{V})$. By definition of the covariant derivative

$$\begin{aligned}
\|\partial_t \partial_t u^\nu\|_p &\leq \|\nabla_t \partial_t u^\nu\|_p + \|\Gamma\|_{C^0(M)} \|\partial_t u^\nu\|_\infty \|\partial_t u^\nu\|_p \\
&\leq c_2 + C_0 T^{1/p} + c_2^2 T^{1/p} \|\Gamma\|_{C^0(M)}.
\end{aligned}$$

Now set $\mu_0 := c_2 + C_0 T^{1/p} + c_2^2 T^{1/p} \|\Gamma\|_{C^0(M)} + (c_1 + c_2) T^{1/p}$. Then proposition 4.1 asserts that for every constant $T' \in (0, T)$ and every integer $k \geq 2$ there is a constant $c_k = c_k(p, \mu_0, T, T', \mathcal{V})$ such that

$$\|u^\nu\|_{\mathcal{W}^{k,p}(Q,\mathbb{R}^N)} \leq c_k$$

where $Q = [-T', 0] \times S^1$. Recall that the inclusion $W^{k,p}(Q) \hookrightarrow C^{k-1}(Q)$ is compact; see e.g. [MS04, B.1.11]. Hence there is a subsequence which converges on Q in the C^k topology. We denote the limit by $u \in C^k(Q)$. Since this is true for every $k \geq 2$ there is a subsequence, still denoted by u^ν, converging on Q to u, uniformly with all derivatives. Since this is true for every compact subcylinder Q of $(-T, 0] \times S^1$, the theorem follows by choosing a diagonal subsequence associated to an exhausting sequence by such Q's. Because, in particular, the convergence is in C^0 and the u^ν take values in M, so does the limit u. By C^k convergence with $k \geq 2$ the limit u satisfies the heat equation (60). $\square$

4.2 An apriori estimate

Theorem 4.5. *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V1) *and a constant $c_0 > 0$. Then there is a constant $C = C(c_0, \mathcal{V}) > 0$ such that the following holds. If $u : \mathbb{R} \times S^1 \to M$ is a smooth solution of (6) such that*

$$\sup_{s\in\mathbb{R}} \mathcal{S}_{\mathcal{V}}(u(s,\cdot)) \le c_0 \tag{69}$$

then $\|\partial_t u\|_\infty \le C$.

The proof of theorem 4.5 is based on the following mean value inequality. For $r > 0$ define the **open parabolic rectangle** $P_r \subset \mathbb{R}^2$ by

$$P_r := (-r^2, 0) \times (-r, r).$$

Lemma 4.6 ([SW03, lemma B.1])**.** *There is a constant $c_1 > 0$ such that the following holds for all $r \in (0, 1]$ and $a \ge 0$. If $w : P_r \to \mathbb{R}$, $(s,t) \mapsto w(s,t)$, is C^1 in the s-variable and C^2 in the t-variable such that*

$$(\partial_t\partial_t - \partial_s)w \ge -aw, \qquad w \ge 0,$$

then

$$w(0) \le \frac{c_1 e^{ar^2}}{r^3} \int_{P_r} w.$$

Corollary 4.7. *Fix two constants $r \in (0, 1]$ and $\mu \ge 0$. Let c_1 be the constant of lemma 4.6. If $F : [-r^2, 0] \to \mathbb{R}$ is a C^2 function satisfying*

$$-F' + \mu F \ge 0, \qquad F \ge 0,$$

then

$$F(0) \le \frac{2c_1 e^{\mu r^2}}{r^2} \int_{-r^2}^{0} F(s)\, ds.$$

Proof. This follows immediately from lemma 4.6 with $w(s,t) := f(s)$. □

Proof of theorem 4.5. The idea is to first derive slicewise L^2 bounds, then verify the differential inequality in lemma 4.6 and apply the lemma using the slicewise bounds on the right hand side. The slicewise bound for $\partial_t u$ follows easily from the assumption

$$c_0 \ge \mathcal{S}_{\mathcal{V}}(u_s) = \frac{1}{2}\|\partial_t u_s\|^2_{L^2(S^1)} - \mathcal{V}(u_s)$$

where $u_s(t) := u(s,t)$. Let C_0 denote the constant in (V0), then this implies

$$\|\partial_t u_s\|^2_{L^2(S^1)} \le 2c_0 + 2\mathcal{V}(u_s) \le 2c_0 + 2C_0 \tag{70}$$

for every $s \in \mathbb{R}$. Consider the pointwise differential inequality given by

$$\begin{aligned}
(\partial_t\partial_t - \partial_s)\,|\partial_t u|^2 &= 2\,|\nabla_t\partial_t u|^2 + 2\langle(\nabla_t\nabla_t - \nabla_s)\partial_t u, \partial_t u\rangle \\
&= 2\,|\nabla_t\partial_t u|^2 - 2\langle\nabla_t \mathrm{grad}\mathcal{V}(u), \partial_t u\rangle \\
&\ge -2C_1\,(1 + |\partial_t u|)\,|\partial_t u| \\
&\ge -C_1 - 3C_1\,|\partial_t u|^2 .
\end{aligned}$$

To obtain the second step we replaced $\nabla_t \partial_t u$ according to the heat equation (6) and used the fact that $\nabla_t \partial_s u = \nabla_s \partial_t u$. The third step is by condition (V1) with constant C_1. Choose $(s_0, t_0) \in \mathbb{R} \times S^1$ and apply lemma 4.6 in the case $r = 1$ and with

$$w(s,t) := \frac{1}{3} + |\partial_t u(s_0 + s, t_0 + t)|^2$$

and $a = 3C_1$ to obtain

$$\begin{aligned} w(0) &\leq c_1 e^a \int_{-1}^{0} \int_{-1}^{+1} \left(\frac{1}{3} + |\partial_t u(s_0 + s, t_0 + t)|^2 \right) dtds \\ &= c_1 e^{3C_1} \left(\frac{2}{3} + 2 \int_{-1}^{0} \|\partial_t u_{s_0+s}\|^2_{L^2(S^1)}\, ds \right). \end{aligned}$$

Theorem 4.5 then follows from the slicewise estimate (70). □

Lemma 4.8. *Fix a constant $c > 0$ and a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0) *with constant $C > 0$. If $u : \mathbb{R} \times S^1 \to M$ is a solution of (6) then*

$$\sup_{s\in\mathbb{R}} \mathcal{S}_{\mathcal{V}}(u(s,\cdot)) \leq c \quad \Rightarrow \quad E(u) \leq c + C.$$

Proof. Let $u_s(t) := u(s,t)$ and choose $T > 0$, then

$$\begin{aligned} E_{[-T,T]}(u) &= \int_{-T}^{T} \int_0^1 |\partial_s u(s,t|^2\ dtds \\ &= -\int_{-T}^{T} \langle \nabla \mathcal{S}_{\mathcal{V}}(u_s), \partial_s u_s \rangle_{L^2} ds \\ &= -\int_{-T}^{T} \frac{d}{ds} \mathcal{S}_{\mathcal{V}}(u_s)\, ds \\ &= \mathcal{S}_{\mathcal{V}}(u_{-T}) - \mathcal{S}_{\mathcal{V}}(u_T). \end{aligned}$$

Here we used the fact that the heat equation (6) is the negative L^2 gradient flow equation for the action functional. Now the crucial property of the action functional is its boundedness from below, namely $\mathcal{S}_{\mathcal{V}}(x) \geq -C$ for every $x \in \mathcal{L}M$ by (V0). Hence $\mathcal{S}_{\mathcal{V}}(u_{-T}) - \mathcal{S}_{\mathcal{V}}(u_T) \leq c + C$ and this proves the lemma. □

4.3 Gradient bounds

Theorem 4.9. *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V2) *and a constant $c_0 > 0$. Then there is a constant $C = C(c_0, \mathcal{V}) > 0$ such that the following holds. If $u : \mathbb{R} \times S^1 \to M$ is a smooth solution of (6) that satisfies (69), i.e. $\sup_{s\in\mathbb{R}} \mathcal{S}_{\mathcal{V}}(u(s,\cdot)) \leq c_0$, then*

$$\begin{aligned} |\partial_s u(s,t)|^2 + |\nabla_t \partial_s u(s,t)|^2 &\leq C E_{[s-1,s]}(u) \\ |\nabla_s \partial_s u(s,t)|^2 + |\nabla_t \nabla_t \partial_s u(s,t)|^2 &\leq C E_{[s-2,s]}(u) \end{aligned}$$

for every $(s,t) \in \mathbb{R} \times S^1$. Here $E_I(u)$ denotes the energy of u over the set $I \times S^1$.

Proof. By theorem 4.5 there is a constant $C_0 = C_0(c_0, \mathcal{V}) > 0$ such that

$$\|\partial_t u\|_\infty \leq C_0.$$

Let $C = C(C_0, \mathcal{V})$ be the constant of theorem 3.3 with this choice of C_0. Observe that $\xi := \partial_s u$ solves the linearized heat equation. Hence theorem 3.3 shows that

$$|\partial_s u(s,t)|^2 \leq C^2 E_{[s-1,s]}(u) \leq C^2(c_0 + c')$$

for every $(s,t) \in \mathbb{R} \times S^1$. Here the last step is by lemma 4.8 and axiom (V0) with constant c'. Use that u solves (6) and satisfies axiom (V0) to obtain that

$$\|\nabla_t \partial_t u\|_\infty \leq \|\partial_s u\|_\infty + \|\mathrm{grad}\mathcal{V}(u)\|_\infty \leq C\sqrt{c_0 + c'} + c'.$$

Now choose C_0 larger than $2C\sqrt{c_0 + c'} + c'$ and let $C = C(C_0, \mathcal{V})$ be the constant of theorem 3.3 with this new choice of C_0. Theorem 3.3 then proves the desired estimate for $|\nabla_t \partial_s u|$. It follows that $\|\nabla_t \partial_s u\|_\infty$ is bounded. Therefore $\|\nabla_t \nabla_t \partial_t u\|_\infty$ is bounded by (6) and axiom (V1). Hence theorem 3.4 applies with a new choice of C_0 and proves the remaining two estimates of theorem 4.9. □

Proof of theorem 1.7. Theorem 4.5, theorem 4.9 and lemma 4.8. Only (V0)–(V1) are used. Use (6) and (V0) to obtain the estimate for $\nabla_t \partial_t u$. □

4.4 Exponential decay

Theorem 4.10. *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V2)*. Suppose $\mathcal{S}_\mathcal{V}$ is Morse and let $a \in \mathbb{R}$ be a regular value of $\mathcal{S}_\mathcal{V}$. Then there exist constants $\delta, c, \rho > 0$ such that the following holds. If $u : \mathbb{R} \times S^1 \to M$ is a smooth solution of (6) that satisfies (69), i.e. $\sup_{s\in\mathbb{R}} \mathcal{S}_\mathcal{V}(u(s,\cdot)) \leq a$, and*

$$E_{\mathbb{R}\setminus[-T_0,T_0]}(u) < \delta \tag{71}$$

for some $T_0 > 0$, then

$$E_{\mathbb{R}\setminus[-T,T]}(u) \leq ce^{-\rho(T-T_0)} E_{\mathbb{R}\setminus[-T_0,T_0]}(u)$$

for every $T \geq T_0 + 1$.

Corollary 4.11. *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V2)*. Suppose $\mathcal{S}_\mathcal{V}$ is Morse and let $x^\pm \in \mathcal{P}(\mathcal{V})$. Then there exist constants $\delta, c, \rho > 0$ such that the following holds. Suppose that $u \in \mathcal{M}(x^-, x^+; \mathcal{V})$ satisfies (71) for some $T_0 > 0$. Then*

$$|\partial_s u(s,t)|^2 + |\nabla_t \partial_s u(s,t)|^2 \leq ce^{-\rho(s-T_0)} E_{\mathbb{R}\setminus[-T_0,T_0]}(u)$$

for every $s \geq T_0 + 2$.

Proof. Theorem 4.9 and theorem 4.10. □

The proof of theorem 4.10 is based on the following lemma which asserts existence of a true critical point nearby an approximate one.

Lemma 4.12 (Critical point nearby approximate one)**.** *Fix a perturbation $\mathcal{V}: \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0) *and let $a \in \mathbb{R}$ be a regular value of $\mathcal{S}_{\mathcal{V}}$. Then, for every $\delta_0 > 0$, there is a constant $\delta_1 > 0$ such that the following is true. Suppose $x : S^1 \to M$ is a smooth loop such that*

$$\mathcal{S}_{\mathcal{V}}(x) \le a, \qquad \left\|\nabla_t \partial_t x + \operatorname{grad}\mathcal{V}(x)\right\|_\infty < \delta_1.$$

Then there is a critical point $x_0 \in \mathcal{P}^a(\mathcal{V})$ and a vector field ξ_0 along x_0 such that $x = \exp_{x_0}(\xi_0)$ and

$$\left\|\xi_0\right\|_\infty + \left\|\nabla_t \xi_0\right\|_\infty + \left\|\nabla_t \nabla_t \xi_0\right\|_\infty \le \delta_0.$$

Proof. First note that

$$\left\|\partial_t x\right\|_2^2 = \int_0^1 \left|\partial_t x(t)\right|^2 dt = 2\mathcal{S}_{\mathcal{V}}(x) + 2\mathcal{V}(x) \le 2(a + C)$$

where C is the constant in (V0). Now, assuming $\delta_1 \le 1$, we have

$$\begin{aligned} \left|\frac{d}{dt}\left|\partial_t x\right|^2\right| &= 2\left|\langle \partial_t x, \nabla_t \partial_t x + \operatorname{grad}\mathcal{V}(x)\rangle - \langle \partial_t x, \operatorname{grad}\mathcal{V}(x)\rangle\right| \\ &\le 2\left(\delta_1 + C\right)\left|\partial_t x\right| \le (1 + C)^2 + \left|\partial_t x\right|^2. \end{aligned}$$

Integrate this inequality to obtain that

$$\left|\partial_t x(t_1)\right|^2 - \left|\partial_t x(t_0)\right|^2 \le (1 + C)^2 + \left\|\partial_t x\right\|_2^2$$

for $t_0, t_1 \in [0,1]$. Integrating again over the interval $0 \le t_0 \le 1$ gives

$$\left\|\partial_t x\right\|_\infty \le \sqrt{(1 + C)^2 + 2\left\|\partial_t x\right\|_2^2} \le c \tag{72}$$

where $c^2 := (1 + C)^2 + 4(a + C)$.

Now suppose that the assertion is wrong. Then there is a constant $\delta_0 > 0$ and a sequence of smooth loops $x_\nu : S^1 \to M$ satisfying

$$\mathcal{S}_{\mathcal{V}}(x_\nu) \le a, \qquad \lim_{\nu \to \infty}\left(\left\|\nabla_t \partial_t x_\nu + \operatorname{grad}\mathcal{V}(x_\nu)\right\|_\infty\right) = 0,$$

but not the conclusion of the lemma for the given constant δ_0. By (V0) we have $\sup_\nu \left\|\nabla_t \partial_t x_\nu\right\|_\infty < \infty$ and (72) implies $\sup_\nu \left\|\partial_t x_\nu\right\|_\infty < \infty$. Hence, by the Arzela–Ascoli theorem, there exists a subsequence, still denoted by x_ν, that converges in the C^1-topology. Let $x_0 \in C^1(S^1, M)$ be the limit. We claim that this subsequence actually converges in the C^2-topology. Then $\nabla_t \partial_t x_0 + \operatorname{grad}\mathcal{V}(x_0) = 0$. Hence $x_0 \in \mathcal{P}^a(\mathcal{V})$ and x_ν converges to x_0 in the C^2-topology. This contradicts our assumption on the sequence x_ν and proves the lemma.

It remains to prove the claim. For simplicity let us assume that M is isometrically embedded in Euclidean space $\mathbb{R}^N$ for some sufficiently large integer N. Since $\sup_\nu \left\|\nabla_t \partial_t x_\nu\right\|_2 < \infty$, the Banach-Alaoglu Theorem asserts existence of a subsequence, still denoted by x_ν, and an element $v \in L^2$ such that $\nabla_t \partial_t x_\nu$ converges to v weakly in L^2. In fact v equals the weak t-derivative of $\partial_t x$. Now $\mathrm{grad}\mathcal{V}(x_\nu)$ converges to $\mathrm{grad}\mathcal{V}(x_0)$ in L^2 and to $-v$ weakly in L^2. But the weak limit equals the strong limit, hence $v = -\mathrm{grad}\mathcal{V}(x_0) \in C^1$. Therefore $\partial_t x_0 \in C^1$ and $\nabla_t \partial_t x_0$ equals the weak t-derivative v of $\partial_t x_0$. Now $x_0 \in C^2$ satisfies

$$\nabla_t \partial_t x_0 + \mathrm{grad}\mathcal{V}(x_0) = 0, \tag{73}$$

because $\nabla_t \partial_t x_\nu$ converges to $v = \nabla_t \partial_t x_0$ weakly in L^2 and to $-\mathrm{grad}\mathcal{V}(x_0)$ strongly in L^2. By induction (73) implies that $x_0 \in C^\infty$. Moreover, it follows using (73) that $\nabla_t \partial_t x_\nu$ converges to $\nabla_t \partial_t x_0$ in C^0 and this proves the claim. □

Proof of theorem 4.10. Given a and $\mathcal{V}$, let $C = C(a, \mathcal{V})$ be the constant of theorem 1.7 and theorem 4.9 with this choice. Let $C_0 > 1$ be the constant in (V0). Then $E(u) \le a + C_0$ by lemma 4.8 and $\|\partial_s u\|_\infty \le CE(u) \le C(a + C_0)$ by theorem 4.9. Hence

$$\|\partial_t u\|_\infty + \|\nabla_t \partial_t u\|_\infty \le c_0$$

by theorem 1.7 and by replacing $\nabla_t \partial_t u$ according to the heat equation (6). Here $c_0 = C(a + 2C_0) + C_0$. Let δ_0 and ρ_0 be the positive constants of theorem 3.10 with this choice of c_0. Choose δ_0 smaller than one quarter the minimal C^0 distance $\kappa = \kappa(a)$ of any two elements of $\mathcal{P}^a(\mathcal{V})$. Let $\delta_1 > 0$ be the constant of lemma 4.12 associated to a and δ_0 and set

$$\delta := \min\left\{ \frac{\delta_0^2}{4C}, \frac{\delta_1^2}{4C} \right\}.$$

Note that δ_0, ρ_0, δ_1, and δ depend only on a, $\mathcal{V}$, and the constant C_0 of axiom (V0). Note furthermore that the vector field along u given by $\xi := \partial_s u$ solves the linear heat equation (43) and that the continuous function $s \mapsto \|\partial_s u_s\|_{L^2(S^1)}$ is bounded.

If $|s| \ge T_0 + 1$, then $E_{[s-1,s]}(u) \le E_{\mathbb{R}\setminus[-T_0,T_0]}(u)$ and it follows that

$$\|\partial_s u_s\|_\infty + \|\nabla_t \partial_s u_s\|_\infty \le \sqrt{CE_{[s-1,s]}(u)} \le \sqrt{C\delta} < \min\{\delta_0, \delta_1\}. \tag{74}$$

Here we used theorem 4.9 in step one, assumption (71) in step two, and the definition of δ in the last step. Hence, by lemma 4.12, there are critical points $x^\pm \in \mathcal{P}^a(\mathcal{V})$ such that

$$u_s = \exp_{x^\pm}(\eta_s^\pm), \qquad \|\eta_s\|_{C^2(S^1)} \le \delta_0$$

whenever $\pm s \ge T_0 + 1$. Although the critical points $x^\pm$ apriori depend on s they are in fact independent, because $\delta_0 < \kappa/4$ and $\mathcal{P}^a(\mathcal{V})$ is a finite set by the Morse condition. Moreover, injectivity of the operators $A_{x^\pm}$ is equivalent to nondegeneracy of the critical points $x^\pm$ and this is true again by the Morse condition. Now theorem 3.10 and remark 3.11 conclude the proof of theorem 4.10. □

Proof of theorem 1.8. We prove exponential decay in three steps.

I) Firstly, the energy of u is finite. In the case (B) this is part of the assumptions. In the case (F) it follows as in the proof of lemma 4.8 for $u : [0,\infty) \times S^1 \to \mathbb{R}$. Namely, let $C_0 > 0$ be the constant in (V0) and set $u_0(t) := u(0,t)$, then $E(u) \leq \mathcal{S}_{\mathcal{V}}(u_0) + C_0$.

II) Secondly, we establish the existence of asymptotic limits. Consider the forward case (F). We claim that $\partial_s u(s,t) \to 0$ as $s \to \infty$, uniformly in t. Let $C > 0$ be the constant in theorem 4.9 and let $s \geq 1$, then

$$|\partial_s u(s,t)| \leq C E_{[s-1,s]}(u) = C \int_{s-1}^{s} \|\partial_s u_\sigma\|_{L^2(S^1)}^2 d\sigma \stackrel{s\to\infty}{\longrightarrow} 0.$$

Here the last step follows by finite energy of u and this proves the claim. Because $\partial_s u_s$ converges to zero in $L^\infty(S^1)$ so does $\nabla_t \partial_t u_s + \operatorname{grad}\mathcal{V}(u_s)$ by (6). Hence it follows from lemma 4.12 that there is a critical point $x^+ \in \mathcal{P}(\mathcal{V})$ and, for every sufficiently large s, there is a smooth vector field ξ_s along x^+ such that

$$u_s = \exp_{x^+}(\xi_s), \qquad \|\xi_s\|_\infty + \|\nabla_t \xi_s\|_\infty + \|\nabla_t \nabla_t \xi_s\|_\infty \stackrel{s\to\infty}{\longrightarrow} 0.$$

(Here we used the fact that – since $\mathcal{S}_{\mathcal{V}}$ is Morse – there are only finitely many elements in $\mathcal{P}(\mathcal{V})$ below any fixed action level.) This and the identities for the maps E_{ij} in (51) imply that

$$\|\partial_s u\|_\infty + \|\partial_t u\|_\infty + \|\nabla_t \partial_t u\|_\infty < \infty. \tag{75}$$

The same arguments apply in case (B) with corresponding asymptotic limit x^-.

III) The third step is to prove exponential decay of the C^k norm of $\partial_s u$. Consider the forward case (F). We prove by induction that for every $k \in \mathbb{N}$ there is a constant $c_k' > 0$ such that

$$\|\partial_s u\|_{W^{k,2}([s,\infty)\times S^1)} \leq c_k' \, \|\partial_s u\|_{L^2([s-k,\infty)\times S^1)}$$

for every $s \geq k$. This estimate, the definition of the energy in (9), and theorem 4.10 with constants $\delta, c, \rho, T_0 > 0$, where T_0 is chosen sufficiently large such that (71) holds true, then show that

$$\|\partial_s u\|_{W^{k,2}([s,\infty)\times S^1)} \leq c_k' \sqrt{E_{[s-k,\infty]}(u)} \leq c_k' \sqrt{c\delta} e^{-\rho(s-k-T_0)/2}$$

whenever $s \geq k + T_0 + 1$. The Sobolev embedding $W^{k,2} \hookrightarrow C^{k-2}$, e.g. on the compact set $[s, s+1] \times S^1$, concludes the proof of forward exponential decay (F).

It remains to carry out the induction argument. It is based on the identity

$$\left(\nabla_s - \nabla_t \nabla_t\right) \partial_s u = R(\partial_s u, \partial_t u)\partial_t u + \mathcal{H}_{\mathcal{V}}(u)\partial_s u \tag{76}$$

– which follows by linearizing the heat equation (6) in the s-direction to obtain that $\partial_s u \in \ker\, \mathcal{D}_u$ in the notation of section 3.4 – and the subsequent estimate.

Proposition 2.13 with $p = 2$ applies[4] by (75) and shows that there is a constant $c' > 0$ with the following significance. If $s_0 \geq 1$ then

$$\begin{aligned} &\|\nabla_s \xi\|_{L^2([s_0,\infty)\times S^1)} + \|\nabla_t \xi\|_{L^2([s_0,\infty)\times S^1)} + \|\nabla_t \nabla_t \xi\|_{L^2([s_0,\infty)\times S^1)} \\ &\leq c' \left(\|\nabla_s \xi - \nabla_t \nabla_t \xi\|_{L^2([s_0-1,\infty)\times S^1)} + \|\xi\|_{L^2([s_0-1,\infty)\times S^1)} \right) \end{aligned} \tag{77}$$

for every $\xi \in \Omega^0([0,\infty) \times S^1)$ of compact support. To see this fix a smooth nondecreasing cutoff function $\beta : \mathbb{R} \to [0,1]$ which equals zero for $s \leq s_0 - 1$ and one for $s \geq s_0$ and whose slope is at most two. Via extension by zero we interpret $\beta\xi$ as a smooth compactly supported vector field along the extended cylinder $u : \mathbb{R} \times S^1 \to M$. Now proposition 2.13 applies to $\beta\xi$ and proves (77). Note that c' depends on the L^∞ norms of $\partial_s \beta$, $\partial_t \beta$, and $\partial_t \partial_t \beta$. We also used lemma 2.12 to deal with the term $\nabla_t \xi$ which appears on the right hand side.

We prove the induction hypothesis in the case $k = 1$. Let $s \geq 1$ and denote by $C_1 > 0$ the constant in (V1). By (77) with $\xi = \partial_s u$ and (76) it follows that

$$\begin{aligned} &\|\nabla_s \partial_s u\|_{L^2([s,\infty)\times S^1)} + \|\nabla_t \partial_s u\|_{L^2([s,\infty)\times S^1)} + \|\nabla_t \nabla_t \partial_s u\|_{L^2([s,\infty)\times S^1)} \\ &\leq c' \left(\|(\nabla_s - \nabla_t \nabla_t)\partial_s u\|_{L^2([s-1,\infty)\times S^1)} + \|\partial_s u\|_{L^2([s-1,\infty)\times S^1)} \right) \\ &= c' \left(\|R(\partial_s u, \partial_t u)\partial_t u + \mathcal{H}_{\mathcal{V}}(u)\partial_s u\|_{L^2([s-1,\infty)\times S^1)} + \|\partial_s u\|_{L^2([s-1,\infty)\times S^1)} \right) \\ &\leq c' \left(\|R\|_\infty \|\partial_t u\|_\infty^2 + 2C_1 + 1 \right) \|\partial_s u\|_{L^2([s-1,\infty)\times S^1)} . \end{aligned}$$

We prove the induction hypothesis for $k = 2$. Assume $s \geq 2$. Then by (77) with $\xi = \nabla_s \partial_s u$ and (76) it follows that

$$\begin{aligned} &\|\nabla_s \nabla_s \partial_s u\|_{L^2([s,\infty)\times S^1)} + \|\nabla_t \nabla_s \partial_s u\|_{L^2([s,\infty)\times S^1)} + \|\nabla_t \nabla_t \nabla_s \partial_s u\|_{L^2([s,\infty)\times S^1)} \\ &\leq c' \Big(\|\nabla_s \left(R(\partial_s u, \partial_t u)\partial_t u + \mathcal{H}_{\mathcal{V}}(u)\partial_s u \right) + [\nabla_s, \nabla_t \nabla_t]\partial_s u\|_{L^2([s-1,\infty)\times S^1)} \\ &\quad + \|\nabla_s \partial_s u\|_{L^2([s-1,\infty)\times S^1)} \Big). \end{aligned}$$

Now use $s \geq 2$, the apriori estimates (75), axiom (V2), and the case $k = 1$ to bound the right hand side by a constant times $\|\partial_s u\|_{L^2([s-2,\infty)\times S^1)}$. An L^2 bound for $\nabla_t \nabla_t \partial_s u$ was obtained earlier in the case $k = 1$ and the identity $\nabla_s \nabla_t \partial_s u = \nabla_t \nabla_s \partial_s u - R(\partial_t u, \partial_s u)\partial_s u$ implies one for $\nabla_s \nabla_t \partial_s u$.

Proving the induction hypothesis in the case $k = 3$ requires additional information: Theorem 4.5 and theorem 4.9 only assume an upper action bound for the heat flow solution. In the case at hand this is provided by $\mathcal{S}_{\mathcal{V}}(u(0,\cdot))$. This reproves (75) and in addition shows that $\|\nabla_t \partial_s u\|_\infty < \infty$. This estimate is crucial, since (77) with $\xi = \nabla_s \nabla_s \partial_s u$ and (76) lead to terms of the form

$$\|R(\nabla_s \partial_s u, \nabla_t \partial_s u)\partial_t u\|_{L^2([s,\infty)\times S^1)},$$

but our induction hypothesis in the case $k = 2$ only provides a C^0 bound for $\partial_s u$. The remaining part of proof follows the same pattern as in the case $k = 2$. Here we use axiom (V3).

[4]Formally add to u *any* smooth half cylinder imposing a uniform limit as $s \to -\infty$.

Fix an integer $k \geq 3$ and assume the induction hypothesis is true for every $\ell \in \{1, \ldots, k\}$. In particular, we have $W^{k,2}$ and C^{k-2} bounds for $\partial_s u$ on the appropriate domains. Apply (77) with $\xi = {\nabla_s}^k \partial_s u$ and (76) to obtain L^2 bounds for ${\nabla_s}^{k+1} \partial_s u$ and $\nabla_t {\nabla_s}^k \partial_s u$. Here we use axiom (V3) and the induction hypothesis for $\ell \in \{1, \ldots, k\}$. A problem of the type encountered in the case $k = 3$ does not arise, since we have C^{k-2} bounds for $\partial_s u$ with $k \geq 3$. To obtain L^2 estimates for the remaining terms of the form ${\nabla_t}^j {\nabla_s}^{k-j} \partial_s u$ with $j \geq 2$ use (76) to treat any $\nabla_t \nabla_t$ for one ∇_s. This reduces the order of the term, hence the induction hypothesis can be applied. This completes the induction step and proves (F). The backward case (B) follows similarly. This proves theorem 1.8. $\square$

Lemma 4.13. *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V3)*, a constant $p > 1$, and nondegenerate critical points $x^{\pm}$ of $\mathcal{S}_{\mathcal{V}}$. If $u \in \mathcal{M}(x^-; x^+; \mathcal{V})$, then the operators $\mathcal{D}_u, \mathcal{D}_u^* : \mathcal{W}_u^{1,p} \to \mathcal{L}_u^p$ are Fredholm and*

$$\operatorname{index} \mathcal{D}_u = \operatorname{ind}_{\mathcal{V}}(x^-) - \operatorname{ind}_{\mathcal{V}}(x^+) = -\operatorname{index} \mathcal{D}_u^*.$$

Proof. By theorem 1.8 on exponential decay u satisfies the assumptions of the Fredholm theorem 1.9. $\square$

4.5 Compactness up to broken trajectories

Proposition 4.14 (Convergence on compact sets)**.** *Assume that the perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ satisfies* (V0)–(V3) *and that $\mathcal{S}_{\mathcal{V}}$ is Morse. Fix critical points $x^{\pm} \in \mathcal{P}(\mathcal{V})$ and a sequence of connecting trajectories $u^\nu \in \mathcal{M}(x^-, x^+; \mathcal{V})$. Then there is a pair $x_0, x_1 \in \mathcal{P}(\mathcal{V})$, a connecting trajectory $u \in \mathcal{M}(x_0, x_1; \mathcal{V})$, and a subsequence, still denoted by u^ν, such that the following hold:*

(i) *The subsequence u^ν converges to u, uniformly with all derivatives on every compact subset of $\mathbb{R} \times S^1$.*

(ii) *For all $s \in \mathbb{R}$ and $T > 0$*

$$\begin{aligned} \mathcal{S}_{\mathcal{V}}\big(u(s,\cdot)\big) &= \lim_{\nu \to \infty} \mathcal{S}_{\mathcal{V}}\big(u^\nu(s,\cdot)\big) \\ E_{[-T,T]}(u) &= \lim_{\nu \to \infty} E_{[-T,T]}(u^\nu). \end{aligned}$$

Proof. Since the flow lines u^ν connect x^- to x^+ and the action $\mathcal{S}_{\mathcal{V}}$ decreases along flow lines, it follows that

$$\sup_{s \in \mathbb{R}} \mathcal{S}_{\mathcal{V}}(u^\nu(s,\cdot)) = \mathcal{S}_{\mathcal{V}}(x^-) =: c_0.$$

Hence by the apriori estimates theorem 4.5 and theorem 4.9 there is a constant $C = C(c_0, \mathcal{V})$ such that

$$|\partial_t u^\nu(s,t)| \leq C, \qquad |\partial_s u^\nu(s,t)| \leq C \sqrt{\mathcal{S}_{\mathcal{V}}(x^-) - \mathcal{S}_{\mathcal{V}}(x^+)},$$

for every $(s,t) \in \mathbb{R} \times S^1$. To obtain the second estimate we used the energy identity (9) for connecting orbits. Now fix a constant $p > 2$ and pick an integer

$\ell \geq 2$. Then the assumptions of theorem 4.3 are satisfied for the sequence u^ν restricted to the cylinder $Z_\ell = (-\ell, \ell] \times S^1$. Hence there is a smooth solution $u : Z_\ell \to M$ of the heat equation (6) and a subsequence, still denoted by u^ν, such that u^ν converges to u, uniformly with all derivatives on the compact subset $[-\ell+1, \ell] \times S^1$ of Z_ℓ. Now (i) follows by choosing a diagonal subsequence associated to the exhausting sequence $Z_2 \subset Z_3 \subset \dots$ of $\mathbb{R} \times S^1$.

To prove (ii) note that

$$\begin{aligned} E_{[-T,T]}(u) &= \lim_{\nu \to \infty} \int_{-T}^{T} \int_0^1 |\partial_s u^\nu|^2 \, dt \, ds \\ &= \lim_{\nu \to \infty} E_{[-T,T]}(u^\nu) \\ &\leq \mathcal{S}_\mathcal{V}(x^-) - \mathcal{S}_\mathcal{V}(x^+) \end{aligned}$$

for every $T > 0$. Here the first step uses that, by (i), the sequence $\partial_s u^\nu$ converges to $\partial_s u$, uniformly on compact sets. The second step is by definition of the energy and the last step is again by the energy identity (9). Hence the limit $u : \mathbb{R} \times S^1 \to M$ has finite energy and so, by theorem 1.8, belongs to the moduli space $\mathcal{M}(x_0, x_1; \mathcal{V})$ for some $x_0, x_1 \in \mathcal{P}(\mathcal{V})$. To prove convergence of the action at time s note that

$$\mathcal{V}\big(u(s, \cdot)\big) = \lim_{\nu \to \infty} \mathcal{V}\big(u^\nu(s, \cdot)\big),$$

because $\mathcal{V}$ is continuous with respect to the C^0 topology on $\mathcal{L}M$ by axiom (V0). Convergence of the action at time s then follows from the fact that $\partial_t u^\nu(s, \cdot)$ converges to $\partial_t u(s, \cdot)$ in $L^\infty(S^1)$. □

Lemma 4.15 (Compactness up to broken trajectories)**.** *Assume that the perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ satisfies* (V0)–(V3) *and that $\mathcal{S}_\mathcal{V}$ is Morse. Fix distinct critical points $x^\pm \in \mathcal{P}(\mathcal{V})$ and let $u^\nu \in \mathcal{M}(x^-, x^+; \mathcal{V})$ be a sequence of connecting trajectories. Then there exist a subsequence, still denoted by u^ν, finitely many critical points $x_0, \dots, x_m$ with $x_0 = x^+$ and $x_m = x^-$, finitely many solutions*

$$u_k \in \mathcal{M}(x_k, x_{k-1}; \mathcal{V}), \qquad \partial_s u_k \not\equiv 0, \qquad k = 1, \dots, m,$$

and finitely many sequences s_k^ν, such that the shifted sequence $u^\nu(s_k^\nu + s, t)$ converges to $u_k(s, t)$, uniformly with all derivatives on every compact subset of $\mathbb{R} \times S^1$. Moreover, these limit solutions satisfy $\sum_{k=1}^m E(u_k) = \mathcal{S}_\mathcal{V}(x^-) - \mathcal{S}_\mathcal{V}(x^+)$.

Proof. In [SW03, Proof of lemma 10.3] replace lemma 10.2 by prop. 4.14. □

5 The implicit function theorem

Throughout this section we fix a smooth perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3) and two nondegenerate critical points $x^{\pm}$ of $\mathcal{S}_{\mathcal{V}}$. The idea to prove the manifold property and the dimension formula in theorem 1.10 is to construct a smooth Banach manifold which contains the moduli space $\mathcal{M}(x^-, x^+; \mathcal{V})$ and then prove these statements locally near each element of the moduli space.

Fix a real number $p > 2$ and denote by

$$\mathcal{B}^{1,p} = \mathcal{B}^{1,p}(x^-, x^+) \tag{78}$$

the space of continuous maps $u : \mathbb{R} \times S^1 \to M$, which satisfy the limit conditions (8), are locally of class $\mathcal{W}^{1,p}$, and satisfy the asymptotic conditions $\xi^- \in \mathcal{W}^{1,p}((-\infty, -T] \times S^1, u^*TM)$ and $\xi^+ \in \mathcal{W}^{1,p}([T, \infty) \times S^1, u^*TM)$ for some sufficiently large $T > 0$. Here $\xi^{\pm}$ are defined pointwise by the identity $\exp_{x^{\pm}(t)} \xi^{\pm}(s,t) = u(s,t)$. For $p > 2$ the space $\mathcal{B}^{1,p}$ carries the structure of a smooth infinite dimensional Banach manifold. The tangent space $T_u\mathcal{B}^{1,p}$ is given by the Banach space $\mathcal{W}^{1,p}_u$ whose norm is defined in (12). Around any *smooth* map u local coordinates are provided by the inverse of the map ${\varphi_u}^{-1} : V_u \to \mathcal{B}^{1,p}$ given by $\xi \mapsto [(s,t) \mapsto \exp_{u(s,t)} \xi(s,t)]$ where $V_u \subset \mathcal{W}^{1,p}_u$ is a sufficiently small neighborhood of zero. By abuse of notation we shall denote this map again by $\xi \mapsto \exp_u \xi$. Moreover, note that if *some* $u \in \mathcal{B}^{1,p}$ satisfies the heat equation (6) almost everywhere, then u is smooth by theorem 1.5, hence $u \in \mathcal{M}(x^-, x^+; \mathcal{V})$.

For $x \in M$ and $\xi \in T_xM$ denote parallel transport with respect to the Levi-Civita connection along the geodesic $\tau \mapsto \exp_x(\tau\xi)$ by

$$\Phi(x, \xi) : T_xM \to T_{\exp_x(\xi)}M.$$

For $u \in \mathcal{B}^{1,p}$ the map $\mathcal{F}_u : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u$ defined by

$$\mathcal{F}_u(\xi) := \Phi(u, \xi)^{-1} \left(\partial_s(\exp_u \xi) - \nabla_t\partial_t(\exp_u \xi) - \operatorname{grad}\mathcal{V}(\exp_u \xi)\right) \tag{79}$$

is induced by pointwise evaluation at (s,t). Its significance lies in the following three facts. Firstly, it is a smooth map between Banach spaces, hence the implicit function theorem for Banach spaces applies. Secondly, the differential $d\mathcal{F}_u(0) : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u$ is given by the linear operator $\mathcal{D}_u$; see [W99, app. A.3]. Thirdly, the map $\xi \mapsto \exp_u \xi$ identifies a neigborhood V of zero in ${\mathcal{F}_u}^{-1}(0)$ with a neigborhood of u in $\mathcal{M}(x^-, x^+; \mathcal{V})$. Now theorem 1.10 follows immediately.

Proof of theorem 1.10. Fix $p > 2$. Then the operator $d\mathcal{F}_u(0) = \mathcal{D}_u : \mathcal{W}^{1,p}_u \to \mathcal{L}^p_u$ is Fredholm by theorem 1.9 and surjective by assumption. Since every surjective Fredholm operator admits a right inverse, the implicit function theorem for Banach spaces, see e.g. [MS04, thm A.3.3], applies to $\mathcal{F}_u$ restricted to a small neighborhood V of zero. It asserts that ${\mathcal{F}_u}^{-1}(0) \cap V$ is a smooth manifold whose tangent space at zero is given by the kernel of $\mathcal{D}_u$. Since $\mathcal{D}_u$ is onto, it follows that $\dim\ker\mathcal{D}_u = \operatorname{index}\mathcal{D}_u$ by definition of the Fredholm index. On the other hand, the Fredholm index equals $\operatorname{ind}_{\mathcal{V}}(x^-) - \operatorname{ind}_{\mathcal{V}}(x^+)$ by theorem 1.9. $\square$

Proof of proposition 1.11. Set $c_* = \frac{1}{2}(\mathcal{S}_\mathcal{V}(x^-) - \mathcal{S}_\mathcal{V}(x^+))$ and identify

$$\widehat{\mathcal{M}}(x^-, x^+; \mathcal{V}) \simeq \mathcal{M}^* := \{u \in \mathcal{M}(x^-, x^+; \mathcal{V}) \mid \mathcal{S}_\mathcal{V}(u(0, \cdot)) = c_*\}.$$

Here we use that the action $\mathcal{S}_\mathcal{V}$ is strictly decreasing along nonconstant (in the s-variable) heat flow trajectories. This is standard and follows from the first variation formula for the functional $\mathcal{S}_\mathcal{V}$; see e.g. [M69, sec. 12]. Now choose a sequence u^ν in $\mathcal{M}^*$. By lemma 4.15 there is a subsequence, still denoted by u^ν, finitely many critical points $x_0 = x^+, x_1, \ldots, x_m = x^-$, finitely many connecting trajectories $u_k \in \mathcal{M}(x_k, x_{k-1}; \mathcal{V})$ and sequences s_k^ν where $k = 1, \ldots, m$, such that each shifted sequence $u^\nu(s_k^\nu + s, t)$ converges to $u_k(s, t)$ in C^∞_{loc}. Note that $m \geq 1$. By the Morse–Smale assumption theorem 1.10 applies to all moduli spaces. Since $\partial_s u_k \not\equiv 0$ and the heat equation (6) is s-shift invariant this implies

$$\mathrm{ind}_\mathcal{V}(x_k) - \mathrm{ind}_\mathcal{V}(x_{k-1}) = \dim \mathcal{M}(x_k, x_{k-1}; \mathcal{V}) \geq 1, \quad \forall k \in \{1, \ldots, m\}.$$

Use these inequalities to obtain that $\mathrm{ind}_\mathcal{V}(x^-) - \mathrm{ind}_\mathcal{V}(x^+) \geq m \geq 1$. But by assumption the index difference is one and therefore $m = 1$. Now this means that the subsequence u^ν converges in C^∞_{loc} to $u := u_1 \in \mathcal{M}(x^-, x^+; \mathcal{V})$. In fact, convergence of the action functional for fixed time $s = 0$, see proposition 4.14 (ii), shows that $u \in \mathcal{M}^*$. Hence $\mathcal{M}^*$ is compact in the C^∞_{loc} topology. On the other hand, the moduli space $\mathcal{M}(x^-, x^+; \mathcal{V})$ is a manifold of dimension one by theorem 1.10. Now the $\mathbb{R}$ action is free and therefore the quotient, hence $\mathcal{M}^*$, is a manifold of dimension zero. But a zero dimensional compact manifold consists of finitely many points. □

The refined implicit function theorem

Proposition 5.1 (The estimate for the right inverse)**.** *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V3) *and nondegenerate critical points $x^\pm$ of $\mathcal{S}_\mathcal{V}$. Assume $u \in \mathcal{M}(x^-; x^+; \mathcal{V})$ and $\mathcal{D}_u$ is onto. Then, for every $p > 1$, there is a positive constant $c = c(p, u)$ invariant under s-shifts of u such that*

$$\|\xi^*\|_{\mathcal{W}^{1,p}_u} \leq c \, \|\mathcal{D}_u \xi^*\|_p \tag{80}$$

for every $\xi^* \in \mathrm{im}\,(\mathcal{D}_u^* : \mathcal{W}^{2,p}_u \to \mathcal{W}^{1,p}_u)$. Here $\mathcal{W}^{2,p}_u := \{\xi \in \mathcal{W}^{1,p}_u \mid \mathcal{D}_u \xi \in \mathcal{W}^{1,p}_u\}$.

Proof of proposition 5.1. The proof of [DS94, lemma 4.5] carries over. We include it for the sake of completeness. Fix $p > 1$ and let $1/q + 1/p = 1$. By lemma 4.13 the operators $\mathcal{D}_u$ and $\mathcal{D}_u^*$ are Fredholm. Since $\mathcal{D}_u$ is onto, the operator $\mathcal{D}_u^*$ is injective by proposition 3.17 and proposition 3.19 (hypothesis 3.15 is satisfied by theorem 1.8 on exponential decay). Hence by the open mapping theorem $\mathcal{D}_u^*$ satisfies the injectivity estimate

$$\|\eta\|_q + \|\nabla_s \eta\|_q + \|\nabla_t \nabla_t \eta\|_q \leq c_1 \, \|\mathcal{D}_u^* \eta\|_q \tag{81}$$

for every $\eta \in \mathcal{W}^{1,q}_u$ and with shift invariant constant $c_1 = c_1(q, u) > 0$. Next observe that

$$\frac{\langle \mathcal{D}_u^* \xi, \mathcal{D}_u^* \eta \rangle}{\|\mathcal{D}_u^* \eta\|_q} = \frac{\langle \mathcal{D}_u \mathcal{D}_u^* \xi, \eta \rangle}{\|\mathcal{D}_u^* \eta\|_q} \leq \|\mathcal{D}_u \mathcal{D}_u^* \xi\|_p \frac{\|\eta\|_q}{\|\mathcal{D}_u^* \eta\|_q} \leq c_1 \, \|\mathcal{D}_u \mathcal{D}_u^* \xi\|_p \tag{82}$$

for all $\xi \in \mathcal{W}_u^{2,p}$ and $\eta \in \mathcal{W}_u^{1,q}$. Here the first step is by definition of the formal adjoint and the second one by Hölder's inequality. The third step is by (81). Now there is a shift invariant constant $c_2 = c_2(p, u) > 0$ such that

$$\|\mathcal{D}_u^* \xi\|_p \leq c_2 \sup_{\eta \in \mathcal{W}_u^{1,q}} \frac{\langle \mathcal{D}_u^* \xi, \mathcal{D}_u^* \eta \rangle}{\|\mathcal{D}_u^* \eta\|_q} \tag{83}$$

for every $\xi \in \mathcal{W}_u^{2,p}$. The argument uses that $\mathcal{D}_u$ is onto and $\dim \ker \mathcal{D}_u < \infty$. The constant c_2 depends also on the choice of an L^2 orthonormal basis of $\ker \mathcal{D}_u$. Full details are given in step 2 of the proof of lemma 4.5 in [DS94]. Now the linear estimate proposition 2.13 for $\xi^* := \mathcal{D}_u^* \xi$ shows that

$$\|\xi^*\|_{\mathcal{W}_u^{1,p}} \leq c_3 \left(\|\mathcal{D}_u \xi^*\|_p + \|\xi^*\|_p \right)$$

where the constant $c_3(p, u)$ is again shift invariant. To estimate the second term in the sum apply (83) and (82) to obtain that $\|\xi^*\|_p \leq c_1 c_2 \|\mathcal{D}_u \xi^*\|_p$. □

Proposition 5.2 (Quadratic estimate)**.** *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V1)*. Let $\iota > 0$ be the injectivity radius of M and fix constants $1 < p < \infty$ and $c_0 > 0$. Then there is a constant $C = C(p, c_0) > 0$ such that the following is true. If $u : \mathbb{R} \times S^1 \to M$ is a smooth map and ξ is a compactly supported smooth vector field along u such that*

$$\|\partial_s u\|_\infty + \|\partial_t u\|_\infty + \|\nabla_t \partial_t u\|_\infty \leq c_0, \quad \|\xi\|_\infty \leq \iota,$$

then

$$\|\mathcal{F}_u(\xi) - \mathcal{F}_u(0) - d\mathcal{F}_u(0)\xi\|_p \leq C \|\xi\|_\infty \|\xi\|_{\mathcal{W}_u^{1,p}} \left(1 + \|\xi\|_{\mathcal{W}_u^{1,p}}\right).$$

Proof. Recall the definition (51) of the maps E_i and E_{ij} and write

$$\mathcal{F}_u(\xi) - \mathcal{F}_u(0) - \left.\frac{d}{d\tau}\right|_{\tau=0} \mathcal{F}_u(\tau\xi) = f(\xi) - g(\xi) - h(\xi)$$

where

$$\begin{aligned}
f(\xi) &:= \Phi(u,\xi)^{-1} \partial_s E(u,\xi) - \partial_s u - \left.\frac{d}{d\tau}\right|_{\tau=0} \Phi(u,\tau\xi)^{-1} \partial_s u \\
&\quad - \left.\frac{d}{d\tau}\right|_{\tau=0} \partial_s E(u,\tau\xi) \\
g(\xi) &:= \Phi(u,\xi)^{-1} \nabla_t \partial_t E(u,\xi) - \nabla_t \partial_t u + (\nabla_2 \Phi(u,0)\xi)\, \nabla_t \partial_t u \\
&\quad - \left.\frac{d}{d\tau}\right|_{\tau=0} \nabla_t \partial_t E(u,\tau\xi) \\
h(\xi) &:= \Phi(u,\xi)^{-1} \operatorname{grad} \mathcal{V}(E(u,\xi)) - \operatorname{grad} \mathcal{V}(u) + (\nabla_2 \Phi(u,0)\xi) \operatorname{grad} \mathcal{V}(u) \\
&\quad - \left.\frac{d}{d\tau}\right|_{\tau=0} \operatorname{grad} \mathcal{V}(E(u,\tau\xi)).
\end{aligned}$$

Here we used that $\Phi(u,0) = \mathbb{1}$. Straightforward calculation using the identities (53) shows that $f(\xi) = f_1(\xi)\nabla_s\xi + f_2(\xi)$ where

$$\begin{aligned} f_1(\xi)\nabla_s\xi &= \left(\Phi(u,\xi)^{-1}E_2(u,\xi) - \mathbb{1}\right)\nabla_s\xi \\ f_2(\xi)\partial_s u &= \left(\Phi(u,\xi)^{-1}E_1(u,\xi) - \mathbb{1} + \nabla_2\Phi(u,0)\xi\right)\partial_s u, \end{aligned}$$

that

$$g = g_1 \circ \nabla_t\partial_t u + g_2 \circ (\partial_t u, \partial_t u) + g_3 \circ \nabla_t\nabla_t\xi + g_4 \circ (\partial_t u, \nabla_t\xi) + g_5 \circ (\nabla_t\xi, \nabla_t\xi)$$

where

$$\begin{aligned} g_1(\xi) &= \Phi(u,\xi)^{-1}E_1(u,\xi) - \mathbb{1} + \nabla_2\Phi(u,0)\xi \\ g_2(\xi) &= \Phi(u,\xi)^{-1}E_{11}(u,\xi) - \left.\frac{d}{d\tau}\right|_{\tau=0} E_{11}(u,\tau\xi) \\ g_3(\xi) &= \Phi(u,\xi)^{-1}E_2(u,\xi) - \mathbb{1} \\ g_4(\xi) &= 2\Phi(u,\xi)^{-1}E_{12}(u,\xi) \\ g_5(\xi) &= \Phi(u,\xi)^{-1}E_{22}(u,\xi), \end{aligned}$$

and that

$$h(\xi) = \Phi(u,\xi)^{-1}\operatorname{grad}\mathcal{V}(E(u,\xi)) - (\mathbb{1} - (\nabla_2\Phi(u,0)\xi))\operatorname{grad}\mathcal{V}(u) - \mathcal{H}_{\mathcal{V}}(u)\xi.$$

Here $\mathcal{H}_{\mathcal{V}}$ denotes the covariant Hessian of $\mathcal{V}$ given by (4). It follows by inspection using the identities (53) that the maps f_2, g_1, g_2, and h together with their first derivative are zero at $\xi = 0$. Therefore there exists a constant $c > 0$ which depends continuously on $|\xi|$ and the constant in (V1) such that

$$|(f_2 + g_1 + g_2 + h)(\xi)| \le c\,|\xi|^2 \left(|\partial_s u| + |\nabla_t\partial_t u| + |\partial_t u|^2 + 1\right)$$

pointwise at every (s,t). Similarly, it follows that the remaining functions are zero at $\xi = 0$ and therefore

$$|(f_1 + g_3 + g_4 + g_5)(\xi)| \le c\,|\xi| \left(|\nabla_s\xi| + |\nabla_t\nabla_t\xi| + |\nabla_t\xi|\,|\partial_t u| + |\nabla_t\xi|^2\right).$$

Take these pointwise estimates to the power p, integrate them over $\mathbb{R}\times S^1$ and pull out L^∞ norms of $\partial_s u, \partial_t u$, and $\nabla_t\partial_t u$ to obtain the conclusion of proposition 5.2. The term $|\xi|\cdot|\nabla_t\xi|^2$ involving a product of first order terms is taken care of by the product estimate lemma 2.14 and remark 2.15. Here we use the fact that the (compact) support of ξ is contained in some set $(a,b]\times S^1$. □

Proof of the refined implicit function theorem 1.12

Assume the result is false. Then there exist constants $p > 2$ and $c_0 > 0$ and a sequence of smooth maps $u_\nu : \mathbb{R}\times S^1 \to M$ such that $\lim_{s\to\pm\infty} u_\nu(s,\cdot) = x^\pm(\cdot)$ exists, uniformly in t, and

$$|\partial_s u_\nu(s,t)| \le \frac{c_0}{1+s^2}, \qquad \|\partial_t u_\nu\|_\infty \le c_0, \qquad \|\nabla_t\partial_t u_\nu\|_\infty \le c_0, \tag{84}$$

for all $(s,t) \in \mathbb{R} \times S^1$ and

$$\left\| \partial_s u_\nu - \nabla_t \partial_t u_\nu - \operatorname{grad} \mathcal{V}(u_\nu) \right\|_p \leq \frac{1}{\nu}, \tag{85}$$

but which does not satisfy the conclusion of theorem 1.12 for $c = \nu$. This means that for every $u_* \in \mathcal{M}(x^-, x^+; \mathcal{V})$ and every $\xi^\nu \in \operatorname{im} \mathcal{D}_{u_*}^* \cap \mathcal{W}_{u_*}$ the following holds. If $u_\nu = \exp_{u_*}(\xi^\nu)$ then

$$\left\| \partial_s u_\nu - \nabla_t \partial_t u_\nu - \operatorname{grad} \mathcal{V}(u_\nu) \right\|_p < \frac{1}{\nu} \left\| \xi^\nu \right\|_{\mathcal{W}}. \tag{86}$$

The **time shift** of a smooth map $u : \mathbb{R} \times S^1$ by $\sigma \in \mathbb{R}$ is defined pointwise by

$$u^\sigma(s,t) := u(s+\sigma, t).$$

Set $a_0 := 2c_0^2$ and observe that

$$\mathcal{S}_\mathcal{V}(x^-) = \lim_{s \to -\infty} \mathcal{S}_\mathcal{V}(u_\nu(s,\cdot)) = \frac{1}{2} \left\| \partial_t u_\nu(s,\cdot) \right\|_2^2 - \mathcal{V}(u_\nu(s,\cdot)) \leq \frac{1}{2} c_0^2 + C_0 \leq a_0.$$

Here we used the assumption on asymptotic $W^{1,2}$ convergence, estimate (84), and our choice of the constant $c_0 > 1$ larger than the constant C_0 in (V0). Now fix a regular value c_* of $\mathcal{S}_\mathcal{V}$ between $\mathcal{S}_\mathcal{V}(x^+)$ and $\mathcal{S}_\mathcal{V}(x^-)$. Here we use that the set $\mathcal{P}^{a_0}(\mathcal{V})$ is finite, because $\mathcal{S}_\mathcal{V}$ is Morse–Smale below level a_0. Applying time shifts, if necessary, we may assume without loss of generality that

$$\mathcal{S}_\mathcal{V}\left(u_\nu(0,\cdot)\right) = c_*. \tag{87}$$

Furthermore we set $\tilde{c}_0 = a$ and let $C_0 = C_0(a, \mathcal{V}) > 0$ be the constant in theorem 1.7 with that choice. Then we have the apriori estimates

$$\left\| \partial_s u \right\|_\infty + \left\| \partial_t u \right\|_\infty + \left\| \nabla_t \partial_t u \right\|_\infty \leq C_0 \tag{88}$$

for all $u \in \mathcal{M}(x, y; \mathcal{V})$ and $x, y \in \mathcal{P}^a(\mathcal{V})$.

Claim. *There is a subsequence, still denoted by u_ν, a constant $C > 0$, a trajectory $u \in \mathcal{M}(x^-, x^+; \mathcal{V})$, and a sequence of times σ_ν such that the sequence η_ν determined by the identity*

$$u_\nu = \exp_{u^{\sigma_\nu}}(\eta_\nu)$$

satisfies $\eta_\nu \in \operatorname{im} \mathcal{D}_{u^{\sigma_\nu}}^* \cap \mathcal{W}_{u^{\sigma_\nu}}$ and

$$\lim_{\nu \to \infty} \left(\left\| \eta_\nu \right\|_\infty + \left\| \eta_\nu \right\|_p \right) = 0, \qquad \left\| \eta_\nu \right\|_\mathcal{W} \leq C. \tag{89}$$

Before we prove the claim we show how it leads to a contradiction. Consider the trajectories $u^{\sigma_\nu} \in \mathcal{M}(x^-, x^+; \mathcal{V})$ and vector fields η_ν provided by the claim. They satisfy the assumptions of the quadratic estimate, proposition 5.2, by (88) and by choosing a further subsequence, if necessary, to achieve that $\|\eta_\nu\|_\infty < \iota$. Set $c_0' = C_0(a, \mathcal{V})$ and let $C_2 = C_2(p, c_0')$ be the constant in proposition 5.2 with that choice. Furthermore, since $\mathcal{M}(x^-, x^+; \mathcal{V})/\mathbb{R}$ is a finite set by proposition 1.11 (and $\mathcal{P}^a(\mathcal{V})$ is a finite set as well) the estimate for the right inverse, proposition 5.1, applies with constant C_1 depending only on p, a, and $\mathcal{V}$. Now by the definition (79) of the map $\mathcal{F}_{\hat{u}}$ and the fact that parallel transport is an isometry we obtain the first step in the following estimate, namely

$$\begin{aligned}
\left\|\partial_s u_\nu - \nabla_t \partial_t u_\nu - \operatorname{grad}\mathcal{V}(u_\nu)\right\|_p &= \left\|\mathcal{F}_{\hat{u}}(\eta_\nu)\right\|_p \\
&\geq \left\|\mathcal{D}_{\hat{u}}\eta_\nu\right\|_p - \left\|\mathcal{F}_{\hat{u}}(\eta_\nu) - \mathcal{F}_{\hat{u}}(0) - d\mathcal{F}_{\hat{u}}(0)\eta_\nu\right\|_p \\
&\geq \left\|\eta_\nu\right\|_{\mathcal{W}} \left(\frac{1}{C_1} - C_2 \left\|\eta_\nu\right\|_\infty \left(1 + \left\|\eta_\nu\right\|_{\mathcal{W}}\right)\right) \\
&\geq \frac{1}{2C_1} \left\|\eta_\nu\right\|_{\mathcal{W}}.
\end{aligned}$$

Step two uses that $\mathcal{F}_{\hat{u}}(0) = \partial_s \hat{u} - \nabla_t \partial_t \hat{u} - \operatorname{grad}\mathcal{V}(\hat{u}) = 0$ and $d\mathcal{F}_{\hat{u}}(0) = \mathcal{D}_{\hat{u}}$. Step three is by proposition 5.1 and proposition 5.2. By (89) the last step holds for sufficiently large ν. For $\nu > 2C_1$ the estimate contradicts (86) and this proves theorem 1.12. It only remains to prove the claim. This takes four steps.

Step 1. *There is a subsequence of u_ν, still denoted by u_ν, and a trajectory $u \in \mathcal{M}(x^-, x^+; \mathcal{V})$ such that*

$$u_\nu = \exp_u(\xi_\nu), \qquad \lim_{\nu \to \infty} \left(\left\|\xi_\nu\right\|_\infty + \left\|\xi_\nu\right\|_p\right) = 0. \tag{90}$$

Proof. We embed the compact Riemannian manifold M isometrically into some Euclidean space $\mathbb{R}^N$ and view any continuous map to M as a map into $\mathbb{R}^N$ taking values in the embedded manifold. By translation we may assume that the embedded M contains the origin. Now L^p and L^∞ norms of u_ν are provided by the ambient Euclidean space. By compactness of M and, in particular, the L^∞ bounds in (84) we obtain on every compact cylindrical domain $Z_T := [-T, T] \times S^1$ the estimates

$$\left\|u_\nu\right\|_{L^p(Z_T)} \leq (2T)^{\frac{1}{p}} \operatorname{diam} M, \quad \left\|\partial_t u_\nu\right\|_{L^p(Z_T)} + \left\|\nabla_t \partial_t u_\nu\right\|_{L^p(Z_T)} \leq 2c_0 (2T)^{\frac{1}{p}},$$

and

$$\left\|\partial_s u_\nu\right\|_r \leq 4c_0 \quad \forall r \in (1, \infty]. \tag{91}$$

The latter follows by the estimate

$$\int_{-\infty}^{\infty} \left(\frac{1}{1+s^2}\right)^r ds \leq 2 + 2\int_1^\infty \frac{1}{s^{2r}}\, ds = \frac{4}{2 - 1/r} < 4$$

whenever $r > 1$. Hence the sequence u_ν is uniformly bounded in $\mathcal{W}^{1,p}(Z_T)$. Thus by the Arzela-Ascoli and the Banach-Alaoglu theorem a suitable subsequence, still denoted by u_ν, converges strongly in C^0 and weakly in $\mathcal{W}^{1,p}$ on every compact cylindrical domain Z_T to some continuous map $u : \mathbb{R} \times S^1 \to M$ which is locally of class $\mathcal{W}^{1,p}$. Hence $\partial_s u_\nu - \nabla_t \partial_t u_\nu - \mathrm{grad}\mathcal{V}(u_\nu)$ converges weakly in L^p to $\partial_s u - \nabla_t \partial_t u - \mathrm{grad}\mathcal{V}(u)$. On the other hand, by (85) it converges to zero in L^p. By uniqueness of limits u satisfies the heat equation (6) almost everywhere. Thus u is smooth by theorem 1.5.

Fix $s \in \mathbb{R}$ and observe that by (84) there are uniform $C^1(S^1)$ bounds for the sequence $\partial_t u_\nu(s, \cdot)$. Hence by Arzela-Ascoli a suitable subsequence, still denoted by $\partial_t u_\nu(s, \cdot)$, converges in $C^0(S^1)$ to $\partial_t u(s, \cdot)$. Thus

$$\lim_{\nu \to \infty} \mathcal{S}_\mathcal{V}(u_\nu(s, \cdot)) = \mathcal{S}_\mathcal{V}(u(s, \cdot))$$

and therefore $\mathcal{S}_\mathcal{V}(u(0, \cdot)) = c_*$ by (87). Recall that $\partial_s u = \nabla_t \partial_t u + \mathrm{grad}\, \mathcal{V}(u)$. When restricted to $s = 0$ this means that the vector field $\partial_s u(0, \cdot)$ is equal to the L^2 gradient of $\mathcal{S}_\mathcal{V}$ at the loop $u(0, \cdot)$. But $\mathcal{S}_\mathcal{V}(u(0, \cdot)) = c_*$ and c_* is a regular value. Hence $\partial_s u(0, \cdot)$ cannot vanish identically.

On the other hand, by (84) and axiom (V0) with constant C_0 it follows exactly as above that

$$\sup_\nu \mathcal{S}_\mathcal{V}(u_\nu(s, \cdot)) = \sup_\nu \frac{1}{2} \left\|\partial_t u_\nu(s, \cdot)\right\|_2^2 - \mathcal{V}(u_\nu) \leq a_0.$$

This shows that all relevant trajectories including relevant limits over s or ν lie in the sublevel set $\mathcal{L}^{a_0} M$ on which $\mathcal{S}_\mathcal{V}$ is Morse–Smale by assumption. In particular, we have that $\sup_{s \in \mathbb{R}} \mathcal{S}_\mathcal{V}(u(s, \cdot)) \leq a_0$ and therefore the energy of u is finite by lemma 4.8. Hence by the exponential decay theorem 1.8 there are critical points $y^\pm \in \mathcal{P}^{a_0}(\mathcal{V})$ such that $u(s, \cdot)$ converges to $y^\pm$ in $C^2(S^1)$, as $s \to \pm\infty$. Moreover, the limits y^- and y^+ are distinct, because the action along a nonconstant trajectory is strictly decreasing and the trajectory is nonconstant because $\partial_s u$ is not identically zero as observed above.

More generally, a standard argument shows the following, see e.g. [SW03, lemma 10.3]. There exist critical points $x^- = x^0, x^1, \ldots, x^\ell = x^+ \in \mathcal{P}^{a_0}(\mathcal{V})$ and trajectories $u^k \in \mathcal{M}(x^{k-1}, x^k; \mathcal{V})$, $\partial_s u^k \not\equiv 0$, for $k \in \{1, \ldots, \ell\}$, a subsequence, still denoted by u_ν, and sequences $s_\nu^k \in \mathbb{R}$, $k \in \{1, \ldots, \ell\}$, such that the shifted sequence $u_\nu(s_\nu^k + s, t)$ converges to $u^k(s, t)$ in an appropriate topology. The point here is that $\partial_s u^k \not\equiv 0$ and therefore the Morse index strictly decreases along the sequence $x^- = x^0, x^1, \ldots, x^\ell = x^+$. Namely, by the Morse–Smale condition each Fredholm operator $\mathcal{D}_{u^k}$ is onto, hence its Fredholm index is equal to the dimension of its kernel. But this is strictly positive because the kernel contains the nonzero element $\partial_s u^k$. On the other hand, by lemma 4.13 the Fredholm index is given by the difference of Morse indices $\mathrm{ind}_\mathcal{V}(x^{k-1}) - \mathrm{ind}_\mathcal{V}(x^k)$. Our assumption that the pair $x^\pm$ has Morse index difference one then implies that $\ell = 1$ and this proves that $u \in \mathcal{M}(x^-, x^+; \mathcal{V})$. The first assertion of step 1.

It remains to prove the second assertion, that is (90). The key fact to prove (90) is that $u_\nu(s, \cdot)$ not only converges in $W^{1,2}(S^1)$ to $x^\pm$, as $s \to \pm\infty$, but that the rate of convergence is independent of ν. More precisely, we prove that for every $\varepsilon > 0$ there is a time $T = T(\varepsilon) > 1$ such that

$$s > T \qquad \Longrightarrow \qquad d\left(u_\nu(s, t), x^+(t)\right) < \varepsilon \tag{92}$$

for all $t \in S^1$ and $\nu \in \mathbb{N}$. Recall that M is embedded isometrically in $\mathbb{R}^N$. By the fundamental theorem of calculus and uniform decay (84) we have that

$$\left|x^+(t) - u_\nu(\sigma,t)\right|_{\mathbb{R}^N} = \left|\int_\sigma^\infty \partial_s u_\nu(s,t)\, ds\right|_{\mathbb{R}^N} \le \int_\sigma^\infty \frac{c_0}{s^2} ds = \frac{c_0}{\sigma} \tag{93}$$

for all $t \in S^1$, $\nu \in \mathbb{N}$, and $\sigma > 1$ sufficiently large. The Riemannian distance d in M and the restriction of the Euclidean distance in $\mathbb{R}^N$ to the compact manifold M are locally equivalent. Hence (93) implies (92). Let $Z_T^+ := [T,\infty)\times S^1$ denote the positive end of the cylinder $\mathbb{R}\times S^1$ and Z_T^- the negative end. Let $\iota > 0$ be the injectivity radius of M. Now fix $\varepsilon \in (0,\iota/2)$ and choose $T = T(\varepsilon) > 0$ such that the ends $u(Z_T^\pm)$ and $u_\nu(Z_T^\pm)$ for all ν are contained in the $(\varepsilon/6)$-neighborhood of $x^\pm(S^1)$. Such T exists by (92). Since u_ν converges to u uniformly on Z_T, there exists $\nu_0 = \nu_0(T(\varepsilon)) \in \mathbb{N}$ such that $\|\xi_\nu\|_{L^\infty(Z_T)} < \varepsilon/3$ for every $\nu \ge \nu_0$. Hence

$$\begin{aligned}
\|\xi_\nu\|_\infty &= \|\xi_\nu\|_{L^\infty(Z_T^-)} + \|\xi_\nu\|_{L^\infty(Z_T)} + \|\xi_\nu\|_{L^\infty(Z_T^+)} \\
&\le \sup_{Z_T^-}\big(d(u_\nu,x^-) + d(x^-,u)\big) + \|\xi_\nu\|_{L^\infty(Z_T)} \\
&\quad + \sup_{Z_T^+}\big(d(u_\nu,x^+) + d(x^+,u)\big) \\
&\le \varepsilon
\end{aligned} \tag{94}$$

for every $\nu \ge \nu_0$. This proves that the L^∞ limit in (90) is zero. To prove that the L^p limit is zero one uses again the decomposition of $\mathbb{R}\times S^1$ into the compact part Z_T and the two ends $Z_T^\pm$. The left hand side of (93) is p-integrable over the ends $Z_T^\pm$. The key fact is that the value of this integral does not depend on ν and converges to zero as $|T| \to \infty$. A similar integral is needed in the case of u. Here the exponential decay theorem 1.8 shows that the integral exists and converges to zero as $|T| \to \infty$. This concludes the proof of step 1. □

Step 2. *Set $\varepsilon_\nu := \|\xi_\nu\|_\infty + \|\xi_\nu\|_p$ and let C_0 be the constant in (88). Then there is a constant $\sigma_0 > 0$ and integer $\nu_0 \ge 1$ such that $\eta = \eta(\sigma,\nu)$ is determined by the identity $u_\nu = \exp_{u^\sigma} \eta$ and satisfies $\|\eta\|_\infty < \iota/2$ for all $\sigma \in [-\sigma_0,\sigma_0]$ and $\nu \ge \nu_0$. Furthermore, there is a constant $c_2 = c_2(a_0,\sigma_0) > 0$ such that*

$$\|\eta\|_\infty \le \varepsilon_\nu + C_0\,|\sigma|\,, \qquad \|\eta\|_p \le 2\varepsilon_\nu + c_2\,|\sigma|$$

and

$$\|\nabla_s\eta\|_p \le c_2, \qquad \|\nabla_t\eta\|_\infty \le c_2, \qquad \|\nabla_t\nabla_t\eta\|_p \le c_2$$

for all $\sigma \in [-\sigma_0,\sigma_0]$ and $\nu \ge \nu_0$.

Proof. Existence of σ_0 and ν_0 follows from the fact that $\eta(\nu,0) = \xi_\nu$, continuity of time shift, and the L^∞ limit in (90). Now denote by L the length functional. Then for every $\sigma \in \mathbb{R}$ and $\gamma(r) := u(s+r\sigma,t)$ for $r \in [0,1]$ we have that

$$d\left(u(s,t),u(s+\sigma,t)\right) \le L(\gamma) = |\sigma|\int_0^1 |\partial_s u(s+r\sigma,t)|\,dr \le |\sigma|\,\|\partial_s u\|_\infty\,. \tag{95}$$

Since $d\left(u_\nu(s,t), u(s,t)\right) = |\xi_\nu(s,t)| \le \varepsilon_\nu$, the first estimate of step 2 follows from $|\eta(s,t)| = d\left(u_\nu(s,t), u(s+\sigma,t)\right)$, the triangle inequality, and (88). To prove the second estimate note that the triangle inequality also implies that

$$\|\eta\|_p^p \le 2^{p-1}\,\|\xi_\nu\|_p^p + 2^{p-1}\int_{-\infty}^{\infty}\int_0^1 d\left(u(s,t), u(s+\sigma,t)\right)^p\,dtds.$$

By theorem 1.8 on exponential decay there are constants $\rho, c_3 > 2$ such that for all $(\tilde{s},t) \in \mathbb{R}\times S^1$ we have that

$$|\partial_s u(\tilde{s},t)| \le c_3 e^{-\rho|\tilde{s}|}, \qquad \|\partial_s u\|_r \le c_3 \quad \forall r > 1. \tag{96}$$

Note that the constants ρ and c_3 depend only on a_0, since the set $\mathcal{P}^{a_0}(\mathcal{V})$ is finite and there are only finitely many elements of $\mathcal{M}(x^-, x^+; \mathcal{V})$ which satisfy (87). By the first inequality in (95) and the first estimate in (96) with $\tilde{s} = s + r\sigma$

$$d\left(u(s,t), u(s+\sigma,t)\right) \le |\sigma|\int_0^1 |\partial_s u(s+r\sigma,t)|\,dr \le |\sigma|\,c_3 e^{\rho\sigma_0} e^{-\rho|s|}.$$

Hence the left hand side is L^p integrable. This concludes the proof of the second estimate of step 2. To prove the next two estimates we differentiate the identity $\exp_{u^\sigma}\eta = u_\nu$ with respect to s and t to obtain that

$$E_1(u^\sigma,\eta)\partial_s u^\sigma + E_2(u^\sigma,\eta)\nabla_s\eta = \partial_s u_\nu \tag{97}$$
$$E_1(u^\sigma,\eta)\partial_t u^\sigma + E_2(u^\sigma,\eta)\nabla_t\eta = \partial_t u_\nu. \tag{98}$$

Here the maps E_i are defined by (51). Since $\|\partial_s u^\sigma\|_p \le c_3$ by (96) and $\|\partial_s u_\nu\|_p \le 4c_0$ by (91), the L^p norm of $\nabla_s\eta$ is uniformly bounded as well. Similarly, since $\|\partial_t u^\sigma\|_\infty \le C_0$ by (88) and $\|\partial_t u_\nu\|_\infty \le c_0$ by (84), the L^∞ norm of $\nabla_t\eta$ is uniformly bounded. To prove the last estimate of step 2 differentiate (98) covariantly with respect to t and abbreviate $E_{ij} = E_{ij}(u^\sigma,\eta)$ to obtain

$$\begin{aligned}
&E_{11}(u^\sigma,\eta)\left(\partial_t u^\sigma, \partial_t u^\sigma\right) + E_{12}(u^\sigma,\eta)\left(\partial_t u^\sigma, \nabla_t\eta\right) + E_1(u^\sigma,\eta)\,\nabla_t\partial_t u^\sigma\\
&+ E_{21}(u^\sigma,\eta)\left(\nabla_t\eta, \partial_t u^\sigma\right) + E_{22}(u^\sigma,\eta)\left(\nabla_t\eta, \nabla_t\eta\right) + E_2(u^\sigma,\eta)\,\nabla_t\nabla_t\eta\\
&+ \operatorname{grad}\mathcal{V}(u_\nu) - \partial_s u_\nu\\
&= \nabla_t\partial_t u_\nu + \operatorname{grad}\mathcal{V}(u_\nu) - \partial_s u_\nu.
\end{aligned}$$

This identity implies a uniform L^p bound for $\nabla_t\nabla_t\eta$ as follows. The right hand side is bounded in L^p by $1/\nu$ and the last term of the left hand side by $4c_0$ according to (91). Since $E_{ij}(u^\sigma,0) = 0$ and we have uniform L^∞ bounds for each of the two linear terms to which $E_{ij}(u^\sigma,\eta)$ is applied, we can estimate the L^p norm by a constant times $\|\eta\|_p$. The only terms left are term three and term seven of the left hand side. By the heat equation (6) their sum equals

$$E_1(u^\sigma,\eta)\,\partial_s u^\sigma - E_1(u^\sigma,\eta)\operatorname{grad}\mathcal{V}(u^\sigma) + \operatorname{grad}\mathcal{V}(u_\nu).$$

Since $\|\partial_s u^\sigma\|_p \le c_3$ by (96), the L^p norm of the first term is uniformly bounded. Consider the remaining two terms as a function f of η. Then $f(0) = 0$, because $E_1(u^\sigma,0) = \mathbb{1}$ and $\eta = 0$ means $u_\nu = u^\sigma$. Hence $\|f\|_p$ is uniformly bounded by a constant times $\|\eta\|_p$. Here we used axiom (V0). This proves step 2. $\square$

Step 3. *For $\sigma \in [-\sigma_0, \sigma_0]$ consider the function $\theta_\nu(\sigma) := -\langle \partial_s u^\sigma, \eta \rangle$ where $\eta = \eta(\sigma, \nu)$ has been defined in step 2 by the identity $u_\nu = \exp_{u^\sigma} \eta$ and where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\mathbb{R} \times S^1)$ inner product. This function has the property that*

$$\theta_\nu(\sigma) = 0 \quad \Longleftrightarrow \quad \eta \in \operatorname{im} \mathcal{D}_{u^\sigma}^*.$$

Moreover, there exist new constants $\sigma_0 > 0$ and $\nu_0 \in \mathbb{N}$ such that

$$|\theta_\nu(0)| \le c_3 \varepsilon_\nu, \qquad \frac{d}{d\sigma} \theta_\nu(\sigma) \ge \frac{\mu}{2},$$

for all $\sigma \in [-\sigma_0, \sigma_0]$ and $\nu \ge \nu_0$ where $\mu := \mathcal{S}_\mathcal{V}(x^-) - \mathcal{S}_\mathcal{V}(x^+) > 0$.

Proof. '$\Leftarrow$' follows by definition of the formal adjoint operator using that $\partial_s u^\sigma \in \ker \mathcal{D}_{u^\sigma}$. We prove '$\Rightarrow$'. The kernel of the linear operator $\mathcal{D}_{u^\sigma}$ is 1-dimensional: It is Fredholm of index one by theorem 1.9 and it is onto by the Morse–Smale condition. This kernel is spanned by the (nonzero) element $\partial_s u^\sigma$. Now consider $\mathcal{D}_{u^\sigma}^*$ on the domain $\mathcal{W}^{2,p}$ and apply proposition 3.19 to obtain that $\mathcal{W}^{1,p} = \ker \mathcal{D}_{u^\sigma} \oplus \operatorname{im} \mathcal{D}_{u^\sigma}^*$. The implication '$\Rightarrow$' now follows immediately by contradiction.

Set $1/q + 1/p = 1$. By (96) and the definition of the sequence $\varepsilon_\nu \to 0$ in step 2 it follows that

$$|\theta_\nu(0)| = |\langle \partial_s u, \xi_\nu \rangle_{L^2}| \le \|\partial_s u\|_q \|\xi_\nu\|_p \le c_3 \varepsilon_\nu.$$

Abbreviate $E_i = E_i(u^\sigma, \eta)$. Then straightforward calculation using the identity (97) for $\nabla_s \eta$ shows that

$$\begin{aligned}
\frac{d}{d\sigma} \theta_\nu(\sigma) &= -\langle \nabla_s \partial_s u^\sigma, \eta \rangle_{L^2} - \langle \partial_s u^\sigma, -\partial_s u^\sigma + \partial_s u^\sigma - E_2^{-1} E_1 \partial_s u^\sigma \rangle_{L^2} \\
&\ge -\|\nabla_s \partial_s u^\sigma\|_q \|\eta\|_p + \|\partial_s u^\sigma\|_2^2 - \|\partial_s u^\sigma\|_q \|\partial_s u^\sigma\|_\infty c_4 \|\eta\|_p \\
&= \|\partial_s u\|_2^2 - \|\eta\|_p \left(\|\nabla_s \partial_s u\|_q + c_4 \|\partial_s u\|_q \|\partial_s u\|_\infty \right) \\
&\ge \|\partial_s u\|_2^2 - (2\varepsilon_\nu + c_2 |\sigma|)(c_5 + c_3^2 c_4)
\end{aligned}$$

for some constant $c_4 = c_4(a_0, \sigma_0) > 0$. The last step is by (96) with constant c_3. We also used that $\|\nabla_s \partial_s u\|_q \le c_5$ for some positive constant $c_5 = c_5(a_0)$, which follows from exponential decay of $\nabla_s \partial_s u$ according to theorem 1.8. The energy identity (9) shows that $\|\partial_s u\|_2^2 = \mu > 0$. Now choose $\sigma_0 > 0$ sufficiently small and ν_0 sufficiently large to conclude the proof of step 3. $\square$

Step 4. *We prove the claim.*

Proof. By step 3 there exists, for every sufficiently large ν, an element $\sigma_\nu \in [-\sigma_0, \sigma_0]$ such that $\theta_\nu(\sigma_\nu) = 0$ and $|\sigma_\nu| \le \varepsilon_\nu(2c_3/\mu)$. Set $\eta_\nu := \eta(\sigma_\nu, \nu)$. Then $\eta_\nu \in \operatorname{im} \mathcal{D}_{u^{\sigma_\nu}}^*$ again by step 3 and

$$\|\eta_\nu\|_\infty + \|\eta_\nu\|_p \le \varepsilon_\nu \left(3 + (c_2 + C_0) 2c_3/\mu\right), \qquad \|\eta_\nu\|_\mathcal{W} \le C,$$

by step 2. This proves (89), hence the claim, and therefore theorem 1.12. $\square$

6 Unique Continuation

To prove unique continuation for the nonlinear heat equation we slightly extend a result of Agmon and Nirenberg [AN67] (to the case $C_1 \neq 0$). This generalization is needed to deal with the nonlinear heat equation (6), since here nonzero order terms appear on the right hand side of (99). For the linear heat equation the original result for $C_1 = 0$ is sufficient.

Theorem 6.1. *Let H be a real Hilbert space and let $A(s) : \operatorname{dom} A(s) \to H$ be a family of symmetric linear operators. Assume that $\zeta : [0,T] \to H$ is continuously differentiable in the weak topology such that $\zeta(s) \in \operatorname{dom} A(s)$ and*

$$\|\zeta'(s) - A(s)\zeta(s)\| \leq c_1 \|\zeta(s)\| + C_1 |\langle A(s)\zeta(s), \zeta(s)\rangle|^{1/2} \tag{99}$$

for every $s \in [0,T]$ and two constants $c_1, C_1 \geq 0$. Here $\zeta'(s) \in H$ denotes the derivative of ζ with respect to s. Assume further that the function $s \mapsto \langle \zeta(s), A(s)\zeta(s)\rangle$ is also continuously differentiable and satisfies

$$\frac{d}{ds}\langle \zeta, A\zeta\rangle - 2\langle \zeta', A\zeta\rangle \geq -c_2 \|A\zeta\| \|\zeta\| - c_3 \|\zeta\|^2 \tag{100}$$

pointwise for every $s \in [0,T]$ and constants $c_2, c_3 > 0$. Then the following holds.
(1) If $\zeta(0) = 0$ then $\zeta(s) = 0$ for all $s \in [0,T]$.
(2) If $\zeta(0) \neq 0$ then $\zeta(s) \neq 0$ for all $s \in [0,T]$ and, moreover,

$$\log \|\zeta(s)\|^2 \geq \log \|\zeta(0)\|^2 - \left(2\frac{\langle \zeta(0), A(0)\zeta(0)\rangle}{\|\zeta(0)\|^2} + \frac{b}{a}\right)\frac{e^{as} - 1}{a} - 2c_1 s$$

where $a = 2{C_1}^2 + c_2$ and $b = 4{c_1}^2 + {c_2}^2/2 + 2c_3$.

Proof. A beautyful exposition in the case $C_1 = 0$ was given by Salamon in [S97, appendix E] in the case $C_1 = 0$. It generalizes easily. A key step is to prove that the function

$$\varphi(s) := \log\|\zeta(s)\|^2 - \int_0^s \frac{2\langle \zeta(\sigma), \zeta'(\sigma) - A(\sigma)\zeta(\sigma)\rangle}{\|\zeta(\sigma)\|^2} d\sigma$$

satisfies the differential inequality

$$\varphi'' + a|\varphi'| + b \geq 0 \tag{101}$$

for two constants $a, b > 0$.

In [S97] it is shown that assumption (100) implies the inequality

$$\varphi'' \geq 2\|\eta - \langle \eta, \xi\rangle \xi\|^2 - \frac{2\|\zeta' - A\zeta\|^2}{\|\zeta\|^2} - 2c_2 \|\eta\| - 2c_3$$

where

$$\xi := \frac{\zeta}{\|\zeta\|}, \qquad \eta := \frac{A\zeta}{\|\zeta\|}.$$

Now it follows by assumption (99) that

$$\frac{2\left\|\zeta' - A\zeta\right\|^2}{\left\|\zeta\right\|^2} \le 4{c_1}^2 + 4{C_1}^2 \frac{|\langle A\zeta,\zeta\rangle|}{\left\|\zeta\right\|^2} = 4{c_1}^2 + 4{C_1}^2 \left|\langle \eta,\xi\rangle\right|$$

and therefore

$$\varphi'' \ge 2\left\|\eta - \langle\eta,\xi\rangle\xi\right\|^2 - 4{c_1}^2 - 4{C_1}^2\left|\langle\eta,\xi\rangle\right| - 2c_2\left\|\eta\right\| - 2c_3.$$

To obtain the inequality (101) it remains to prove that

$$2\left\|\eta - \langle\eta,\xi\rangle\xi\right\|^2 - 4{c_1}^2 - 4{C_1}^2\left|\langle\eta,\xi\rangle\right| - 2c_2\left\|\eta\right\| - 2c_3 \ge -a\left|\varphi'\right| - b.$$

Since $\varphi' = 2\langle\xi,\eta\rangle$ this is equivalent to

$$c_2\left\|\eta\right\| \le \left\|\eta - \langle\eta,\xi\rangle\xi\right\|^2 + (a - 2{C_1}^2)\left|\langle\eta,\xi\rangle\right| + (b/2 - 2{c_1}^2 - c_3).$$

Abbreviate

$$u := \left\|\eta - \langle\eta,\xi\rangle\xi\right\|^2, \qquad v := \left|\langle\eta,\xi\rangle\right|,$$

then $\|\eta\|^2 = u^2 + v^2$ and the desired inequality has the form

$$c_2\sqrt{u^2+v^2} \le u^2 + (a - 2{C_1}^2)v + (b/2 - 2{c_1}^2 - c_3).$$

Since $c_2\sqrt{u^2+v^2} \le c_2 u + c_2 v \le u^2 + c_2 v + {c_2}^2/4$ this is satisfies with

$$a = 2{C_1}^2 + c_2, \qquad b = 4{c_1}^2 + {c_2}^2/2 + 2c_3.$$

This proves the inequality (101). The remaining part of the proof of theorem 6.1 carries over from [S97] unchanged. □

6.1 Linear equation

Unique continuation for the linearized heat equation is used to prove proposition 7.5 on transversality of the universal section and the unstable manifold theorem 8.1.

Proposition 6.2. *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V2) *and two constants $a < b$. Let $u : [a,b] \times S^1 \to M$ be a smooth map and let $\xi = \xi(s,t)$ be a smooth vector field along u such that $\mathcal{D}_u\xi = 0$ or $\mathcal{D}_u^*\xi = 0$, where the operators are defined by (57) and (58), respectively. Abbreviate $\xi(s,\cdot)$ by $\xi(s)$. Then the following is true.*

(a) *If $\xi(s_*) = 0$ for some s_*, then $\xi(s) = 0$ for all $s \in [a,b]$.*

(b) *If $\xi(s_*) \ne 0$ for some s_*, then $\xi(s) \ne 0$ for all $s \in [a,b]$.*

Proof. We represent $\mathcal{D}_u$ by the operator $D_{A+C} = \frac{d}{ds} + A(s) + C(s)$ given by (59). Here the family $A(s)$ consists of self-adjoint operators on the Hilbert space $H := L^2(S^1,\mathbb{R}^n)$ with dense domain W; see (ii) and (iv) in section 3.4. The space

W has been defined prior to (59). Recall that if the vector bundle $u^*TM \to [a,b] \times S^1$ is trivial then $W = W^{2,2}(S^1, \mathbb{R}^n)$ and otherwise some boundary condition enters. In either case $W =: \operatorname{dom} A(s)$ is independent of s.

(b) Let $\xi \in \ker D_{A+C}$ satisfy $\xi(s_*) \neq 0$. Assume by contradiction that $\xi(s_0) = 0$ for some $s_0 \in [a,b]$. Now if $s_0 > s_*$, then replace $\xi(s)$ by $\xi(s+s_*)$ and set $T = b - s_*$ and $s_1 = s_0 - s_*$, otherwise replace $\xi(s)$ by $\xi(-s+s_*)$ and set $T = -a+s_*$ and $s_1 = -s_0+s_*$. Hence we may assume without loss of generality that $\xi \in \ker D_{A+C}$ maps $[0,T]$ to H and satisfies $\xi(0) \neq 0$ and $\xi(s_1) = 0$ for some $s_1 \in (0,T]$.

Next we check that the conditions in theorem 6.1 are satisfied: Firstly, the vector field ξ is smooth by assumption. Secondly, the family $A(s)$ consists of self-adjoint operators by (ii) in section 3.4. Thirdly, the function $s \mapsto \langle \xi(s), A(s)\xi(s)\rangle$ is continuously differentiable. Here we use the first condition in axiom (V2), which tells that the Hessian $\mathcal{H}_\mathcal{V}$ is a zeroth order operator, and the fact that by compactness of the domain the vector fields $\partial_t u$, $\partial_s u$, $\nabla_t \partial_s u$, and $\nabla_t \nabla_t \partial_s u$ are bounded in $L^\infty([0,T] \times S^1)$ by a constant $c_T > 0$. Next assumption (99) is satisfied with $C_1 = 0$, because

$$\|\xi'(s) - A(s)\xi(s)\| = \|C(s)\xi(s)\| \leq c_T' \|\xi(s)\|$$

where the constant $c_T' = \sup_{[0,T]\times S^1} \|C(s,t)\|_{\mathcal{L}(\mathbb{R}^n)}$ is finite by compactness of the domain. To verify the inequality (100) note that its left hand side is given by $\langle \xi(s), A'(s)\xi(s)\rangle$; see [AN67, Rmk. in sec. 1] and [S97, Rmk. F.3]. Now

$$\begin{aligned} \langle \xi(s), A'(s)\xi(s)\rangle &\geq -\|\xi(s)\| \, \|A'(s)\xi(s)\| \\ &\geq -c_T'' \|\xi(s)\| \left(\|\xi(s)\| + \|\partial_t \xi(s)\|\right). \end{aligned}$$

where the second step is by straightforward calculation of $A'(s)$. Replacing $\|\partial_t \xi(s)\|$ according to the elliptic estimate for $A(s)$ yields (100).

Now the Agmon-Nirenberg theorem 6.1 applies and part (2) tells that $\xi(s) \neq 0$ for all $s \in [0,T]$. This contradiction proves (b) for elements in the kernel of $\mathcal{D}_u$. The same argument covers the case of the operator $\mathcal{D}_u^*$, since it is represented by $-D_{-A-C}$ according to remark 3.16.

(a) This follows either by a time reversing argument (see proof of the Agmon-Nirenberg Theorem in [S97]) and application of (b) or by a line of argument analoguous to the proof of (b) given above, where in the final step part (2) of theorem 6.1 is replaced by part (1). □

6.2 Nonlinear equation

Unique continuation for the nonlinear heat equation is used to prove the unstable manifold theorem 8.1.

Theorem 6.3 (Unique Continuation for compact cylindrical domains)**.** *Fix two constants $a < b$ and a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0) *and* (V1)*. If two smooth solutions $u, v : [a,b] \times S^1 \to M$ of the heat equation (6) coincide along one loop, then $u = v$.*

Proof. Abbreviate $u_s = u(s,\cdot)$ and assume $u_\sigma = v_\sigma : S^1 \to M$ for some $\sigma \in [a,b]$. Moreover, we may assume without loss of generality that $\partial_s u$ is nonzero at some point (s,t). Otherwise u coincides with a critical point x of the action functional $\mathcal{S}_\mathcal{V}$ and, since $v_\sigma = u_\sigma = x$, so does v and we are done. It follows similarly that $\partial_s v$ is nonzero somewhere. Hence

$$\delta := \frac{\iota}{2 + \|\partial_s u\|_\infty + \|\partial_s v\|_\infty} \in (0, \iota/2). \tag{102}$$

Here $\iota > 0$ denotes the injectivity radius of our compact Riemannian manifold.

The first step is to prove that the restrictions of u and v to $[\sigma-\delta, \sigma+\delta] \times S^1$ are equal. (In fact we should take the intersection with $[a,b] \times S^1$, but *suppress this throughout* for simplicity of notation.) The key idea is to express the difference of u and v near σ with respect to geodesic normal coordinates based at u_σ and show that this difference ζ and a suitable operator A satisfy the requirements of theorem 6.1 (with nonzero constant C_1). Then, since $\zeta(\sigma) = 0$, part (1) of the theorem shows that $\zeta = 0$ and therefore $u = v$ on $[\sigma - \delta, \sigma + \delta] \times S^1$.

Once the above has been achieved we successively restrict u and v to cylinders of the form $[\sigma + (2k-1)\delta, \sigma + (2k+1)\delta] \times S^1$, where $k \in \mathbb{Z}$, and use that u and v coincide along one of the two boundary components to conclude by the same argument as above that $u = v$ on each of these cylinders. Due to compactness of Z the same constants c_1 and C_1 can be chosen in (99) for *all* cylinders. After finitely many steps the union of these cylinders covers $[a,b] \times S^1$ and this proves the theorem.

It remains to carry out the first step. Consider the interval $I = [\sigma - \delta, \sigma + \delta]$ and the cylinder

$$Z = I \times S^1 = [\sigma - \delta, \sigma + \delta] \times S^1.$$

From now on u and v are restricted to the domain Z. Note that the Riemannian distance between $u(\sigma,t)$ and $u(s,t)$ is less than half the injectivity radius ι for every $(s,t) \in Z$. Hence the identities

$$u(s,t) = \exp_{u(\sigma,t)} \xi(s,t), \qquad v(s,t) = \exp_{u(\sigma,t)} \eta(s,t)$$

for $(s,t) \in Z$ uniquely determine smooth families of vector fields ξ and η along the loop u_σ. The domain of ξ and η is Z, they satisfy the estimates

$$\|\xi\|_\infty < \frac{\iota}{2}, \qquad \|\eta\|_\infty < \frac{\iota}{2},$$

and $\xi(\sigma,t) = 0 = \eta(\sigma,t)$ for every $t \in S^1$. Moreover, since $\xi(s,t)$ and $\eta(s,t)$ live in the same tangent space $T_{u(\sigma,t)}M$ their difference $\zeta = \xi - \eta$ is well defined. Now consider the Hilbert space $H = L^2(S^1, {u_\sigma}^*TM)$ and the symmetric differential operator $A = \nabla_t \nabla_t$ with domain $W = W^{2,2}(S^1, {u_\sigma}^*TM)$. Here ∇_t denotes the covariant derivative along the loop u_σ. Hence the operator A is independent of s and condition (100) in the Agmon-Nirenberg theorem 6.1 is vacuous. If we can verify condition (99) as well, then $\zeta(\sigma) = 0$ implies that $\zeta(s) = 0$ for every $s \in I$ by theorem 6.1 (1). Since ζ is smooth, this means that on Z we have

$\xi = \eta$ pointwise and therefore $u = v$.
It remains to verify (99). Use (51) to obtain the identities

$$\begin{aligned} \partial_s u &= E_2(u_\sigma, \xi)\partial_s \xi \\ \nabla_t \partial_t u &= E_{11}(u_\sigma, \xi)\big(\partial_t u_\sigma, \partial_t u_\sigma\big) + 2E_{12}(u_\sigma, \xi)\big(\partial_t u_\sigma, \nabla_t \xi\big) \\ &\quad + E_1(u_\sigma, \xi)\nabla_t \partial_t u_\sigma + E_{22}(u_\sigma, \xi)\big(\nabla_t \xi, \nabla_t \xi\big) + E_2(u_\sigma, \xi)\nabla_t \nabla_t \xi \end{aligned} \tag{103}$$

pointwise for $(s,t) \in Z$ and similarly for v and η. To obtain the second identity we used the symmetry property (52) of E_{12}. Now consider the heat equation (6) and replace $\partial_s u$ and $\nabla_t \partial_t u$ according to (103), then solve for $\partial_s \xi - \nabla_t \nabla_t \xi$. Do the same for v and η to obtain a similar expression for $-\partial_s \eta + \nabla_t \nabla_t \eta$. Add both expressions to get the pointwise identity

$$\begin{aligned} &\big(\partial_s - \nabla_t \nabla_t\big)\big(\xi - \eta\big) \\ &= \big(E_2(u_\sigma, \xi)^{-1} E_{11}(u_\sigma, \xi) - E_2(u_\sigma, \eta)^{-1} E_{11}(u_\sigma, \eta)\big)\left(\partial_t u_\sigma, \partial_t u_\sigma\right) \\ &\quad + \big(E_2(u_\sigma, \xi)^{-1} E_1(u_\sigma, \xi) - E_2(u_\sigma, \eta)^{-1} E_1(u_\sigma, \eta)\big)\nabla_t \partial_t u_\sigma \\ &\quad + 2\left(E_2(u_\sigma, \xi)^{-1} E_{21}(u_\sigma, \xi)\nabla_t \xi - E_2(u_\sigma, \eta)^{-1} E_{21}(u_\sigma, \eta)\nabla_t \eta\right)\partial_t u_\sigma \\ &\quad + E_2(u_\sigma, \xi)^{-1} \operatorname{grad} \mathcal{V}(\exp_{u_\sigma} \xi) - E_2(u_\sigma, \eta)^{-1} \operatorname{grad} \mathcal{V}(\exp_{u_\sigma} \eta) \\ &\quad + E_2(u_\sigma, \xi)^{-1} E_{22}(u_\sigma, \xi)\big(\nabla_t \xi, \nabla_t \xi\big) - E_2(u_\sigma, \eta)^{-1} E_{22}(u_\sigma, \eta)\big(\nabla_t \eta, \nabla_t \eta\big). \end{aligned}$$

Now by compactness of the domain Z there is a constant $C > 0$ such that

$$\|\partial_t u_\sigma\|_{L^\infty(S^1)} \le \|\partial_t u\|_{L^\infty(Z)} < C, \qquad \|\nabla_t \partial_t u_\sigma\|_{L^\infty(S^1)} < C.$$

Moreover, since the maps E_i and E_{ij} are uniformly continuous on the radius $\iota/2$ disk tangent bundle $\mathcal{O} \subset TM$ in which ξ and η take their values, there exists a constant $c_1 > 0$ such that

$$\begin{aligned} &\left|\partial_s(\xi - \eta) - \nabla_t \nabla_t (\xi - \eta)\right| \\ &\le \left(c_1 C^2 + c_1 C\right)|\xi - \eta| \\ &\quad + 2C\left|E_2(u_\sigma, \xi)^{-1} E_{21}(u_\sigma, \xi)\nabla_t \xi - E_2(u_\sigma, \eta)^{-1} E_{21}(u_\sigma, \eta)\nabla_t \eta\right| \\ &\quad + \left|E_2(u_\sigma, \xi)^{-1} \operatorname{grad} \mathcal{V}(\exp_{u_\sigma} \xi) - E_2(u_\sigma, \eta)^{-1} \operatorname{grad} \mathcal{V}(\exp_{u_\sigma} \eta)\right| \\ &\quad + \left|E_2(u_\sigma, \xi)^{-1} E_{22}(u_\sigma, \xi)\big(\nabla_t \xi, \nabla_t \xi\big) - E_2(u_\sigma, \eta)^{-1} E_{22}(u_\sigma, \eta)\big(\nabla_t \eta, \nabla_t \eta\big)\right| \end{aligned}$$

pointwise for $(s,t) \in Z$. It remains to estimate the last three terms in the sum. First we estimate term three. Use linearity and the symmetry property (52) of E_{22} to obtain the first identity in the pointwise estimate

$$\begin{aligned} &\left|E_2(u_\sigma, \xi)^{-1} E_{22}(u_\sigma, \xi)\big(\nabla_t \xi, \nabla_t \xi\big) - E_2(u_\sigma, \eta)^{-1} E_{22}(u_\sigma, \eta)\big(\nabla_t \eta, \nabla_t \eta\big)\right| \\ &= \Big|E_2(u_\sigma, \xi)^{-1} E_{22}(u_\sigma, \xi)\big(\nabla_t \xi - \nabla_t \eta, \nabla_t \xi\big) \\ &\quad + E_2(u_\sigma, \eta)^{-1} E_{22}(u_\sigma, \eta)\big(\nabla_t \xi - \nabla_t \eta, \nabla_t \eta\big) \\ &\quad + \left(E_2(u_\sigma, \xi)^{-1} E_{22}(u_\sigma, \xi) - E_2(u_\sigma, \eta)^{-1} E_{22}(u_\sigma, \eta)\right)\big(\nabla_t \xi, \nabla_t \eta\big)\Big| \\ &\le \left\|{E_2}^{-1} E_{22}\right\|_{L^\infty(\mathcal{O})}\left(\|\nabla_t \xi\|_\infty + \|\nabla_t \eta\|_\infty\right)|\nabla_t(\xi - \eta)| \\ &\quad + c_1 \|\nabla_t \xi\|_\infty \|\nabla_t \eta\|_\infty |\xi - \eta| \\ &\le \mu_1 |\nabla_t(\xi - \eta)| + \mu_2 |\xi - \eta| \end{aligned}$$

where $\mu_1 = 2{c_2}^2 C(1+c_2)$, $\mu_2 = c_1 {c_2}^2 C^2 (1+c_2)^2$, and the constant $c_2 > 0$ is chosen sufficiently large such that for $j = 0, 1$ we have

$$\|E_j\|_{L^\infty(\mathcal{O})} + \left\|{E_2}^{-1}\right\|_{L^\infty(\mathcal{O})} + \left\|{E_2}^{-1} E_{22}\right\|_{L^\infty(\mathcal{O})} + \left\|{E_2}^{-1} E_{21}\right\|_{L^\infty(\mathcal{O})} \le c_2.$$

Moreover, we used that by the first identity in (51)

$$\nabla_t \xi = E_2(u_\sigma, \xi)^{-1} \left(\partial_t u - E_1(u_\sigma, \xi) \partial_t u_\sigma\right).$$

Hence $\|\nabla_t \xi\|_\infty \le c_2 C(1+c_2)$ and similarly for $\nabla_t \eta$. Next we estimate term one. Replace $\nabla_t \xi$ by $\nabla_t \xi - \nabla_t \eta + \nabla_t \eta$, then similarly as above we obtain that

$$\begin{aligned}
&2C \left| E_2(u_\sigma, \xi)^{-1} E_{21}(u_\sigma, \xi) \nabla_t \xi - E_2(u_\sigma, \eta)^{-1} E_{21}(u_\sigma, \eta) \nabla_t \eta \right| \\
&\le 2c_2 C \left|\nabla_t(\xi - \eta)\right| + 2c_1 c_2 C^2 (1+c_2) \left|\xi - \eta\right|
\end{aligned}$$

pointwise for $(s,t) \in Z$. Next rewrite term two setting $X := \eta - \xi$ and replacing η accordingly to obtain pointwise at $(s,t) \in Z$ the identity

$$\begin{aligned}
&E_2(u_\sigma, \xi)^{-1} \mathrm{grad} \mathcal{V}(\exp_{u_\sigma} \xi) - E_2(u_\sigma, \xi + X)^{-1} \mathrm{grad} \mathcal{V}(\exp_{u_\sigma} \xi + X) \\
&=: f(X) \\
&= f(0) + \frac{d}{d\tau} f(\tau X) \\
&= \frac{d}{d\tau} \left(E_2(u_\sigma, \xi + \tau X)^{-1} \mathrm{grad} \mathcal{V}(\exp_{u_\sigma} \xi + \tau X) \right)
\end{aligned}$$

for some $\tau \in [0,1]$. Since $f(0) = 0$, this implies that

$$\begin{aligned}
|f(X)| &\le \left\|{E_2}^{-1} E_{22}\right\|_{L^\infty(\mathcal{O})} |X| \cdot \left\|{E_2}^{-1}\right\|_{L^\infty(\mathcal{O})} \left|\mathrm{grad}\, \mathcal{V}(\exp_{u_\sigma}(\xi + \tau X))\right| \\
&\quad + \left\|{E_2}^{-1}\right\|_{L^\infty(\mathcal{O})} \left|\nabla_\tau \mathrm{grad}\, \mathcal{V}(\exp_{u_\sigma}(\xi + \tau X))\right| \\
&\le c_2^2 C_0 |X| + c_2^2 C_1 \Big(|X| + \|X_s\|_{L^1(S^1)} \Big)
\end{aligned}$$

pointwise at $(s,t) \in Z$. Here C_0 and C_1 denote the constants in axiom (V0) and (V1), respectively. To obtain the final step we applied the first estimate in axiom (V1) to the curve $\tau \mapsto \exp_{u_\sigma}(\xi_s + \tau X_s)$ in the loop space $\mathcal{L}M$. Now replace X by $\eta - \xi$.

Putting things together we have proved that due to compactness of the domain Z there exists a positive constant $\mu = \mu(Z, g)$ such that for every $s \in I$

$$\|\zeta'(s) - A\zeta(s)\| \le \mu \left(\|\zeta(s)\| + \|\nabla_t \zeta(s)\| \right).$$

Here the norm is in $L^2(S^1, {u_\sigma}^* TM)$. Now by integration by parts

$$\|\nabla_t \zeta\|^2 = \langle \nabla_t \zeta, \nabla_t \zeta \rangle = -\langle A\zeta, \zeta \rangle \le |\langle A\zeta, \zeta \rangle|.$$

Hence (99) is satisfied and this concludes the proof of theorem 6.3. $\square$

In the proof of the unstable manifold theorem 8.1 we use backward unique continuation for the nonlinear heat equation.

Theorem 6.4 (Forward and backward unique continuation)**.** *Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies* (V0)–(V1)*.*

(F) *Let u and v be smooth solutions of the heat equation (6) defined on the forward halfcylinder $[0, \infty) \times S^1$. If u and v agree along the loop at $s = 0$, then $u = v$.*

(B) *Let u and v be smooth solutions of the heat equation (6) defined on the backward halfcylinder $(-\infty, 0] \times S^1$. Assume further that*

$$\sup_{s \in (-\infty,0]} \mathcal{S}_{\mathcal{V}}\big(u(s,\cdot)\big) \le c_0, \qquad \sup_{s \in (-\infty,0]} \mathcal{S}_{\mathcal{V}}\big(v(s,\cdot)\big) \le c_0,$$

for some constant $c_0 > 0$. Then the following is true. If u and v agree along the loop at $s = 0$, then $u = v$.

Proof. The idea is the same as in the proof of theorem 6.3, namely to decompose the halfcylinder into small cylinders of width δ and then show $u = v$ on each piece (by the method developed in the first step of the proof of theorem 6.3). The only additional problem is noncompactness of the domain. One way to deal with this is to choose the same width for each piece (in order to arrive at any given time s in finitely many steps). Here we need uniform bounds for $|\partial_s u|$ and $|\partial_s v|$. Once we have these we can define δ again by (102). Check the proof of theorem 6.3 to see that the only further ingredients in proving $u = v$ on each small cylinder are uniform bounds for the first two t-derivatives of u and of v. Hence to complete the proof it remains to show that

$$\|\partial_s u\|_\infty + \|\partial_t u\|_\infty + \|\nabla_t \partial_t u\|_\infty + \|\partial_s v\|_\infty + \|\partial_t v\|_\infty + \|\nabla_t \partial_t v\|_\infty \le C$$

for some constant $C > 0$.

ad (F) Let C_0 be the constant in axiom (V0) and observe that $\mathcal{S}_{\mathcal{V}} \ge -C_0$. Now by theorem 4.9 with constant C_1, more precisely, by checking its proof

$$\begin{aligned} |\partial_s u(s,t)|^2 &\le C_1 E_{[s-1,s]}(u) \\ &= C_1 \left(\mathcal{S}_{\mathcal{V}}(u_{s-1}) - \mathcal{S}_{\mathcal{V}}(u_s)\right) \\ &\le C_1 \left(\mathcal{S}_{\mathcal{V}}(u_0) + C_0\right) \end{aligned}$$

for $(s,t) \in [1, \infty) \times S^1$. In the second and the last step we used that u is a negative gradient flow line and the action decreases along u. Note that the proof of theorem 4.9 shows that the estimate at a point depends on its past. This is why we get the above estimate only on $[1, \infty) \times S^1$. However, the missing part $[0,1] \times S^1$ is compact and u is smooth. Hence $\|\partial_s u\|_\infty \le C$ and

$$\|\nabla_t \partial_t u\|_\infty \le \|\partial_s u\|_\infty + \|\mathrm{grad}\mathcal{V}(u)\|_\infty \le C + C_0.$$

Here we used the heat equation (6) and axiom (V0) with constant C_0. It follows similarly by (checking the proof of) theorem 4.5 that $|\partial_t u(s,t)|$ is uniformly bounded on $[1, \infty) \times S^1$. The corresponding estimates for v are analoguous.

ad (B) The proof of the L^∞ estimates follows the same steps as in (F). We even get all estimates right away on the whole backward halfcylinder, because this halfcylinder contains the past of each of its points. □

7 Transversality

In section 7.1 we construct a separable Banach space Y of abstract perturbations satisfying axioms (V0)–(V3). In section 7.2 we fix a perturbation $\mathcal{V}$ such that (V0)–(V3) hold and $\mathcal{S}_\mathcal{V}$ is Morse. Then we choose a closed L^2 neighborhood U of the critical points of the function $\mathcal{S}_\mathcal{V}$ and – given a regular value a – we define a separable Banach manifold $\mathcal{O}^a = \mathcal{O}^a(\mathcal{V}, U)$ of admissible perturbations v. They have the property that their support lies in the sublevel set $\mathcal{L}^a M$ and is disjoint to U. Furthermore, the functions $\mathcal{S}_\mathcal{V}$ and $\mathcal{S}_{\mathcal{V}+v}$ do have the same critical points on the *whole* loop space $\mathcal{L}M$ and their sublevel sets with respect to a coincide. For such a triple $(\mathcal{V}, a, U)$ we prove in section 7.3 that there is a residual subset $\mathcal{O}^a_{reg} \subset \mathcal{O}^a$ of regular perturbations v. They have the property that $\mathcal{S}_{\mathcal{V}+v}$ is Morse–Smale below level a and this proves theorem 1.13. The crucial step is to prove proposition 7.5 on surjectivity of the universal section $\mathcal{F}$. Here unique continuation for the linear heat equation enters. A further key ingredient in the 'no return' part of the proof is the (negative) gradient flow property which implies that the functional is strictly decreasing along nonconstant heat flow solutions.

7.1 The universal Banach space of perturbations

We fix, once and for all, the following data.

- **a)** A dense sequence $\left(x_i\right)_{i\in\mathbb{N}}$ in $\mathcal{L}M = C^\infty(S^1, M)$.
- **b)** For every x_i a dense sequence $\left(\eta^{ij}\right)_{j\in\mathbb{N}}$ in $C^\infty(S^1, x_i^*TM)$.
- **c)** A smooth cutoff function $\rho : \mathbb{R} \to [0,1]$ such that $\rho = 1$ on $[-1,1]$ and $\rho = 0$ outside $[-4,4]$ and such that $\|\rho'\|_\infty < 1$. Then set $\rho_{1/k}(r) = \rho(rk^2)$ for $k \in \mathbb{N}$ (Figure 1).

Moreover, let $\iota > 0$ denote the injectivity radius of the closed Riemannian manifold M and fix a smooth cutoff function β such that $\beta = 1$ on $[-(\iota/2)^2, (\iota/2)^2]$ and $\beta = 1$ outside $[-\iota^2, \iota^2]$ (Figure 2).

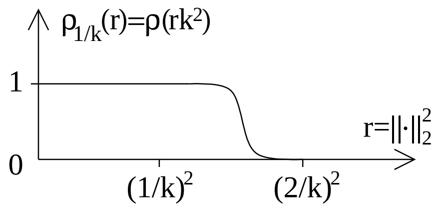

Figure 1: The cutoff function $\rho_{1/k}$

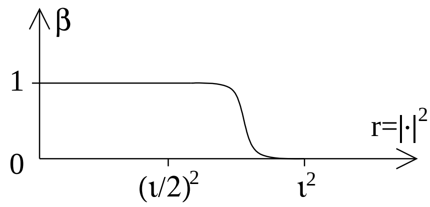

Figure 2: The cutoff function β

Then for any choice of $i, j, k \in \mathbb{N}$ there is a smooth function on the loop space given by

$$\mathcal{V}_\ell(x) = \mathcal{V}_{ijk}(x) = \rho_{1/k}\left(\|x - x_i\|^2_{L^2}\right)\int_0^1 V^{ij}(t, x(t))\, dt, \tag{104}$$

where V^{ij} is the smooth function on $S^1 \times M$ defined by

$$V^{ij}(t,q) := \begin{cases} \beta\big(|\xi^i_q(t)|^2\big)\left\langle \xi^i_q(t), \eta^{ij}(t)\right\rangle & , |\xi^i_q(t)| < \iota, \\ 0 & , \text{else.} \end{cases}$$

Here the vector $\xi^i_q(t)$ is determined by the identity

$$q = \exp_{x_i(t)} \xi^i_q(t)$$

whenever the Riemannian distance between q and $x_i(t)$ is less than ι. To simplify notation we fixed a bijection $\ell : \mathbb{N}^3 \to \mathbb{N}_0$. Observe that the support of $\mathcal{V}_{ijk}$ is contained in the L^2 ball of radius $2/k$ about x_i. Each function $\mathcal{V}_\ell : \mathcal{L}M \to \mathbb{R}$ is uniformly continuous with respect to the C^0 topology and satisfies (V0)–(V3). This follows by compactness of M, smoothness of the potential V, and by the identity

$$\begin{aligned} \langle \mathrm{grad}\mathcal{V}(u), \partial_s u\rangle_{L^2} &= \frac{d}{ds}\mathcal{V}(u) \\ &= 2\rho'\left(\|u - x_0\|^2_2\right)\left(\int_0^1 V_t(u(s,t))\, dt\right)\langle u - x_0, \partial_s u\rangle_{L^2} \\ &\quad + \rho\left(\|u - x_0\|^2_2\right)\langle \nabla V(u), \partial_s u\rangle_{L^2} \end{aligned}$$

which determines $\mathrm{grad}\mathcal{V}$. Here $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s,\cdot)$ is any smooth map.

Given $\mathcal{V}_\ell$, we fix a constant $C^0_\ell \geq 1$ which is greater than its constant of uniform continuity and for which (V0) holds true. Then we fix a constant $C^1_\ell \geq C^0_\ell$ for which both estimates in (V1) hold true and a constant $C^2_\ell \geq C^1_\ell$ to cover the three estimates of (V2). Furthermore, for every integer $i \geq 3$, we choose a constant $C^i_\ell \geq C^{i-1}_\ell$ that covers all estimates in (V3) with $k' + \ell' = i$ (here k' and ℓ' denote the integers k and ℓ that appear in (V3)). To summarize, for each integer $\ell \geq 0$ we have fixed a sequence of constants

$$1 \leq C^0_\ell \leq C^1_\ell \leq \ldots \leq C^\ell_\ell \leq \ldots \qquad \forall \ell \in \mathbb{N}_0. \tag{105}$$

The *universal space of perturbations* is the normed linear space

$$Y = \left\{ v_\lambda := \sum_{\ell=0}^\infty \lambda_\ell \mathcal{V}_\ell \;\middle|\; \lambda = (\lambda_\ell) \subset \mathbb{R} \text{ and } \|v_\lambda\| := \sum_{\ell=0}^\infty |\lambda_\ell| C^\ell_\ell < \infty \right\}. \tag{106}$$

Proposition 7.1. *The universal space Y of perturbations is a separable Banach space and every $v_\lambda \in Y$ satisfies the axioms* (V0)–(V3).

Proof. The map $v_\lambda \mapsto (\lambda_\ell C_\ell^\ell)_{\ell\in\mathbb{N}_0}$ provides an isomorphism from Y to the separable Banach space ℓ^1 of absolutely summable real sequences. This proves that Y is a separable Banach space. That every element $v_\lambda = \sum \lambda_\ell \mathcal{V}_\ell$ of Y satisfies (V0)–(V3) follows readily from the corresponding property of the generators $\mathcal{V}_\ell$. To explain the idea we give the proof of the second estimate in (V2), namely

$$\begin{aligned}
|\nabla_t\nabla_s \mathrm{grad} v_\lambda(u)| &\le \sum_{\ell=0}^\infty |\lambda_\ell| \cdot |\nabla_t\nabla_s \mathrm{grad}\mathcal{V}_\ell(u)| \\
&\le \left(|\lambda_0|\, C_0^2 + |\lambda_1|\, C_1^2 + \sum_{\ell=2}^\infty |\lambda_\ell|\, C_\ell^2 \right) f(u) \\
&\le \left(|\lambda_0|\, C_0^2 + |\lambda_1|\, C_1^2 + \|v_\lambda\| \right) f(u)
\end{aligned}$$

for every smooth map $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s,\cdot)$ and every $(s,t) \in \mathbb{R}\times S^1$. We abbreviated $f(u) = (|\nabla_t\partial_s u| + (1+|\partial_t u|)(|\partial_s u| + \|\partial_s u\|_{L^1}))$. Step two uses the second estimate in (V2) for each $\mathcal{V}_\ell$ with constant C_ℓ^2. Step three follows from $C_\ell^k \le C_\ell^\ell$ whenever $\ell \ge k$, see (105). The remaining estimates in (V0)–(V3) follow by the same argument. Continuity of v_λ with respect to the C^0 topology follows similarly using uniform continuity of the functions $\mathcal{V}_\ell$. □

7.2 Admissible perturbations

Throughout we fix a perturbation $\mathcal{V}$ that satisfies (V0)–(V3) and such that $\mathcal{S}_\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ is Morse. The idea to prove the transversality theorem 1.13 is to perturb $\mathcal{S}_\mathcal{V}$ outside some neigborhood U of its critical points in such a way that no new critical points arise. To achieve this we fix for every critical point x a closed L^2 neighborhood U_x such that $U_x \cap U_y = \emptyset$ whenever $x \ne y$. This is possible, because on any sublevel set there are only finitely many critical points ($\mathcal{S}_\mathcal{V}$ is Morse and satisfies the Palais-Smale condition; see e.g. [W02, app. A]). Set

$$U = U(\mathcal{V}) := \bigcup_{x\in\mathcal{P}(\mathcal{V})} U_x$$

and consider the Banach space of perturbations Y given by (106). We are interested in the subset of those perturbations supported away from U, namely

$$Y(\mathcal{V},U) := \left\{ v_\lambda = \sum_{\ell=0}^\infty \lambda_\ell \mathcal{V}_\ell \in Y \;\middle|\; \mathrm{supp}\mathcal{V}_\ell \cap U \ne \emptyset \;\Rightarrow\; \lambda_\ell = 0 \right\}.$$

Lemma 7.2. *$Y(\mathcal{V},U)$ is a closed subspace of the separable Banach space Y.*

Proof. Let $\alpha,\beta\in\mathbb{R}$ and let v_λ and $v_{\tilde\lambda}$ be elements of $Y(\mathcal{V},U)$. By definition of $Y(\mathcal{V},U)$ the following is true for every $\ell\in\mathbb{N}_0$. If $\mathrm{supp}\mathcal{V}_\ell \cap U \ne \emptyset$, then $\lambda_\ell = 0$ and $\tilde\lambda_\ell = 0$. Hence $\alpha\lambda_\ell + \beta\tilde\lambda_\ell = 0$ and therefore $\alpha v_\lambda + \beta v_{\tilde\lambda} \in Y(\mathcal{V},U)$. To see that the subspace $Y(\mathcal{V},U)$ is closed let $v_\lambda^i = \sum \lambda_\ell^i \mathcal{V}_\ell$ be a sequence in $Y(\mathcal{V},U)$ which converges to some element $v_\lambda = \sum \lambda_\ell \mathcal{V}_\ell$ of Y. This means that $\lambda_\ell^i \to \lambda_\ell$

as $i \to \infty$, for every ℓ. Now assume $\mathrm{supp}\mathcal{V}_\ell \cap U \neq \emptyset$. It follows that $\lambda_\ell^i = 0$, because $v_\lambda^i \in Y(\mathcal{V}, U)$, and this is true for all i. Hence the limit λ_ℓ is zero and therefore $v_\lambda \in Y(\mathcal{V}, U)$. □

Given a constant b, we denote by c_b the largest critical value of $\mathcal{S}_\mathcal{V}$ which is smaller or equal to b; that is

$$c_b = c_b(\mathcal{V}) := \max_{x \in \mathcal{P}^b(\mathcal{V})} \mathcal{S}_\mathcal{V}(x). \tag{107}$$

Now we consider those perturbations supported in $\{\mathcal{S}_\mathcal{V} < c_b\}$ but not in U, namely

$$Y^b(\mathcal{V}, U) := \left\{ \sum_{\ell=0}^\infty \lambda_\ell \mathcal{V}_\ell \in Y(\mathcal{V}, U) \;\Big|\; \mathrm{supp}\mathcal{V}_\ell \cap \{\mathcal{S}_\mathcal{V} \geq c_b\} \neq \emptyset \;\Rightarrow\; \lambda_\ell = 0 \right\}.$$

Lemma 7.3. *$Y^b(\mathcal{V}, U)$ is a closed subspace of the separable Banach space Y.*

Proof. Same arguments as in proof of lemma 7.2. □

Now fix a regular value a of $\mathcal{S}_\mathcal{V}$ and consider the positive constant given by

$$\kappa_a = \kappa_a(\mathcal{V}, U) := \inf_{x \in \mathcal{L}^a M \setminus U} \|\mathrm{grad}\mathcal{S}_\mathcal{V}(x)\|_2 > 0, \quad \mathcal{L}^a M = \{\mathcal{S}_\mathcal{V} \leq a\}.$$

To prove the strict inequality assume by contradiction that $\kappa_a = 0$. Then by Palais-Smale there exists a sequence $(x_k) \subset \mathcal{L}^a M \setminus U$ converging in the $W^{1,2}$ and therefore in the L^2 topology to a critical point $x \in \mathcal{L}^a M$. Hence $x \in U$ and U is a neighborhood of x, both by definition of U. But this contradicts the fact that $x_k \notin U$ for every $k \in \mathbb{N}$.
Our next step is to avoid creating new critical points outside U by admitting only perturbations supported in $\mathcal{L}^a M \setminus U$ with sufficiently small L^2 gradient. Simultaneously we achieve that the sublevel set $\{\mathcal{S}_\mathcal{V} \leq a\}$ will not change under these perturbations. More precisely, the set of *admissible perturbations* is given by the open ball of radius

$$r_a = r_a(\mathcal{V}, U) := \frac{1}{2} \min\{\kappa_a, a - c_a\} > 0$$

in the Banach space $Y^a(\mathcal{V}, U)$, namely

$$\mathcal{O}^a = \mathcal{O}^a(\mathcal{V}, U) := \{v_\lambda \in Y^a(\mathcal{V}, U) : \|v_\lambda\| < r_a\}. \tag{108}$$

This is a separable Banach manifold by lemma 7.3. The following lemma asserts that $\mathcal{S}_\mathcal{V}$ and $\mathcal{S}_{\mathcal{V}+v_\lambda}$ have the same sublevel sets with respect to a and the same critical points on the *whole* loop space, whenever $v_\lambda \in \mathcal{O}^a$. This proves the first part of theorem 1.13.

Lemma 7.4. *For $\mathcal{V}$, U, and a as above the following is true. If $v_\lambda \in \mathcal{O}^a$, then*

$$\mathcal{P}(\mathcal{V}) = \mathcal{P}(\mathcal{V} + v_\lambda), \qquad \{\mathcal{S}_\mathcal{V} \leq a\} = \{\mathcal{S}_{\mathcal{V}+v_\lambda} \leq a\}.$$

Proof. Fix a perturbation $v_\lambda \in \mathcal{O}^a$. To prove the first assertion of the lemma we show that on the set $U = U(\mathcal{V})$ the functionals $\mathcal{S}_\mathcal{V}$ and $\mathcal{S}_{\mathcal{V}+v_\lambda}$ coincide and that outside U they have no critical points at all. To see this observe that $\mathcal{S}_{\mathcal{V}+v_\lambda} = \mathcal{S}_\mathcal{V} - v_\lambda$ and that v_λ lies in $Y^a(\mathcal{V}, U)$. In particular, the support of v_λ is disjoint to U and therefore $\mathcal{S}_{\mathcal{V}+v_\lambda} = \mathcal{S}_\mathcal{V}$ on U. Now recall that by definition of U there are no critical points of $\mathcal{S}_\mathcal{V}$ outside U. Moreover, since the support of v_λ is contained in $\mathcal{L}^a M \setminus U$, it remains to prove that the perturbed functional $\mathcal{S}_{\mathcal{V}+v_\lambda}$ does not admit any critical point on $\mathcal{L}^a M \setminus U$. Assume by contradiction that it does admit a critical point $x \in \mathcal{L}^a M \setminus U$. Then

$$0 = \mathrm{grad}\mathcal{S}_{\mathcal{V}+v_\lambda}(x) = \mathrm{grad}\mathcal{S}_\mathcal{V}(x) - \mathrm{grad}v_\lambda(x).$$

Hence $\|\mathrm{grad}v_\lambda(x)\|_2 = \|\mathrm{grad}\mathcal{S}_\mathcal{V}(x)\|_2 \geq \kappa_a$ by definition of κ_a. On the other hand, since $v_\lambda = \sum \lambda_\ell \mathcal{V}_\ell$ it follows that

$$\begin{aligned}
\|\mathrm{grad}v_\lambda(x)\|_2 &\leq \sum_{\ell=0}^{\infty} |\lambda_\ell| \, \|\mathrm{grad}\mathcal{V}_\ell(x)\|_\infty \\
&\leq \sum_{\ell=0}^{\infty} |\lambda_\ell| \, C_\ell^0 \\
&\leq \|v_\lambda\| < \frac{\kappa_a}{2}.
\end{aligned}$$

Here we used $\|\cdot\|_2 \leq \|\cdot\|_\infty$, axiom (V0) for $\mathcal{V}_\ell$ and the fact that $C_\ell^0 \leq C_\ell^\ell$ by (105). The last line is by definition of the norm and $\mathcal{O}^a$.

We prove the second assertion of the lemma. First we prove the inclusion $\supset$. It is easy to see that $\mathcal{S}_{\mathcal{V}+v_\lambda}(x) \leq a$ implies $\mathcal{S}_\mathcal{V}(x) \leq a$. Assume by contradiction that $\mathcal{S}_\mathcal{V}(x) > a$, then $v_\lambda(x) = 0$, because $\mathrm{supp}\, v_\lambda \subset \mathcal{L}^a M$. Hence $\mathcal{S}_\mathcal{V}(x) = \mathcal{S}_\mathcal{V}(x) - v_\lambda(x) = \mathcal{S}_{\mathcal{V}+v_\lambda}(x) \leq a$.
To prove the inclusion $\subset$ assume that $\mathcal{S}_\mathcal{V}(x) \leq a$. Now there are two cases, namely $\mathcal{S}_\mathcal{V}(x) \geq c_a$ and $\mathcal{S}_\mathcal{V}(x) < c_a$. In the first case $v_\lambda(x) = 0$, because $\mathrm{supp}\, v_\lambda \subset \{\mathcal{S}_\mathcal{V} < c_a\}$, and therefore $\mathcal{S}_{\mathcal{V}+v_\lambda}(x) = \mathcal{S}_\mathcal{V}(x) - v_\lambda(x) = \mathcal{S}_\mathcal{V}(x) \leq a$. In the second case observe that

$$|v_\lambda(x)| \leq \sum_{\ell=0}^{\infty} |\lambda_\ell \mathcal{V}_\ell(x)| \leq \sum_{\ell=0}^{\infty} |\lambda_\ell| \, C_\ell^0 \leq \sum_{\ell=0}^{\infty} |\lambda_\ell| \, C_\ell^\ell = \|v_\lambda\| < a - c_a.$$

Here we used axiom (V0) for $\mathcal{V}_\ell$, the fact that $C_\ell^0 \leq C_\ell^\ell$ by (105), the definition of the norm in (106) and the assumption that $v_\lambda \in \mathcal{O}^a$. Hence $\mathcal{S}_{\mathcal{V}+v_\lambda}(x) = \mathcal{S}_\mathcal{V}(x) - v_\lambda(x) < c_a + |v_\lambda(x)| < a$. This concludes the proof of lemma 7.4. $\square$

7.3 Surjectivity

Proof of theorem 1.13. Assume that the perturbation $\mathcal{V}$ satisfies (V0)–(V3) and the function $\mathcal{S}_\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ is Morse. Fix a neighborhood U of the critical points of $\mathcal{S}_\mathcal{V}$ as in the previous section and a regular value a. For $\mathcal{O}^a = \mathcal{O}^a(\mathcal{V}, U)$ given by (108) the first part of theorem 1.13 is true by lemma 7.4. To prove the second part fix in addition two distinct critical points $x, y \in \mathcal{P}^a(\mathcal{V})$ and a constant $p > 2$. We denote by $\mathcal{B}^{1,p}_{x,y}$ the smooth Banach manifold of cylinders between x and y defined near (78) in section 5. This manifold is separable and admits a countable atlas. For $\mathcal{O}^a = \mathcal{O}^a(\mathcal{V}, U)$ given by (108) consider the smooth Banach space bundle

$$\mathcal{E}^p \to \mathcal{B}^{1,p}_{x,y} \times \mathcal{O}^a$$

whose fibre over (u, v_λ) are the L^p vector fields along u. The formula

$$\mathcal{F}(u, v_\lambda) = \partial_s u - \nabla_t \partial_t u - \operatorname{grad}\big(\mathcal{V} + v_\lambda\big)(u) \tag{109}$$

defines a smooth section of this bundle. Its zero set

$$\mathcal{Z} = \mathcal{Z}(x, y, a) = \mathcal{F}^{-1}(0)$$

is called the *universal moduli space*. It does not depend on $p > 2$, since all solutions of the heat equation (6) are smooth by theorem 1.5. We claim that zero is a regular value of $\mathcal{F}$. This means, by definition, that $d\mathcal{F}(u, v_\lambda)$ is onto and $\ker d\mathcal{F}(u, v_\lambda)$ admits a topological complement, whenever $\mathcal{F}(u, v_\lambda) = 0$. Surjectivity is the content of proposition 7.5 below and existence of a topological complement follows (see e.g. [W02, prop. 3.3]) from surjectivity and the fact that by theorem 1.9 and theorem 1.8 the operator $\mathcal{D}_u$ is Fredholm. Hence $\mathcal{Z}$ is a smooth Banach manifold by the implicit function theorem. Now consider the projection onto the second factor

$$\pi : \mathcal{Z} \to \mathcal{O}^a.$$

By standard Thom-Smale transversality theory (see e.g. [MS04, lemma A.3.6]) π is a smooth Fredholm map whose index is given by the Fredholm index of $\mathcal{D}_u$. This index is equal to the difference of the Morse indices of x and y again by theorem 1.9. Since $\mathcal{Z}$ is separable and admits a countable atlas, we can apply the Sard-Smale theorem [Sm73] to countably many coordinate representations of π. It follows that the set of regular values of π is residual in $\mathcal{O}^a$. We denote this set by $\mathcal{O}^a_{reg}(x, y) = \mathcal{O}^a_{reg}(x, y; \mathcal{V}, U)$ and observe that

$$\mathcal{O}^a_{reg}(x, y) = \{v_\lambda \in \mathcal{O}^a \mid \mathcal{D}_u \text{ onto } \forall u \in \mathcal{M}(x, y; \mathcal{V} + v_\lambda)\}$$

again by standard transversality theory; see e.g. [W02, prop. 3.4]. Then

$$\mathcal{O}^a_{reg} = \mathcal{O}^a_{reg}(\mathcal{V}, U) := \bigcap_{x,y \in \mathcal{P}^a(\mathcal{V})} \mathcal{O}^a_{reg}(x, y)$$

is a residual subset of $\mathcal{O}^a$, since it consists of a finite intersection of residual subsets. This proves theorem 1.13 up to proposition 7.5. □

Proposition 7.5 (Surjectivity)**.** *Let $\mathcal{V}$, U, a, x, y, and $p > 2$ be as in the proof of theorem 1.13 and consider the section $\mathcal{F}$ given by (109). Then the following is true. The linearization*

$$d\mathcal{F}(u, v_\lambda) : \mathcal{W}^{1,p}_u \times Y^a(\mathcal{V}, U) \to \mathcal{L}^p_u$$

is onto at every zero (u, v_λ) of $\mathcal{F}$.

Proof. Fix $q > 1$ such that $1/p + 1/q = 1$. By the regularity theorem 1.5 the map u is smooth and by theorem 1.8 on exponential decay all derivatives of $\partial_s u$ are bounded. Now the linearized operator is given by

$$\begin{aligned} d\mathcal{F}(u, v_\lambda)\,(\xi, \hat{\mathcal{V}}) &= d\mathcal{F}_{v_\lambda}(u)\,\xi + d\mathcal{F}_u(v_\lambda)\,\hat{\mathcal{V}} \\ &= \mathcal{D}_u \xi - \mathrm{grad}\hat{\mathcal{V}}(u) \end{aligned}$$

where $\mathcal{F}_{v_\lambda}(u) := \mathcal{F}(u, v_\lambda) =: \mathcal{F}_u(v_\lambda)$. By theorem 1.8 the Fredholm theorem 1.9 applies and shows that the operator $\mathcal{D}_u$ is Fredholm. Moreover, the second operator

$$Y^a(\mathcal{V}, U) \to \mathcal{L}_u^p : \hat{\mathcal{V}} \mapsto -\mathrm{grad}\hat{\mathcal{V}}(u)$$

is bounded (for each $\mathcal{V}_\ell$ use the last condition in (V0) with constant $C_\ell^0 \le C_\ell^\ell$). Hence the range of $d\mathcal{F}(u, v_\lambda)$ is closed by standard arguments; see e.g. [W02, proposition 3.3]. It remains to prove that it is dense. We use that density of the range is equivalent to *triviality of its annihilator*: By definition this means that, given $\eta \in \mathcal{L}_u^q$, then

$$\langle \eta, \mathcal{D}_u \xi \rangle = 0, \qquad \forall \xi \in \mathcal{W}_u^{1,p}, \tag{110}$$

and

$$\langle \eta, \mathrm{grad}\hat{\mathcal{V}}(u) \rangle = 0, \qquad \forall \hat{\mathcal{V}} \in Y^a(\mathcal{V}, U), \tag{111}$$

imply that $\eta = 0$.

Assume by contradiction that $\eta \in \mathcal{L}_u^q$ satisfies (110) and $\eta \neq 0$. In five steps we derive a contradiction to (111). Steps 1–3 are preparatory, in step 4 we construct a model perturbation $\mathcal{V}_\varepsilon$ violating (111) and in step 5 we approximate $\mathcal{V}_\varepsilon$ by the fundamental perturbations $\mathcal{V}_{ijk}$ of the form (104). To start with observe that η is smooth by (110) and theorem 3.1. Furthermore, integration by parts shows that $\mathcal{D}_u^* \eta = 0$ pointwise. Throughout we use the notation $\eta_s(t) = \eta(s,t)$, hence $\eta_s \in \Omega(S^1, u_s^* TM)$.

Step 1. (Unique Continuation) *$\eta_s \neq 0$ and $\partial_s u_s \neq 0$ for every $s \in \mathbb{R}$.*

Because η is smooth, nonzero, and $\mathcal{D}_u^* \eta = 0$, proposition 6.2 on unique continuation shows that $\eta_s \neq 0$ for every $s \in \mathbb{R}$. Next observe that $\partial_s u$ is smooth, because u is smooth, and that $0 = \frac{d}{ds}\mathcal{F}_{v_\lambda}(u) = \mathcal{D}_u \partial_s u$. Since u connects different critical points, the derivative $\partial_s u$ cannot vanish everywhere on $\mathbb{R} \times S^1$. Hence $\xi(s) := \partial_s u_s \neq 0$ for every $s \in \mathbb{R}$ by proposition 6.2. This proves step 1.

Step 2. (Slicewise Orthogonal) *$\langle \eta_s, \partial_s u_s \rangle = 0$ for every $s \in \mathbb{R}$.*

Straightforward calculation shows that

$$\begin{aligned} \frac{d}{ds}\langle \eta_s, \partial_s u_s \rangle &= \langle \nabla_s \eta_s, \partial_s u_s \rangle + \langle \eta_s, \nabla_s \partial_s u_s \rangle \\ &= \langle -\nabla_t \nabla_t \eta_s - R(\eta_s, \partial_t u_s)\partial_t u_s - \mathcal{H}_{\mathcal{V}+v_\lambda}(u_s)\eta_s, \partial_s u_s \rangle \\ &\quad + \langle \eta_s, \nabla_t \nabla_t \partial_s u_s - R(\partial_s u_s, \partial_t u_s)\partial_t u_s - \mathcal{H}_{\mathcal{V}+v_\lambda}(u_s)\partial_s u_s \rangle \\ &= 0. \end{aligned}$$

In the second equality we replaced $\nabla_s \eta_s$ according to the identity $\mathcal{D}_u^* \eta = 0$ and (58) and $\nabla_s \partial_s u_s$ according to $\mathcal{D}_u \partial_s u = 0$ and (57). The last step is by integration by parts, symmetry of the Hessian $\mathcal{H}$, and the first Bianchi identity for the curvature operator R. It follows that $\langle \eta_s, \partial_s u_s \rangle$ is constant in s. Now this constant, say c, must be zero since

$$\int_{-\infty}^{\infty} c \, ds = \int_{-\infty}^{\infty} \langle \eta_s, \partial_s u_s \rangle \, ds = \langle \eta, \partial_s u \rangle$$

and the inner product on the right hand side is finite, because $\eta \in \mathcal{L}_u^q$ and $\partial_s u \in \mathcal{L}_u^p$ where $1/p + 1/q = 1$. This proves step 2.

Observe that η_s and $\partial_s u_s$ are linearly independent for every $s \in \mathbb{R}$ as a consequence of step 1 and step 2.

Step 3. (No Return) *Assume the loop u_{s_0} is different from the asymptotic limits x and y and let $\delta > 0$. Then there exists $\varepsilon > 0$ such that for every $s \in \mathbb{R}$*

$$\left\| u_s - u_{s_0} \right\|_2 < 3\varepsilon \quad \Longrightarrow \quad s \in (s_0 - \delta, s_0 + \delta).$$

In words, once s leaves a given δ-interval about s_0 the loops u_s cannot return to some L^2 ε-neighborhood of u_{s_0}.

Key ingredients in the proof are smoothness of u, existence of asymptotic limits, and the gradient flow property. Recall the footnote in remark 1.3 concerning the difference of loops $u_s - u_{s_0}$. Now assume by contradiction that there is a sequence of positive reals $\varepsilon_i \to 0$ and a sequence of reals s_i which satisfy $\|u_{s_i} - u_{s_0}\|_2 < 3\varepsilon_i$ and $s_i \notin (s_0 - \delta, s_0 + \delta)$. In particular, it follows that

$$u_{s_i} \xrightarrow{L^2} u_{s_0} \quad \text{as } i \to \infty. \tag{112}$$

Assume first that the sequence s_i is unbounded. Hence we can choose a subsequence, without changing notation, such that s_i converges to $+\infty$ or $-\infty$. In either case u_{s_i} converges to one of the critical points x or y and the convergence is in $C^0(S^1)$ by theorem 1.8. By (112) and uniqueness of limits it follows that u_{s_0} equals one of the critical points x, y, but this contradicts our assumption.

Assume now that the sequence s_i is bounded. Then we can choose a subsequence, without changing notation, such that s_i converges to some element $s_1 \notin (s_0 - \delta, s_0 + \delta)$. Since u is smooth, it follows that u_{s_i} converges to u_{s_1} in $C^0(S^1)$. Again by uniqueness of limits $u_{s_1} = u_{s_0}$. On the other hand, the action functional is strictly decreasing along nonconstant negative gradient flow lines. Therefore $s_1 = s_0$ and this contradiction concludes the proof of step 3.

Step 4. *There exists a time s_0 such that $\mathcal{S}_{\mathcal{V}}(u_{s_0}) < c_a$, where c_a is the largest critical value below a. Furthermore there exist a positive constant ε and a smooth function $\mathcal{V}_0 : \mathcal{L}M \to \mathbb{R}$ supported in the L^2 ball of radius 2ε about u_{s_0} such that*

$$\mathcal{V}_0(u_{s_0}) = 0, \qquad d\mathcal{V}_0(u_{s_0})\eta_{s_0} = \left\|\eta_{s_0}\right\|_2^2, \qquad \langle \mathrm{grad}\mathcal{V}_0(u), \eta \rangle \neq 0.$$

Recall that the asymptotic limits x and y are different and the closed L^2 neighborhoods U_x and U_y were chosen in the first paragraph of section 7 to be disjoint. Moreover, both x and y are not above level c_a and $\mathcal{S}_{\mathcal{V}}(u_s)$ is strictly decreasing in s. Therefore there exists a time s_0 such that u_{s_0} lies not in U and strictly below level c_a.

Observe that the graph $t \mapsto (t, u_{s_0}(t))$ of the loop u_{s_0} is embedded in $S^1 \times M$. Now we define a smooth function V on $S^1 \times M$ supported near the graph as follows. Denote by $\iota > 0$ the injectivity radius of the closed Riemannian manifold M. Pick a smooth cutoff function $\beta : \mathbb{R} \to [0,1]$ such that $\beta = 1$ on $[-(\iota/2)^2, (\iota/2)^2]$ and $\beta = 0$ outside $[-\iota^2, \iota^2]$; see Figure 2. Then define

$$V_t(q) := V(t,q) := \begin{cases} \beta\big(|\xi_q(t)|^2\big) \left\langle \xi_q(t), \eta_{s_0}(t) \right\rangle & , |\xi_q(t)| < \iota, \\ 0 & , \text{else}, \end{cases} \tag{113}$$

where the vector $\xi_q(t)$ is determined by the identity

$$q = \exp_{u_{s_0}(t)} \xi_q(t)$$

whenever the Riemannian distance between q and $u_{s_0}(t)$ is less than ι. Note that the function V vanishes on the graph of the loop u_{s_0}.

Since all maps involved are smooth, we can choose a constant $\delta > 0$ sufficiently small such that for every $s \in (s_0 - \delta, s_0 + \delta)$ it holds

i) $d_{C^0}(u_s, u_{s_0}) = \|\xi_s\|_\infty < \frac{1}{2}\iota$, where ξ_s is uniquely determined by the identity $u_s = \exp_{u_{s_0}} \xi_s$ pointwise for every $t \in S^1$,

ii) $\langle E_2(u_{s_0}, \xi_s)^{-1}\eta_s, \eta_{s_0}\rangle \geq \frac{1}{2}\mu_0$, where $\mu_0 := \|\eta_{s_0}\|_2^2 > 0$,

iii) $\frac{1}{2}\mu_1 \leq \frac{\|u_s - u_{s_0}\|_2}{|s-s_0|} \leq \frac{3}{2}\mu_1$, where $\mu_1 := \|\partial_s u_{s_0}\|_2 > 0$.

Let $s \in (s_0 - \delta, s_0 + \delta)$, then

$$\begin{aligned} dV_t(u_s)\,\eta_s &= \tfrac{d}{dr}\big|_{r=0} V_t(\exp_{u_s} r\eta_s) \\ &= 2\beta'(|\xi_s|^2) \left\langle \xi_s, E_2(u_{s_0}, \xi_s)^{-1}\eta_s \right\rangle \cdot \left\langle \xi_s, \eta_{s_0} \right\rangle \\ &\quad + \beta(|\xi_s|^2) \left\langle E_2(u_{s_0}, \xi_s)^{-1}\eta_s, \eta_{s_0} \right\rangle \\ &= \left\langle E_2(u_{s_0}, \xi_s)^{-1}\eta_s, \eta_{s_0} \right\rangle \end{aligned} \tag{114}$$

pointwise for every $t \in S^1$. The final step uses i) and the definition of β. Note that $dV_t(u_{s_0})\,\eta_{s_0} = |\eta_{s_0}|^2$ pointwise.

Integrating V along a loop defines a smooth function on the loop space which vanishes on u_{s_0}. Next we cut it off with respect to the L^2 distance. Fix a smooth cutoff function $\rho : \mathbb{R} \to [0,1]$ such that $\rho = 1$ on $[-1,1]$, $\rho = 0$ outside $[-4,4]$, and $\|\rho'\|_\infty < 1$. Then, for the constant δ fixed above, choose $\varepsilon > 0$ according to step 3 (No Return) and set $\rho_\varepsilon(r) = \rho(r/\varepsilon^2)$; see Figure 1 with $\varepsilon = 1/k$. Note that $\|\rho_\varepsilon'\|_\infty < \varepsilon^{-2}$. Observe that we can choose $\varepsilon > 0$ smaller and the assertion of step 3 remains true. Now define a smooth function on $\mathcal{L}M$ by

$$\mathcal{V}_0(x) := \rho_\varepsilon\left(\|x - u_{s_0}\|_2^2\right) \int_0^1 V(t, x(t))\, dt$$

where V is given by (113). The function $\mathcal{V}_0$ vanishes on the loop u_{s_0} and satisfies

$$\begin{aligned} d\mathcal{V}_0(u_s)\,\eta_s &= \tfrac{d}{dr}\big|_{r=0}\mathcal{V}_0(\exp_{u_s} r\eta_s) \\ &= 2\rho_\varepsilon'\big(\|u_s - u_{s_0}\|_2^2\big)\ \langle u_s - u_{s_0}, \eta_s\rangle \int_0^1 V_t(u_s(t))\, dt \\ &\quad + \rho_\varepsilon\big(\|u_s - u_{s_0}\|_2^2\big) \int_0^1 dV_t(u_s(t))\,\eta_s(t)\, dt. \end{aligned}$$

Hence $d\mathcal{V}_0(u_{s_0})\eta_{s_0} = \|\eta_{s_0}\|_2^2$.

To prove the final assertion of step 4 observe that $s \notin (s_0 - \delta, s_0 + \delta)$ implies $\|u_s - u_{s_0}\|_2 \geq 3\varepsilon$, by step 3, and therefore $u_s \notin \operatorname{supp} \mathcal{V}_0$. It follows that

$$\begin{aligned} \langle \operatorname{grad} \mathcal{V}_0(u), \eta\rangle &= \int_{s_0-\delta}^{s_0+\delta} d\mathcal{V}_0(u_s)\eta_s\, ds \\ &= \int_{s_0-\delta}^{s_0+\delta} 2\rho_\varepsilon'\big(\|u_s - u_{s_0}\|_2^2\big)\langle u_s - u_{s_0}, \eta_s\rangle\langle \xi_s, \eta_{s_0}\rangle\, ds \\ &\quad + \int_{s_0-\delta}^{s_0+\delta} \rho_\varepsilon\big(\|u_s - u_{s_0}\|_2^2\big)\langle E_2(u_{s_0}, \xi_s)^{-1}\eta_s, \eta_{s_0}\rangle\, ds. \end{aligned} \tag{115}$$

We shall estimate the two terms in the sum separately. Let $s_2 > s_0$ be such that $\|u_{s_2} - u_{s_0}\|_2 = \varepsilon$ and $\|u_s - u_{s_0}\|_2 < \varepsilon$ whenever $s \in (s_0, s_2)$. In other words s_2 is the *forward* exit time of u_s with respect to the L^2 ball of radius ε about u_{s_0}. Let $s_1 < s_0$ be the corresponding *backward* exit time; see Figure 3. Then, by ii) and $\rho_\varepsilon \geq 0$, it holds that

$$\begin{aligned} &\int_{s_0-\delta}^{s_0+\delta} \rho_\varepsilon\big(\|u_s - u_{s_0}\|_2^2\big)\langle E_2(u_{s_0}, \xi_s)^{-1}\eta_s, \eta_{s_0}\rangle\, ds \\ &\geq \int_{s_1}^{s_2} 1 \cdot \frac{\mu_0}{2}\, ds = \frac{\mu_0}{2}\,(s_2 - s_0 + s_0 - s_1) \\ &\geq \frac{\mu_0}{3\mu_1}\,(\|u_{s_2} - u_{s_0}\|_2 + \|u_{s_0} - u_{s_1}\|_2) = \frac{2\mu_0}{3\mu_1}\,\varepsilon. \end{aligned}$$

Here the second inequality uses iii). To estimate the other term in (115) let σ_1 be the time of first entry into the L^2 ball of radius 2ε starting from $s_0 - \delta$ and let σ_2 be the corresponding time when time runs backwards and we start from $s_0 + \delta$; see Figure 3. Then

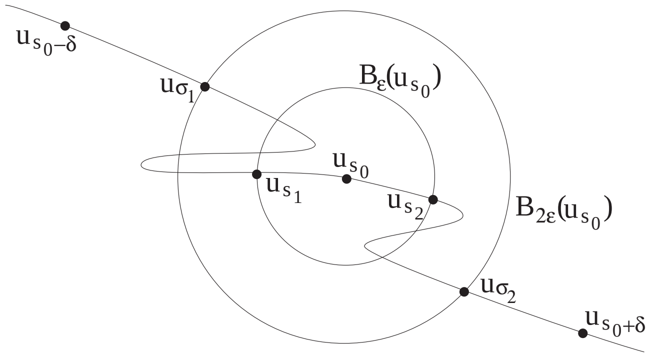

Figure 3: Exit times s_1, s_2 and entry times σ_1, σ_2

$$
\begin{aligned}
&\int_{s_0-\delta}^{s_0+\delta} 2\rho_\varepsilon'\big(\|u_s - u_{s_0}\|_2^2\big)\langle u_s - u_{s_0}, \eta_s\rangle\langle \xi_s, \eta_{s_0}\rangle\, ds\\
&\geq -2\int_{\sigma_1}^{\sigma_2} \|\rho_\varepsilon'\|_\infty \, |\langle u_s - u_{s_0}, \eta_s\rangle| \cdot |\langle \xi_s, \eta_{s_0}\rangle|\, ds\\
&\geq -2c_1c_2\varepsilon^{-2}\int_{\sigma_1}^{\sigma_2} (s-s_0)^4\, ds\\
&= -\frac{2c_1c_2}{5\varepsilon^2}\,(\sigma_2 - s_0 + s_0 - \sigma_1)^5 \geq -\frac{2c_1c_2 8^5}{5\mu_1^5}\varepsilon^3.
\end{aligned}
$$

It remains to explain the second and the final inequality. In the final one we use that by iii) there is the estimate $\sigma_2 - s_0 \leq 2\|u_{\sigma_2} - u_{s_0}\|_2/\mu_1 = 4\varepsilon/\mu_1$ and similarly for $s_0 - \sigma_1$. The second inequality is based on the geometric fact that $\partial_s u$ and η are slicewise orthogonal by step 2: Let $f(s) = \langle u_s - u_{s_0}, \eta_s\rangle$ and $h(s) = \langle \xi_s, \eta_{s_0}\rangle$, then $f(s_0) = h(s_0) = 0$ and

$$
\begin{aligned}
f'(s) &= \langle \partial_s u_s, \eta_s\rangle + \langle u_s - u_{s_0}, \nabla_s \eta_s\rangle = \langle u_s - u_{s_0}, \nabla_s\eta_s\rangle\\
h'(s) &= \langle E_2(u_{s_0}, \xi_s)^{-1}\partial_s u_s, \eta_{s_0}\rangle.
\end{aligned}
$$

Hence $f'(s_0) = h'(s_0) = 0$ and there exist constants $c_1 = c_1(f) > 0$ and $c_2 = c_2(h) > 0$ depending continuously on δ such that for every $s \in (s_0-\delta, s_0+\delta)$

$$
|f(s)| \leq c_1(s-s_0)^2, \qquad |h(s)| \leq c_2(s-s_0)^2.
$$

This proves the second inequality. Now choose $\varepsilon > 0$ sufficiently small such that $\varepsilon^2 < \mu_0\mu_1^4/c_1c_2$. This implies that $\langle \operatorname{grad}\mathcal{V}_0(u), \eta\rangle > 0$. Choosing ε again smaller we may assume without loss of generality that the L^2 ball of radius 3ε about u_{s_0} is disjoint from U and contained in $\{\mathcal{S}_\mathcal{V} < c_a\}$, that 3ε is smaller than the injectivity radius ι of M, and that $\varepsilon = 1/k$ for some integer k. This proves step 4.

Step 5. *Given k as in the line above, there exist positive integers i and j such that the function $\hat{\mathcal{V}} := \mathcal{V}_{ijk}$ given by (104) is element of $Y^a(\mathcal{V}, U)$ and satisfies*

$$
\langle \operatorname{grad}\mathcal{V}_{ijk}(u), \eta\rangle > 0.
$$

This contradicts (111) and thereby proves proposition 7.5.

Denote $\varepsilon = 1/k$ and let s_0 be the time determined in step 4. In section 7.1 we have fixed a dense sequence (x_i) in $C^\infty(S^1, M)$ and for each i a dense sequence (η^{ij}) in $C^\infty(S^1, x_i^*TM)$. Choose a subsequence, still denoted by (x_i), such that

$$x_i \to u_{s_0} \qquad \text{as } i \to \infty.$$

Now we may assume without loss of generality that every x_i lies in $B_\varepsilon(u_{s_0})$ the L^2 ball of radius ε about u_{s_0}. Hence $B_{2\varepsilon}(x_i) \subset B_{3\varepsilon}(u_{s_0})$. Let $\xi^i_{s_0}$ be defined by the identity $u_{s_0} = \exp_{x_i} \xi^i_{s_0}$ pointwise for every $t \in S^1$. Choose a diagonal subsequence, denoted by (η^{ii}), such that

$$\Phi_{x_i}(\xi^i_{s_0})\eta^{ii} \to \eta_{s_0} \qquad \text{as } i \to \infty.$$

Here $\Phi_x(\xi)$ is parallel transport from x to $\exp_x \xi$ along $\tau \mapsto \exp_x \tau\xi$ pointwise for every $t \in S^1$. Let $(\mathcal{V}_{iik})_{i \in \mathbb{N}}$ be the corresponding sequence of functions where each $\mathcal{V}_{iik}$ is given by (104). The sequence is contained in $Y^a(\mathcal{V}, U)$, because

$$\operatorname{supp} \mathcal{V}_{iik} \subset B_{2/k}(x_i) = B_{2\varepsilon}(x_i) \subset B_{3\varepsilon}(u_{s_0}) \subset \{\mathcal{S}_\mathcal{V} < c_a\} \tag{116}$$

and $B_{3\varepsilon}(u_{s_0}) \cap U = \emptyset$. This uses our choice of ε right before step 5.

Now recall that the constant $\delta > 0$ has been chosen in the proof of step 4 in order to exclude any return of the trajectory $s \mapsto u_s$ to the ball $B_{3\varepsilon}(u_{s_0})$ once s has left the interval $(s_0 - \delta, s_0 + \delta)$. Together with (116) this shows that $\mathcal{V}_{iik}(u_s) = 0$, whenever $s \notin (s_0 - \delta, s_0 + \delta)$. Therefore

$$\begin{aligned}\langle \operatorname{grad} \mathcal{V}_{iik}(u), \eta \rangle &= \int_{s_0-\delta}^{s_0+\delta} 2\rho'_{1/k}\big(\|u_s - x_i\|_2^2\big) \langle u_s - x_i, \eta_s \rangle \langle \xi^i_s, \eta^{ii} \rangle \, ds \\ &\quad + \int_{s_0-\delta}^{s_0+\delta} \rho_{1/k}\big(\|u_s - x_i\|_2^2\big) \langle E_2(x_i, \xi^i_s)^{-1} \eta_s, \eta^{ii} \rangle \, ds\end{aligned}$$

where ξ^i_s is determined by $u_s = \exp_{x_i} \xi^i_s$. Now the right hand side converges, as $i \to \infty$, to the right hand side of (115), which equals $\langle \operatorname{grad} \mathcal{V}_0(u), \eta \rangle > 0$. This proves step 5 and proposition 7.5. $\square$

8 Heat flow homology

In section 8.1 we define the unstable manifold of a critical point x of the action functional $\mathcal{S}_\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ as the set of endpoints at time zero of all backward halfcylinders solving the heat equation (6) and emanating from x at $-\infty$. The main result is theorem 8.1 saying that if x is nondegenerate, then this is a submanifold of the loop space and its dimension equals the Morse index of x.

Section 8.2 puts together the results proved so far to construct the Morse complex for the negative L^2 gradient of the action functional on the loop space.

8.1 The unstable manifold theorem

Fix a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ that satisfies (V0)–(V3) and let Z^- be the **backward halfcylinder** $(-\infty, 0] \times S^1$. Given a critical point x of the action functional $\mathcal{S}_\mathcal{V}$ the moduli space

$$\mathcal{M}^-(x; \mathcal{V}) \tag{117}$$

is, by definition, the set of all smooth solutions $u^- : Z^- \to M$ of the heat equation (6) such that $u^-(s,t) \to x(t)$ as $s \to -\infty$, uniformly in $t \in S^1$. Note that the moduli space is not empty, since it contains the stationary solution $u^-(s,t) = x(t)$. The **unstable manifold of x** is defined by

$$W^u(x; \mathcal{V}) = \{u^-(0, \cdot) \mid u^- \in \mathcal{M}^-(x; \mathcal{V})\}.$$

Theorem 8.1. *Let $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ be a perturbation that satisfies* (V0)–(V3)*. If x is a nondegenerate critical point of the action functional $\mathcal{S}_\mathcal{V}$, then the unstable manifold $W^u(x; \mathcal{V})$ is a smooth contractible embedded submanifold of the loop space and its dimension is equal to the Morse index of x.*

The idea to prove theorem 8.1 is to first show in proposition 8.2 that nondegeneracy of x implies that the moduli space $\mathcal{M}^-(x; \mathcal{V})$ is a smooth manifold of the desired dimension. A crucial ingredient is proposition 8.3 on surjectivity of the operator $\mathcal{D}_{u^-} : \mathcal{W}^{1,p} \to \mathcal{L}^p$ whenever $u^- \in \mathcal{M}^-(x; \mathcal{V})$ and $p \geq 2$. Here the operator $\mathcal{D}_{u^-}$ given by (57) arises by linearizing the heat equation at the backward trajectory u^-. A further key result to prove theorem 8.1 is unique continuation for the linear and the nonlinear heat equation, proposition 6.2 and theorem 6.4. Namely, unique continuation implies that the evaluation map

$$ev_0 : \mathcal{M}^-(x; \mathcal{V}) \to \mathcal{L}M, \qquad u^- \mapsto u^-(0, \cdot)$$

is an injective immersion. It is even an embedding by the gradient flow property.

Proposition 8.2 (Moduli space)**.** *Let $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ be a perturbation satisfying* (V0)–(V3) *and suppose that x is a nondegenerate critical point of $\mathcal{S}_\mathcal{V}$. Then the moduli space $\mathcal{M}^-(x; \mathcal{V})$ is a smooth contractible manifold of dimension* $\mathrm{ind}_\mathcal{V}(x)$*. Its tangent space at u^- is equal to the vector space X^- given by (118).*

Proposition 8.3 (Surjectivity). *Fix a constant $p > 2$, a perturbation $\mathcal{V}$ that satisfies* (V0)–(V3)*, and a nondegenerate critical point x of $\mathcal{S}_\mathcal{V}$. If $u^- \in \mathcal{M}^-(x;\mathcal{V})$, then the operator $\mathcal{D}_{u^-} : \mathcal{W}^{1,p} \to \mathcal{L}^p$ is onto and its kernel is given by*

$$\begin{aligned} X^- := \Bigl\{\xi \in C^\infty(Z^-, {u^-}^*TM) \mid\ & \mathcal{D}_{u^-}\xi = 0,\ \exists c, \delta > 0\ \forall s \le 0 : \\ & \|\xi_s\|_\infty + \|\nabla_t \xi_s\|_\infty + \|\nabla_t\nabla_t\xi_s\|_\infty + \|\nabla_s\xi_s\|_\infty \le ce^{\delta s}\Bigr\}. \end{aligned} \tag{118}$$

Moreover, the dimension of X^- is equal to the Morse index of x.

Proposition 8.3 is in fact a corollary of theorem 8.5 below which asserts surjectivity in the special case of a stationary solution $u^-(s,t) = x(t)$, where x is a nondegenerate critical point of $\mathcal{S}_\mathcal{V}$. The idea is that if a solution u^- is nearby the stationary solution x in the $\mathcal{W}^{1,p}$ topology, then the corresponding linearizations $\mathcal{D}_{u^-}$ and $\mathcal{D}_x$ are close in the operator norm topology. But surjectivity is an open condition with respect to the norm topology. The case of a general solution reduces to the nearby case by shifting the s-variable.

Remark 8.4. Abbreviate $H = L^2(S^1,\mathbb{R}^n)$ and $W = W^{2,2}(S^1,\mathbb{R}^n)$ and consider the operator

$$A_S = -\frac{d^2}{dt^2} - S : H \to H$$

with dense domain W. Here we assume that $S : W \to H$ is a symmetric and compact linear operator. Under these assumptions it is well known (see (ii) in section 3.4) that A_S is self-adjoint and that its Morse index $\mathrm{ind}(A_S)$, that is the dimension of the negative eigenspace E^- of A_S, is finite.

Theorem 8.5. *Let S and A_S be as in remark 8.4. Fix $p \ge 2$ and assume that the linear operator $S : W^{1,p}(S^1,\mathbb{R}^n) \to L^p(S^1,\mathbb{R}^n)$ is bounded with bound c_S. Then the following is true. If A_S is injective, then the operator*

$$D = \partial_s - \partial_t\partial_t - S : \mathcal{W}^{1,p}(Z^-,\mathbb{R}^n) \to L^p(Z^-,\mathbb{R}^n)$$

is onto. In the case $p = 2$ the map $E^- \to \ker D$, $v \mapsto e^{-sA_S}v$ is an isomorpism.

Proof of theorem 8.5. The proof takes four steps. Step 1 proves the theorem for $p = 2$. The proof by Salamon [S99, lemma 2.4 step 1] of the corresponding result in Floer theory carries over with minor but important modifications. These are due to the fact that our domain Z^- does have a boundary. The proof uses the theory of semigroups. We recall the details for convenience of the reader. The generalization of surjectivity in step 4 to $p > 2$ follows an argument due to Donaldson [Do02]. It uses the case $p = 2$ and the estimates provided by step 2 and step 3. Again we follow the presentation in [S99, lemma 2.4 steps 2–4] up to minor but subtle modifications. One subtlety is related to the parabolic estimate of step 2. Here in contrast to the elliptic case the domain needs to be increased only towards the *past.* Hence the estimates of step 3 work precisely for the *backward* halfcylinder. Throughout the proof, unless indicated differently, the domain of all spaces is the backward halfcylinder Z^- and the target is $\mathbb{R}^n$.

Step 1. *The theorem is true for $p = 2$.*

The operator A_S is unbounded and self-adjoint on the Hilbert space H with dense domain W. Denote the negative and positive eigenspaces of A_S by E^- and E^+, respectively. Note that $\dim E^- < \infty$ by remark 8.4. By assumption A_S is injective, hence zero is not an eigenvalue and there is a splitting $H = E^- \oplus E^+$. Denote by $P^\pm : H \to E^\pm$ the orthogonal projections and set $A^\pm = A_S|_{E^\pm}$. The self-adjoint negative semidefinite operators A^- and $-A^+$ generate contraction semigroups on E^- and E^+, respectively, by the Hille-Yosida theorem; see e.g. [ReS75, sec. X.8 ex. 1]. We denote them by $s \mapsto e^{sA^-}$ and $s \mapsto e^{-sA^+}$, respectively. Both are defined for $s \geq 0$. Define the map $K : \mathbb{R} \to \mathcal{L}(H)$ by

$$K(s) = \begin{cases} -e^{-sA^-}P^-, & \text{for } s \leq 0, \\ e^{-sA^+}P^+, & \text{for } s > 0. \end{cases}$$

This function is strongly continuous for $s \neq 0$ and satisfies

$$\|K(s)\|_{\mathcal{L}(H)} \leq e^{-\delta|s|} \tag{119}$$

where $\delta = \min\{-\lambda^-, \lambda^+\} > 0$. Here λ^- denotes the largest eigenvalue of A^- and λ^+ the smallest eigenvalue of A^+. Abbreviate $\mathbb{R}^- = (-\infty, 0]$. For $\eta \in L^2(\mathbb{R}^-, H)$ consider the operator

$$(Q\eta)(s) := \int_{-\infty}^0 K(s-\sigma)\eta(\sigma)\, d\sigma.$$

Now the operator Q maps $L^2(\mathbb{R}^-, H)$ to the intersection of Banach spaces $W^{1,2}(\mathbb{R}^-, H) \cap L^2(\mathbb{R}^-, W)$ and it is a right inverse of D. To prove the latter set $\xi := Q\eta$. Then $\xi = \xi^- + \xi^+$, where

$$\xi^+(s) = \int_{-\infty}^s e^{-(s-\sigma)A^+}P^+\eta(\sigma)\, d\sigma, \qquad \xi^-(s) = -\int_s^0 e^{-(s-\sigma)A^-}P^-\eta(\sigma)\, d\sigma.$$

Calculation shows that $D\xi^\pm = P^\pm\eta$ pointwise for every $s \in \mathbb{R}^-$. It follows that

$$DQ\eta = D\xi = D\xi^- + D\xi^+ = P^-\eta + P^+\eta = \eta.$$

Since the space $W^{1,2}(\mathbb{R}^-, H) \cap L^2(\mathbb{R}^-, W)$ agrees with $\mathcal{W}^{1,2}$, this proves that Q is a right inverse of D. Hence Q is injective and D is onto. To calculate the kernel of D fix $\xi \in \mathcal{W}^{1,2}$ and set $\eta := D\xi$. Then by straightforward calculation

$$\begin{aligned}
(QD\xi)(s) &= (Q\eta)(s) = \xi^+(s) + \xi^-(s) \\
&= \int_{-\infty}^s \frac{d}{d\sigma}\left(e^{-(s-\sigma)A^+}P^+\xi(\sigma)\right) d\sigma - \int_s^0 \frac{d}{d\sigma}\left(e^{-(s-\sigma)A^-}P^-\xi(\sigma)\right) d\sigma \\
&= P^+\xi(s) - e^{-sA^-}P^-\xi(0) + P^-\xi(s) \\
&= \xi(s) - e^{-sA^-}P^-\xi(0).
\end{aligned}$$

To obtain the third identity replace $\eta(\sigma)$ in $\xi^{\pm}(s)$ by $\xi'(\sigma)+A_S\xi(\sigma)$ and use the fact that $A^{\pm}P^{\pm} = P^{\pm}A_S$. Now observe that $\xi \in \ker D$ is equivalent to $D\xi \in \ker Q$, because Q is injective. But $QD\xi = 0$ means that $\xi(s) = e^{-sA^-}P^-\xi(0)$ for every $s \in \mathbb{R}^-$. This shows that the map

$$E^- \to \ker\left[D : \mathcal{W}^{1,2} \to L^2\right] : v_k \mapsto e^{-s\lambda_k} v_k \tag{120}$$

induces an isomorphism. Here $v_1, \ldots, v_N$ is an orthonormal basis of E^- consisting of eigenvectors of A_S with eigenvalues $\lambda_1, \ldots, \lambda_N$ and where $N = \mathrm{ind}(A_S)$.

Step 2. *Fix a constant $p \geq 2$. Then there is a constant $c_1 = c_1(p, c_S)$ such that*

$$\|\xi\|_{\mathcal{W}^{1,p}([-1,0]\times S^1)} \leq c_1 \left(\|D\xi\|_{L^p([-3,0]\times S^1)} + \|\xi\|_{L^2([-3,0]\times S^1)}\right)$$

for $\xi \in C^\infty([-3,0]\times S^1)$. Moreover, if $\xi \in \mathcal{W}^{1,2}$ and $D\xi \in L^p_{loc}$, then $\xi \in \mathcal{W}^{1,p}_{loc}$.

Choose a smooth compactly supported cutoff function $\rho : (-2, 0] \to [0, 1]$ such that $\rho = 1$ on $[-1, 0]$ and $\|\partial_s\rho\|_\infty \leq 2$. Now apply proposition 2.13 for the backward halfcylinder Z^-, Euclidean space $\mathbb{R}^n$, covariant derivatives replaced by partial derivatives, and with constant c to the function $\rho\xi$ to obtain that

$$\|\xi\|_{\mathcal{W}^{1,p}([-1,0]\times S^1)} \leq c \left(2\,\|(\partial_s - \partial_t\partial_t)\xi\|_{L^p([-2,0]\times S^1)} + \|\xi\|_{L^p([-2,0]\times S^1)}\right)$$

for every $\xi \in C^\infty([-2, 0] \times S^1)$. To obtain the first estimate in step 3 for the backward *half* cylinder it will be crucial that the domain on the right hand side does not extend to the future. Now write $\partial_s - \partial_t\partial_t = D + S$ and use that the operator $S : W^{1,p}(S^1) \to L^p(S^1)$ is bounded to obtain that

$$\begin{aligned}\|\xi\|_{\mathcal{W}^{1,p}([-1,0]\times S^1)} \leq c\Big(&\|D\xi\|_{L^p([-2,0]\times S^1)} + (1 + c_S)\,\|\xi\|_{L^p([-2,0]\times S^1)}\\ &+ c_S\,\|\partial_t\xi\|_{L^p([-2,0]\times S^1)}\Big)\end{aligned}$$

for every $\xi \in C^\infty([-2, 0] \times S^1)$ and some constant $\tilde{c} = \tilde{c}(p, c, c_S)$. Now integrate the estimate in lemma 2.12 over $s \in [-2, 0]$ and chose $\delta > 0$ sufficiently small in order to throw the arising term $\partial_t\partial_t\xi$ to the left hand side. It follows that

$$\|\xi\|_{\mathcal{W}^{1,p}([-1,0]\times S^1)} \leq \tilde{c}\Big(\|D\xi\|_{L^p([-2,0]\times S^1)} + \|\xi\|_{L^p([-2,0]\times S^1)}\Big) \tag{121}$$

for every $\xi \in C^\infty([-2, 0] \times S^1)$ and some constant $\tilde{c} = \tilde{c}(p, c, c_S)$. It remains to replace the L^p norm of ξ by the L^2 norm. Since $p \geq 2$, there is the Sobolev inequality $\|\xi\|_{L^p} \leq c_p\|\xi\|_{W^{1,2}}$ for $\xi \in W^{1,2}$; see e.g. [LL97, theorem 8.5 (ii)] for the domain $\mathbb{R}^2$. The first step is to replace the last term in (121) according to the Sobolev inequality. Then use (121) with $p = 2$ and on *increased* domains to complete the proof of the estimate in step 2 (use Hölder's inequality to estimate the L^2 norm of $D\xi$ by the L^p norm).

To conclude the proof of step 2 assume $\xi \in \mathcal{W}^{1,2}$, then of course $\xi \in L^2$ and $D\xi \in L^2$. If in addition $D\xi$ is locally L^p integrable, then the estimate of step 2 which we just proved shows that $\xi \in \mathcal{W}^{1,p}_{loc}$.

Step 3. *Fix a constant $p \geq 2$ and consider the norm*

$$\|\xi\|_{2;p} = \left(\int_{-\infty}^{0} \|\xi(s,\cdot)\|_{L^2(S^1)}^p \, ds \right)^{1/p}.$$

Then there exist constants c_2 and c_3 both depending on p and c_S such that the following is true. If $\xi \in \mathcal{W}^{1,2}$ and $D\xi \in L^p$, then $\xi \in \mathcal{W}^{1,p}$ and

$$\|\xi\|_{\mathcal{W}^{1,p}} \leq c_2 \left(\|D\xi\|_{L^p} + \|\xi\|_{2;p} \right), \qquad \|QD\xi\|_{2;p} \leq c_3 \|D\xi\|_{L^p}.$$

Fix $\xi \in \mathcal{W}^{1,2}$ such that $D\xi \in L^p$. Then $\xi \in \mathcal{W}^{1,p}_{loc}$ by step 2. Moreover, the estimate of step 2 implies that

$$\|\xi\|_{\mathcal{W}^{1,p}([k,k+1]\times S^1)}^p \leq 3^{p/2-1} 2^p {c_1}^p \int_{k-2}^{k+1} \left(\|D\xi\|_{L^p(S^1)}^p + \|\xi\|_{L^2(S^1)}^p \right) ds$$

for every integer $k < 0$; see [S99, lemma 2.4 step 3] for details. Now take the sum over all such k to obtain the first estimate of step 3.

Next observe that $\eta := D\xi$ lies in $L^2(\mathbb{R}^-, H)$ and in $L^p(\mathbb{R}^-, H)$. Here $H = L^2(S^1)$ and we used that by Hölder's inequality

$$\|\cdot\|_{L^2(S^1)} \leq \|\cdot\|_{L^p(S^1)}. \tag{122}$$

Since η is in the domain $L^2(\mathbb{R}^-, H)$ of the operator Q from step 1, we obtain

$$QD\xi = Q\eta = K * \eta.$$

Now Young's inequality applies to $K * \eta$, because $\eta \in L^p(\mathbb{R}^-, H)$. Hence

$$\|K * \eta\|_{2;p} \leq \|K\|_{L^1(\mathbb{R}^-, \mathcal{L}(H))} \|\eta\|_{L^p(\mathbb{R}^-, H)} \leq C \|D\xi\|_{L^p} \tag{123}$$

where C depends on the constant δ in estimate (119) for the norm of K; see [S99]. The last step uses (122) again. This proves the second estimate of step 3.

It remains to prove that $\xi \in \mathcal{W}^{1,p}$. The two estimates of step 3 imply that

$$\|\xi\|_{\mathcal{W}^{1,p}} \leq c_2 \left((1 + c_3) \|D\xi\|_{L^p} + \|\xi - QD\xi\|_{2;p} \right).$$

To see that the right hand side is finite recall that $D\xi \in L^p$ by assumption and $\xi - QD\xi$ lies in the kernel of $D : \mathcal{W}^{1,2} \to L^2$ by (the proof of) step 1. Moreover, by (120) every element of this kernel is a finite sum of functions of the form $\xi_k = e^{-s\lambda_k} v_k$ and $\|\xi_k\|_{2;p} < \infty$ by calculation.

Step 4. *The theorem is true for $p > 2$.*

Fix $p > 2$ and set $X^- := \ker[D : \mathcal{W}^{1,2} \to L^2]$. Then the linear operator

$$\pi : \mathcal{W}^{1,p} \to \left(X^-, \|\cdot\|_{2;p} \right), \qquad \xi \mapsto \xi - QD\xi,$$

is well defined, bounded and of finite rank, hence compact. To prove this observe that π is well defined on the dense subset $C_0^\infty(Z^-)$ of $\mathcal{W}^{1,p}$. Since $C_0^\infty(Z^-)$ is also dense in $\mathcal{W}^{1,2}$, step 1 shows that $\xi - QD\xi \in X^-$. To see that π is bounded on $C_0^\infty(Z^-)$ let $\xi \in C_0^\infty(Z^-)$. Then

$$\|\pi\xi\|_{2;p} = \|\xi - QD\xi\|_{2;p} \leq \|\xi\|_p + c_3 \|D\xi\|_p \leq (1 + c_3c_4) \|\xi\|_{\mathcal{W}^{1,p}}$$

by definition of π, the triangle inequality, the estimate (122), and the second estimate of step 3. The last inequality follows from the estimate

$$\|D\xi\|_{L^p} \leq \|\partial_s\xi\|_{L^p} + \|\partial_t\partial_t\xi\|_{L^p} + \|S\xi\|_{L^p} \leq c_4 \|\xi\|_{\mathcal{W}^{1,p}}$$

with suitable constant $c_4 = c_4(p, c_S)$. Here we used that $\|S\|_p \leq c_S(\|\xi\|_p + \|\partial_t\xi\|_p)$ by boundedness of S. Now being bounded on a dense subset the operator π extends to a bounded linear operator on $\mathcal{W}^{1,p}$. The rank of π is finite, because the dimension of its target X^- is equal to the Morse index of A_S by step 1.

To prove that $D : \mathcal{W}^{1,p} \to L^p$ is onto we show first that the range is closed and then that it is dense. By the two estimates of step 3 we have that

$$\|\xi\|_{\mathcal{W}^{1,p}} \leq c_2 \left((1 + c_3) \|D\xi\|_{L^p} + \|\pi\xi\|_{2;p}\right)$$

for every $\xi \in C_0^\infty$, hence for every $\xi \in \mathcal{W}^{1,p}$ by density. Since π is compact, the range of D is closed by the abstract closed range lemma. To prove density of the range fix $\eta \in L^p \cap L^2$ and note that the subset $L^p \cap L^2$ is dense in L^p, because it contains the dense subset C_0^∞ of L^p. Now by surjectivity of D in the case $p = 2$ (step 1) and since $\eta \in L^2$, there exists an element $\xi \in \mathcal{W}^{1,2}$ such that $D\xi = \eta$. But then $\xi \in \mathcal{W}^{1,p}$ by step 3, because $D\xi = \eta \in L^p$ by the choice of η. Hence η is in the range of $D : \mathcal{W}^{1,p} \to L^p$. This proves theorem 8.5. □

Proof of proposition 8.3. The arguments in the proof of proposition 3.17 show that the kernel of $\mathcal{D}_{u^-} : \mathcal{W}^{1,p} \to \mathcal{L}^p$ is equal to X^- and X^- does not depend on p. On the other hand, for $p = 2$ the dimension of the kernel is equal to the Morse index of x by theorem 8.5. Surjectivity of $\mathcal{D}_{u^-}$ follows in three stages.

The stationary case. Consider the stationary solution $u^-(s,t) = x(t)$, then $\mathcal{D}_x$ is onto by theorem 8.5. To see this represent $\mathcal{D}_x$ with respect to an orthonormal frame along x; see section 3.4.

The nearby case. Surjectivity is preserved under small perturbations with respect to the operator norm. Moreover, the operator family $\mathcal{D}_{u^-}$ depends continuously on u^- with respect to the $\mathcal{W}^{1,p}$ topology (here we use $p > 2$). Hence, if $u^- \in \mathcal{M}^-(x; \mathcal{V})$ satisfies $u^- = \exp_x(\eta)$ and $\|\eta\|_{\mathcal{W}^{1,p}}$ is sufficiently small, it follows that $\mathcal{D}_{u^-}$ is onto.

The general case. Given $u \in \mathcal{M}^-(x; \mathcal{V})$ and $\sigma < 0$, consider the shifted solution $u^\sigma(s,t) := u(s + \sigma, t)$. Then $(\mathcal{D}_u\xi)^\sigma = \mathcal{D}_{u^\sigma}\xi^\sigma$ by shift invariance of the linear heat equation. This means that surjectivity of $\mathcal{D}_u$ is equivalent to surjectivity of $\mathcal{D}_{u^\sigma}$. But the latter is true by the nearby case above, because u^σ converges to x in the $\mathcal{W}^{1,p}$ topology as $\sigma \to -\infty$. To see this apply theorem 4.10 (B) on exponential decay to u and note that $u^\sigma(0,t) = u(\sigma, t)$. □

Proof of proposition 8.2. The proof follows the same (standard) pattern as the proof of theorem 1.10; see also the introduction to section 5. The first key step is the definition of a Banach manifold $\mathcal{B} = \mathcal{B}_x^{1,p}$ of backward halfcylinders emanating from x such that $\mathcal{B}$ contains the moduli space $\mathcal{M}^-(x;\mathcal{V})$ whenever $p > 2$. The second key step is to define a smooth map $\mathcal{F}_{u^-}$ between Banach spaces as in (79). Its significance lies in the fact that its zeroes correspond precisely to the elements of the moduli space near u^- and that $d\mathcal{F}_{u^-}(0) = \mathcal{D}_{u^-}$. By proposition 8.3 this operator is surjective and the dimension of its kernel is equal to the Morse index of x. Hence $\mathcal{M}^-(x;\mathcal{V})$ is locally near u^- modeled on $\ker \mathcal{D}_{u^-}$ by the implicit function theorem for Banach spaces. To see that the moduli space is a contractible manifold observe that backward time shift provides a *contraction*

$$\begin{aligned} h : \mathcal{M}^-(x;\mathcal{V}) \times [0,1] &\to \mathcal{M}^-(x;\mathcal{V}) \\ (u,r) &\mapsto u(\cdot - \sqrt{r/(1-r)}, \cdot) \end{aligned}$$

onto the stationary solution x. This means that h is continuous and satisfies $h(u,0) = u$ and $h(u,1) = x$ for every $u \in \mathcal{M}^-(x;\mathcal{V})$. □

Proof of theorem 8.1. We abbreviate $\mathcal{M}^- = \mathcal{M}^-(x;\mathcal{V})$ and $W^u = W^u(x;\mathcal{V})$. Recall that the moduli space $\mathcal{M}^-$ is a smooth manifold of dimension equal to $\mathrm{ind}_{\mathcal{V}}(x)$ by proposition 8.2 and, furthermore, by definition the unstable manifold W^u is equal to the image of the evaluation map $ev_0 : \mathcal{M}^- \to \mathcal{L}M$. We use the notation $ev_0(u) =: u_0$, hence $u_0(t) = u(0,t)$. It remains to prove that ev_0 and its linearization are injective and that ev_0 is a homeomorphism onto W^u.

To prove that ev_0 is injective let $u, v \in \mathcal{M}^-$ and assume that $ev_0(u) = ev_0(v)$, that is $u_0 = v_0$. Hence $u = v$ by theorem 6.4 on backward unique continuation.

We prove that the linearization $d(ev_0)_u$ of ev_0 at $u \in \mathcal{M}^-$ is injective. Let $\xi, \eta \in T_u\mathcal{M}^-$. Hence $\mathcal{D}_u\xi = 0 = \mathcal{D}_u\eta$ by proposition 8.2. Now assume that $d(ev_0)_u\xi = d(ev_0)_u\eta$. This means that $\xi_0 = \eta_0$. Therefore $\xi = \eta$ by application of proposition 6.2 (a) on linear unique continuation to the vector field $\xi - \eta$.

We prove that $ev_0 : \mathcal{M}^- \to \mathcal{L}M$ is a homeomorphism onto its image. Fix $u \in \mathcal{M}^-$ and recall that every immersion is locally an embedding. Hence there is an open disk D in $\mathcal{M}^-$ containing u such that $ev_0|_D : D \to \mathcal{L}M$ is an embedding. It remains to prove that there is an open neighborhood U of $u_0 = ev_0(u)$ in $\mathcal{L}M$ such that

$$U \cap W^u = U \cap ev_0(D). \tag{124}$$

Now there are two cases. In case one u is constant in s and therefore $u \equiv x$. Here we exploit the (negative) gradient flow property that the restricted function $\mathcal{S}_{\mathcal{V}}|_{W^u}$ takes on its maximum precisely at the critical point x. Case two is the complementary case in which u depends on s. Here we use a convergence argument based on the compactness theorem 4.3.

Case 1: $u \equiv x$. Set $c = \mathcal{S}_{\mathcal{V}}(x)$, then a set U having the desired property (124) is given by

$$U := \{c - \varepsilon < \mathcal{S}_{\mathcal{V}} < c + \varepsilon\},$$

where

$$2\varepsilon := \min_{u \in \mathrm{cl}D \setminus D} \left(\mathcal{S}_{\mathcal{V}}(x) - \mathcal{S}_{\mathcal{V}}(u_0) \right).$$

Here the compact set $\mathrm{cl}D \setminus D$ is the topological boundary of the open disc D. Note that the elements of $W^u \setminus ev_0(D)$ have action at most $c - 2\varepsilon$.

Case 2: $u \not\equiv x$. Assume by contradiction that there is no U which satisfies (124). Then there is a sequence $\gamma^\nu \in W^u \setminus ev_0(D)$ that converges to u_0 in $\mathcal{L}M$ as $\nu \to \infty$. Note that $\gamma^\nu = ev_0(u^\nu)$ where $u^\nu \in \mathcal{M}^- \setminus D$. In particular, each heat trajectory u^ν converges in backward time asymptotically to x. Thus we obtain that

$$\sup_{s \in (-\infty, 0]} \mathcal{S}_{\mathcal{V}}(u_s^\nu) \le \mathcal{S}_{\mathcal{V}}(x) =: c$$

for every ν. Together with the energy identity this implies that

$$\begin{aligned} E(u^\nu) &= \mathcal{S}_{\mathcal{V}}(x) - \mathcal{S}_{\mathcal{V}}(u_0^\nu) \\ &= c - \tfrac{1}{2} \left\| \partial_t u_0^\nu \right\|_{L^2(S^1)}^2 + \mathcal{V}(u_0^\nu) \\ &\le c + C_0 \end{aligned}$$

where $C_0 > 1$ is the constant in axiom (V0). Adapting the proofs of the apriori theorem 4.5 and the gradient bound theorem 4.9 to cover the case of backward *half* cylinders it follows that there is a constant $C = C(c, \mathcal{V}) > 0$ such that

$$\left\| \partial_t u^\nu \right\|_\infty \le C,$$

and

$$\left\| \partial_s u^\nu \right\|_\infty \le C \sqrt{E(u^\nu)} \le C(c + C_0)$$

for every ν. Here the norms are taken on the domain $(-\infty, 0] \times S^1$. Adapting also the proof of the compactness theorem 4.3 we obtain – in view of the uniform apriori L^∞ bounds for $\partial_t u^\nu$ and $\partial_s u^\nu$ just derived –the existence of a smooth heat flow solution $v : (-\infty, 0] \times S^1 \to M$ and a subsequence, still denoted by u^ν, such that u^ν converges to v in C^∞_{loc}. In particular, this implies that $u_0 = v_0$ and that $\partial_t u_s^\nu$ converges to $\partial_t v_s$, as $\nu \to \infty$, uniformly with all derivatives on S^1 and for each s. This and our earlier uniform action bound for u_s^ν show that

$$\mathcal{S}_{\mathcal{V}}(v_s) = \lim_{\nu \to \infty} \mathcal{S}_{\mathcal{V}}(u_s^\nu) \le c$$

for every s. To summarize,we have two backward flow lines u and v defined on $(-\infty, 0] \times S^1$ along which the action is bounded from above by c and which coincide along the loop $u_0 = v_0$. Hence theorem 6.4 (B) on backward unique continuation asserts that $u = v$. Because u^ν converges to v in C^∞_{loc}, this means that $u^\nu \in D$ whenever ν is sufficiently large. For such ν we arrive at the contradiction $\gamma^\nu = ev_0(u^\nu) \in ev_0(D)$ and this proves theorem 8.1. □

8.2 The Morse complex

Assume that the action $\mathcal{S}_V$ is a Morse function on the loop space. This is true for a generic potential $V \in C^\infty(S^1 \times M)$ by [W02]. Fix a regular value a of $\mathcal{S}_V$ and, furthermore, for each critical point $x \in \mathcal{P}^a(V)$ fix an **orientation** $\langle x \rangle$ of the tangent space at x to the (finite dimensional) unstable manifold $W^u(x; V)$. By $\nu = \nu(V, a)$ we denote a **choice of orientations** for all $x \in \mathcal{P}^a(V)$. The **Morse chain groups** are the $\mathbb{Z}$-modules

$$\mathrm{CM}_k^a = \mathrm{CM}_k^a(V, \nu) := \bigoplus_{\substack{x \in \mathcal{P}^a(V) \\ \mathrm{ind}_V(x) = k}} \mathbb{Z}\,\langle x \rangle, \qquad k \in \mathbb{Z}.$$

These modules are finitely generated and graded by the Morse index. We set $C_k^a = \{0\}$ whenever the direct sum is taken over the empty set. We define

$$\mathrm{CM}_*^a := \bigoplus_{k=0}^{N} \mathrm{CM}_k^a$$

where N is the largest Morse index of an element of the finite set $\mathcal{P}^a(V)$.

Set $\mathcal{V}(x) = \int_0^1 V_t(x(t))\,dt$ and note that $\mathcal{V}$ satisfies (V0)–(V3). Now consider the associated set of admissible perturbations $\mathcal{O}^a$ of $\mathcal{V}$ defined by (108) and the dense subset $\mathcal{O}^a_{reg}$ of regular perturbations provided by theorem 1.13. (The ambient Banach space Y given by (106) provides the metric on $\mathcal{O}^a$.) Now for any $v \in \mathcal{O}^a_{reg}$ we have the following key facts: The functionals $\mathcal{S}_V$ and $\mathcal{S}_{V+v}$ coincide near their critical points and have the same sublevel set with respect to a. Moreover, the perturbed functional $\mathcal{S}_{V+v}$ is Morse-Smale below level a. Here and throughout we sometimes denote $\mathcal{V} + v$ in abuse of notation by $V + v$ to emphasize that we are actually perturbing a *geometric* potential.

To define the Morse boundary operator ∂ on CM_*^a it suffices to define it on the set of generators $\mathcal{P}^a(V)$ and then extend linearly. Fix a regular perturbation $v \in \mathcal{O}^a_{reg}$. Note that each chosen orientation $\langle x \rangle$ orients the *perturbed* unstable manifold $W^u(x; V + v)$. This is because the tangent spaces at x to $W^u(x; V)$ and $W^u(x; V + v)$ coincide (v is not supported near x) and unstable manifolds are finite dimensional and contractible, hence orientable, by theorem 8.1. Now given two critical points $x^\pm$ of action less than a, consider the heat moduli space $\mathcal{M}(x^-, x^+; V + v)$ of solutions u of the heat equation (6) with $\mathcal{V}$ replaced by $\mathcal{V} + v$ and subject to the boundary condition (8). Jointly with D. Salamon we proved in [SW03, ch. 11] that a choice of orientations for all unstable manifolds determines a system of **coherent orientations** on the heat moduli spaces in the sense of Floer–Hofer [FH93].

From now on we assume that $x^\pm$ are of *Morse index difference one*. In this case $\mathcal{M}(x^-, x^+; V + v)$ is a smooth 1-dimensional manifold by theorem 1.10 and its quotient $\mathcal{M}(x^-, x^+; V + v)/\mathbb{R}$ by the (free) time shift action consists of finitely many points by proposition 1.11. For $[u] \in \mathcal{M}(x^-, x^+; V + v)/\mathbb{R}$ time shift naturally induces an orientation of the corresponding component of $\mathcal{M}(x^-, x^+; V + v)$; compare [SW03] and note that $\partial_s u \in \ker \mathcal{D}_u = \det(\mathcal{D}_u)$. We set $n_u = +1$, if the time shift orientation coincides with the coherent orientation, and we set $n_u = -1$ otherwise. One calls n_u the **characteristic sign** of the heat trajectory u. It depends on the orientations $\langle x^- \rangle$ and $\langle x^+ \rangle$. Consider the (finite) sum of characteristic signs corresponding to all heat trajectories from x^- to x^+, namely

$$n(x^-, x^+) := \sum_{[u]\in\mathcal{M}(x^-,x^+;V+v)/\mathbb{R}} n_u.$$

If the sum runs over the empty set, we set $n(x^-, x^+) = 0$. For $x \in \mathcal{P}^a(V)$ define the **Morse boundary operator $\boldsymbol{\partial = \partial(V, a, \nu, v)}$** by the (finite) sum

$$\partial x := \sum_{\substack{y\in\mathcal{P}^a(V)\\ \mathrm{ind}_V(x)-\mathrm{ind}_V(y)=1}} n(x, y)\, y.$$

Set $\partial x = 0$, if the sum runs over the empty set.

Theorem 8.6 (Boundary operator and homology)**.** *Let $V \in C^\infty(S^1 \times M)$ be a potential such that $\mathcal{S}_V$ is Morse and let a be a regular value of $\mathcal{S}_V$. Take a choice of orientations $\nu = \nu(V, a)$ and fix a regular perturbation $v \in \mathcal{O}^a_{reg}$. Then $\partial = \partial(V, a, \nu, v)$ satisfies $\partial \circ \partial = 0$ and* **Morse** *or* **heat flow homology** *is defined by*

$$\mathrm{HM}^a_*(\mathcal{L}M, \mathcal{S}_V) := \frac{\ker \partial(V, a, \nu, v)}{\mathrm{im}\, \partial(V, a, \nu, v)}.$$

The right hand side is independent of $\nu(V, a)$ and the regular perturbation v.

Proof. The main result of [SW03] is that for each heat flow line u between critical points of Morse index difference one there is precisely one Floer trajectory in the loop space of the cotangent bundle between corresponding critical points of the symplectic action functional; see [SW03, cor. 10.4 (ii)]. Moreover, we proved that the characteristic sign of u coincides with the characteristic sign of the corresponding Floer trajectory. In other words, both chain complexes are *equal* (up to natural identification). Hence $\partial \circ \partial = 0$ follows immediately from the well known analogue for the Floer boundary operator; see e.g. [F89b, S99]. (The required, but in case of our nongeometric potentials $\mathcal{V}$ slightly nonstandard apriori C^0 estimate is provided by [SW03, thm. 5.1] with $\varepsilon = 1$.)

The fact that heat flow homology is independent of the choice of orientations $\nu(V, a)$ and the regular perturbation v follows from the homotopy argument which is standard in Floer theory; see again e.g. [F89b, S99]. Here it is crucial to observe that our admissible perturbations $v \in \mathcal{O}^a$ are supported away from the level set $\{\mathcal{S}_V = a\}$ on which the L^2 gradient of $\mathcal{S}_V$ (hence of $\mathcal{S}_{V+v}$) is nonvanishing and inward pointing with respect to $\mathcal{L}^a M$. Likewise independence follows by theorem 1.14. $\square$

9 Homology of the loop space

9.1 The forward semiflow

Consider the Hilbert manifold of loops in M given by

$$\Lambda M := W^{1,2}(S^1, M).$$

In this section we prove that the Cauchy problem for the heat equation is solvable in forward time for any initial value in ΛM. This gives rise to a C^1 semiflow $\varphi : (0,\infty) \times \Lambda M \to \Lambda M$ which is continuous on $[0,\infty) \times \Lambda M$.

It is convenient to fix an isometric embedding of the Riemannian manifold M into some Euclidean space $\mathbb{R}^N$ using Nash's theorem. We denote by P the corresponding second fundamental form. From now on we view loops in M as loops in $\mathbb{R}^N$ taking values in M. Given a smooth such loop x, recall that the covariant derivative $\nabla_t \partial_t x$ is given by taking the derivative $\partial_t \partial_t x$ in the ambient vector space $\mathbb{R}^N$ and subtracting the component normal to M. This normal component is pointwise given by $P(x)\,(\partial_t x, \partial_t x)$. In these terms the heat equation (6) reads

$$\partial_s u - \partial_t \partial_t u = -P(u)\,(\partial_t u, \partial_t u) + \operatorname{grad} \mathcal{V}(u) =: \phi(u) \tag{125}$$

for maps $u : \mathbb{R} \times S^1 \to \mathbb{R}^N$ taking values in M. Changing perspective and abusing notation we interpret this pde as a Cauchy problem

$$\frac{d}{ds} u = \Delta u + \phi(u), \qquad u(0) = \gamma \in \Lambda M, \tag{126}$$

for maps $u : [0,\infty) \to \Lambda M$ and where $\Delta := \partial_t \partial_t$. The case $\mathcal{V} = 0$ is the harmonic map flow introduced by Eells and Sampson [ES64] in 1964. To prove short time existence they applied the method of successive approximation. However, in the mean time the elegant theory of abstract evolution equations in Banach spaces has been developed; an excellent reference is the book by Henry [He81]. We will use this theory to construct the semiflow.

Hence the next step is to interpret the pde above as an evolution equation for maps $s \mapsto u(s)$ from an interval $[0,T]$ to some Banach *space* X. We set

$$X := W^{1,2}(S^1, \mathbb{R}^N), \qquad Y := L^1(S^1, \mathbb{R}^N).$$

Note that according to our convention X contains ΛM. Note also that the perturbation $\operatorname{grad} \mathcal{V}$ is defined only on the subset of smooth loops $\mathcal{L}M \subset X$. Since $\mathcal{L}M$ is dense in ΛM we extend $\mathcal{V}$ continuously to ΛM. Next we identify a compact neighborhood U of M in $\mathbb{R}^N$ with a neighborhood of the zero section of the normal bundle $pr : \nu_M \to M$ of M in $\mathbb{R}^N$. Now fix a sufficiently small neighborhood $\mathcal{U}$ of ΛM in X such that the following is true. If $z \in \mathcal{U}$, then $z(t) \in U$ for every $t \in S^1$. Hence the projection

$$\pi : \mathcal{U} \to \Lambda M, \qquad (\pi z)(t) := pr(z(t)), \quad \forall t \in S^1, \tag{127}$$

is well defined. It provides a canonical means to define the desired extensions. Namely, these are given for $z \in \mathcal{U}$ by $\operatorname{grad} \mathcal{V}(\pi z)$ and $P(\pi z)\,(\partial_t(\pi z), \partial_t(\pi z))$.

Now the task at hand is to prove that the Cauchy problem

$$\frac{d}{ds} u = \Delta u + f(u), \qquad f := \phi \circ \pi, \qquad u(0) = \gamma \in \Lambda M, \tag{128}$$

admits a unique short time solution $u : [0,T] \to \mathcal{U} \subset X$. Note that $f : \mathcal{U} \to Y$.

Definition 9.1. A *solution of the Cauchy problem* (128) is a continuous map $u : [0,T] \to Y$ with $u(0) = \gamma$ such that for $s \in (0,T]$ we have $u(s) \in \mathcal{U} \cap W^{2,1}(S^1)$, the derivative $\frac{du}{ds}(s)$ exists in Y, the differential equation is satisfied, and the map $f \circ u : (0,T] \to Y$ is locally Hölder continuous and bounded.

It is convenient to reformulate (128) as an integral equation, namely

$$u(s) = e^{s\Delta}\gamma + \int_0^s e^{(s-\sigma)\Delta} f(u(\sigma))\, d\sigma =: (\Psi_\gamma u)(s). \tag{129}$$

Definition 9.2. A *solution of the integral equation with initial value* $\gamma \in \Lambda M$ is a continuous map $u : (0,T] \to \mathcal{U}$ satisfying (129) such that the map $f \circ u : (0,T] \to Y$ is continuous and bounded.

Both notions of solution are equivalent by lemma 9.11. (See the following subsection for more information on the analytic semigroup $e^{s\Delta}$.) To prove theorem 9.3 on local existence und uniqueness of a solution to (129) we set up a complete metric space Z on which Ψ_γ acts as a strict contraction. Now Ψ_γ admits a unique fixed point by the Banach contraction mapping principle. But fixed points of Ψ_γ correspond precisely to solutions of the integral equation.

The next task, theorem 9.12, is to establish higher regularity of u. Here we exploit the integral representation (129) of u in combination with the fact that analytic semigroups are extremely regularizing. By the method of bootstrapping we improve regularity step by step. Once we arrive at the point where u – now viewed again as a map from $(0,T] \times S^1$ to $\mathbb{R}^N$ – is locally of class $\mathcal{W}^{1,p}$ for some $p > 2$ we are done. Namely, this implies that u takes values in M and therefore satisfies the earnest heat equation (125). But in this case our previous regularity theorem 4.2 asserts smoothness. To summarize, if $u(0) = \gamma$ is in ΛM, then all existing future loops $u(s)$ remain in the set $\mathcal{L}M$ of smooth loops *in* M. In particular, we don't need to worry any more that $u(s)$ might leave $\mathcal{U}$.

Certainly this good news greatly enhances chances that solutions actually exist globally, that is for all positive times. This is the content of theorem 9.13. In theorem 9.14 we analyze the asymptotic behavior of $u(s)$ as $s \to \infty$. Theorem 9.15 asserts that the dependence of the solution u of (126) on the initial value $\gamma \in \Lambda M$ is of class C^1. Then

$$\varphi : (0,\infty) \times \Lambda M \to \Lambda M, \quad (s,\gamma) \mapsto u(s), \tag{130}$$

provides the desired C^1 semiflow. It extends continuously to 0 by theorem 9.3.

In the main proofs below we follow the line of argument presented in [He81] for fractional power spaces. We provide details in our setting for completeness. Certain facts concerning semigroups are collected in remark 9.6 without proof.

9.1.1 Local existence and uniqueness

Theorem 9.3. *Fix a perturbation $\mathcal{V}$ that satisfies* (V0)–(V1) *and a loop $\gamma \in \Lambda M$. Then there exists a time $T = T(\gamma) \in (0, 1]$ and a unique solution $u \in C^0([0, T], X)$ taking values in $\mathcal{U}$ of the integral equation (129) such that $u(0) = \gamma$. This solution is continuously differentiable as a map $u : (0, T] \to Y$ and satisfies*

$$u(s) \in W^{2,1}(S^1, \mathbb{R}^N), \qquad \frac{d}{ds}u(s) = \Delta u(s) + f(u(s)) \text{ in } Y,$$

for every $s \in (0, T]$ and where $f = \phi \circ \pi$; see (125) and (127).

The proof of this and subsequent results rests on the fact that the operator $-\Delta$ generates an analytic semigroup on $L^p(S^1)$.[5] First we define this notion and recall a key proposition. Then we collect further relevant facts in remark 9.6.

Definition 9.4. A *strongly continuous semigroup* on a Banach space Z is a family $T = \{T(s)\}_{s \geq 0}$ of continuous linear operators on Z satisfying

$$s, t \geq 0 \quad \Rightarrow \quad T(0) = \mathbb{1},\ T(s + \sigma) = T(s)T(\sigma), \tag{131}$$

$$z \in Z \quad \Rightarrow \quad T(s)z \to z \text{ as } s \to 0^+. \tag{132}$$

The infinitesimal generator L of this semigroup is defined as follows $Lz := \lim_{s \to 0^+} \frac{1}{s}(T(s)z - z)$, its domain $\operatorname{dom} L$ consisting of all $z \in Z$ for which this limit exists in Z. One usually writes $T(s) = e^{sL}$. If T satisfies in addition

$$z \in Z \quad \Rightarrow \quad s \mapsto T(s)z \text{ is real analytic on } (0, \infty) \tag{133}$$

we call it an *analytic semigroup.*

Proposition 9.5. *Let $\{e^{sL}\}_{s \geq 0}$ be an analytic semigroup on a Banach space Z with infinitesimal generator L. Then*

$$s > 0,\ z \in Z \quad \Rightarrow \quad e^{sL}z \in \operatorname{dom} L, \quad \frac{d}{ds}e^{sL}z = Le^{sL}z, \tag{134}$$

and there is a constant C such that

$$\left\| Le^{sL}z \right\|_Z \leq \frac{C}{s} \left\| z \right\|_Z, \quad \text{for } s \in (0, 1]. \tag{135}$$

A strongly continuous semigroup still shares similar properties when *restricted to the domain of L*. Namely, for $z \in \operatorname{dom} L$ the map $(0, \infty) \to Z : s \mapsto e^{sL}z$ is continuously differentiable and

$$s > 0,\ z \in \operatorname{dom} L \quad \Rightarrow \quad \frac{d}{ds}e^{sL}z = Le^{sL}z. \tag{136}$$

Moreover, there are constants $\mu \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\left\| e^{sL} \right\|_{\mathcal{L}(Z)} \leq \mu e^{s\omega}, \qquad \forall s \geq 0. \tag{137}$$

[5] We abbreviate $L^p := L^p(S^1) := L^p(S^1, \mathbb{R}^N)$ and similarly for Sobolev spaces.

Remark 9.6 (The semigroup $e^{s\Delta}$)**.** Fix constants $p \in [1,\infty)$ and $q \in (1,\infty)$.

(a) The Laplacian $\Delta = \partial_t \partial_t$ on $L^p(S^1)$ with dense domain $W^{2,p}$ generates an analytic semigroup on $L^p(S^1)$.

(b) There is a constant $C = C(p) > 0$ such that

$$\left\| e^{s\Delta} \right\|_{\mathcal{L}(L^1, W^{1,p})} \le C s^{-(1-\frac{1}{2p})}, \qquad \forall s \in (0,1]. \tag{138}$$

The estimate continues to hold on a larger interval $(0,T]$ on the expense that C in addition depends on T.

(c) There is a constant $\mu = \mu \ge 1$ such that

$$\left\| e^{s\Delta} \right\|_{\mathcal{L}(L^p)} \le \mu, \qquad \forall s \ge 0. \tag{139}$$

(d) The Laplacian $\Delta = \partial_t \partial_t$ on $W^{k,q}(S^1)$ with dense domain $W^{k+2,q}$ generates a strongly continuous semigroup on $W^{k,q}(S^1)$ for each integer $k > 0$.

(e) Suppose $p \ge q$ and fix integers $k \ge \ell \ge 0$, then

$$\left\| e^{s\Delta} \right\|_{\mathcal{L}(W^{\ell,q}, W^{k,p})} \le C s^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}(k-\ell)}, \quad \forall s \in (0,1] \tag{140}$$

for some constant $C = C(p,q,k,\ell) > 0$. The estimate holds on a larger interval $(0,T]$ on the expense that C in addition depends on T.

Some comments are in order. For (136) and (137) see section I.1 in [DK92]. Concerning part (a) of remark 9.6 see chapter 13 in [Ta96]. The estimate in part (b) follows by applying Gagliardo-Nirenberg interpolation, see e.g. [MS04, prop. B.1.18], to (135) for $X = L^1$ and (139) for $p = 1$. In case of the larger interval $(0,T]$ use that $e^{(k+s)\Delta}$ equals the composition of bounded operators $e^{\Delta} \cdots e^{\Delta} e^{s\Delta}$ for $k \in \mathbb{N}$. Part (c) follows from the fact that for an analytic semigroup the constant ω in (137) equals the spectral bound of the infinitesimal generator. Hence in the case at hand we obtain that $\omega = \sup \operatorname{spec} \Delta = 0$. For part (d) and (e) see table (1.1.15) in [Ta96, ch. 15]. Alternatively to see (d) observe that by continuity of $e^{s\Delta}$ and strong continuity (132) the operators ∂_t and $e^{s\Delta}$ commute, then use (a).

Lemma 9.7. *Fix a constant $T > 0$ and a continuous bounded map $\tilde{f} : (0,T] \to L^1(S^1)$. For $s \in [0,T]$ set*

$$F(s) = \int_0^s e^{(s-\sigma)\Delta} \tilde{f}(\sigma)\, d\sigma. \tag{141}$$

Then the following is true.

(a) *If $p > 1$, then $F \in C^0([0,T], W^{1,p}(S^1))$ and $F(0) = 0$.*

(b) *If $q \geq 1$ and $\tilde{f}$ is, in addition, locally Hölder continuous as a map $(0,T] \to L^q(S^1)$, then $F \in C^1((0,T], L^q(S^1))$ and*

$$F(s) \in W^{2,q}(S^1), \qquad \frac{d}{ds}F(s) = \Delta F(s) + \tilde{f}(s) \text{ in } L^q(S^1),$$

for every $s \in (0,T]$.

Proof. The proof in Henry [He81, lemma 3.2.1] carries over step by step. Nevertheless, we provide full details in our situation, because the result is fundamental for all aspects of the construction of the C^1 forward semiflow (130).

ad (a) The trouble with the integral F is that the integrand has a singularity at $\sigma = s$. The key idea is to define a family of maps F_δ avoiding the singularity and for which continuity is easy to see. Then it suffices to prove uniform convergence of F_δ to F as $\delta \to 0$.

The whole argument is based on the estimate (138) whose right hand side is integrable over $(0,T]$, since the exponent of s is strictly larger than -1. Now we choose $\delta \in (0,T)$ and define $F_\delta : [0,T] \to W^{1,p}$ by

$$F_\delta(s) = \begin{cases} 0 & , s \in [0,\delta], \\ \int_0^{s-\delta} e^{(s-\sigma)\Delta} \tilde{f}(\sigma)\, d\sigma & , s \in [\delta, T]. \end{cases}$$

Set $K_1 := \sup \|\tilde{f}\|_1$. By (138) the integrand of F_δ takes values in $W^{1,p}$ and is integrable. Hence F_δ is well defined and takes values in $W^{1,p}$ itself. Next choose $\delta \leq s \leq s+h \leq T$, then the difference

$$F_\delta(s+h) - F_\delta(s) = \left(e^{h\Delta} - \mathbb{1}\right) F_\delta(s) + \int_{s-\delta}^{s+h-\delta} e^{(s+h-\sigma)\Delta} \tilde{f}(\sigma)\, d\sigma$$

converges to zero in $W^{1,p}$ as $h \to 0$. This proves continuity of F_δ. Here we used that $\{e^{h\Delta}\}_{h\geq 0}$ is a strongly continuous semigroup on $W^{1,p}$ by remark 9.6 (d) and a short calculation to see that the $W^{1,p}$ norm of the second term is bounded above by $2pCK_1((h+\delta)^{1/2p} - \delta^{1/2p})$. Now another calculation shows that

$$\begin{aligned} \|F(s) - F_\delta(s)\|_{1,p} &\leq 2pCK_1 \begin{cases} s^{1/2p} & , s \in [0,\delta], \\ \delta^{1/2p} & , s \in [\delta, T], \end{cases} \\ &\leq 2pCK_1 \delta^{1/2p}, \end{aligned}$$

and the right hand tends to zero as $\delta \to 0$, uniformly in $s \in [0,T]$. Continuity of each map $F_\delta : [0,T] \to W^{1,p}$ with $F_\delta(0) = 0$ and uniform convergence of F_δ to F in $W^{1,p}$ show that the limit $F : [0,T] \to W^{1,p}$ is continuous and $F(0) = 0$.

ad (b) The proof has four steps. Fix $q \geq 1$.

Step 1. *Fix $\delta \in (0,T)$. If $s \in [\delta, T]$, then $F_\delta(s) \in W^{2,q}$ and*

$$\Delta F_\delta(s) = \int_0^{s-\delta} \Delta e^{(s-\sigma)\Delta} \left(\tilde{f}(\sigma) - \tilde{f}(s)\right) d\sigma + \left(e^{s\Delta} - e^{\delta\Delta}\right) \tilde{f}(s).$$

Throughout the proof of step 1 we set $L := \Delta$. For $s = \delta$ the statement is trivial, since $F_\delta(\delta) = 0$. Hence fix $s \in (\delta, T]$. Now recall that L generates an analytic semigroup on L^q by remark 9.6 (a). Hence $e^{(\tau-\sigma)L}\tilde{f}(\sigma)$ lies in the domain $W^{2,q}$ of L for all $0 < \sigma < \tau$ by (134). Therefore the Riemann sums for $F_\delta(s)$ which we denote by

$$\sum_{\sigma_j \le s-\delta} e^{(s-\sigma_j)L}\tilde{f}(\sigma_j)\,\Delta\sigma_j$$

are in L^q. Now in L^q the following limit exists and is given by

$$\lim_{\Delta\sigma\to 0} L \sum_{\sigma \le s-\delta} e^{(s-\sigma)L}\tilde{f}(\sigma)\,\Delta\sigma = \int_0^{s-\delta} Le^{(s-\sigma)L}\tilde{f}(\sigma)\,d\sigma. \tag{142}$$

To see this pull L through the finite sum and then observe that

$$\left\|Le^{(s-\sigma)L}\tilde{f}(\sigma)\right\|_q \le \left\|e^{(s-\sigma)L}\tilde{f}(\sigma)\right\|_{2,q} \le \left\|e^{(s-\sigma)L}\right\|_{\mathcal{L}(L^1,W^{2,q})}\left\|\tilde{f}(\sigma)\right\|_1.$$

Hence by (140) with constant C we obtain the bound

$$\left\|Le^{(s-\sigma)L}\tilde{f}(\sigma)\right\|_q \le CK_1(s-\sigma)^{-3/2-1/2q} \le CK_1\delta^{-3/2-1/2q}$$

for all $\sigma \in (0, s-\delta)$. This proves (142). Now since the operator L is closed it follows by (142) that $F_\delta(s)$ is in the domain $W^{2,q}$ of L and

$$\begin{aligned} LF_\delta(s) &= \int_0^{s-\delta} Le^{(s-\sigma)L}\tilde{f}(\sigma)\,d\sigma \\ &= \int_0^{s-\delta} Le^{(s-\sigma)L}\left(\tilde{f}(\sigma)-\tilde{f}(s)\right)d\sigma + \left(e^{sL}-e^{\delta L}\right)\tilde{f}(s). \end{aligned} \tag{143}$$

To obtain the last step we used the identity $-Le^{(s-\sigma)L} = \frac{d}{d\sigma}e^{(s-\sigma)L}$ and the fundamental theorem of calculus. This proves step 1.

Step 2. *If $s \in (0,T]$, then $F(s) \in W^{2,q}$ and*

$$\Delta F(s) = \int_0^s \Delta e^{(s-\sigma)\Delta}\left(\tilde{f}(\sigma)-\tilde{f}(s)\right)d\sigma + \left(e^{s\Delta}-\mathbb{1}\right)\tilde{f}(s).$$

Moreover, on each closed subinterval $[a,b] \subset (0,T]$ we have uniform convergence in L^q of ΔF_δ to ΔF, as $\delta \to 0$.

Again by (140) with constant C' we obtain that

$$\begin{aligned} \left\|\Delta e^{(s-\sigma)\Delta}\left(\tilde{f}(\sigma)-\tilde{f}(s)\right)\right\|_q &\le \left\|e^{(s-\sigma)\Delta}\left(\tilde{f}(\sigma)-\tilde{f}(s)\right)\right\|_{2,q} \\ &\le \left\|e^{(s-\sigma)\Delta}\right\|_{\mathcal{L}(L^q,W^{2,q})}\left\|\left(\tilde{f}(\sigma)-\tilde{f}(s)\right)\right\|_q \\ &\le cC'(s-\sigma)^{-1+\alpha} \end{aligned}$$

where $c > 0$ and $\alpha > 0$ are, respectively, the Hölder constant and the Hölder exponent for $\tilde{f}$ whenever σ is sufficiently close to s. This estimate shows that the left hand side is integrable over $\sigma \in (0, s)$. Hence by the formula in step 1 we obtain that the following limit exists in L^q, namely

$$\Delta F_\delta(s) \to \int_0^s \Delta e^{(s-\sigma)\Delta} \left(\tilde{f}(\sigma) - \tilde{f}(s)\right) d\sigma + \left(e^{s\Delta} - \mathbb{1}\right) \tilde{f}(s), \quad \text{as } \delta \to 0,$$

where for the last term we used (132) with $Z = L^q$. Recall that $F_\delta(s)$ converges in L^q to $F(s)$, as $\delta \to 0$, by part (a). Thus again by closedness of the operator Δ it follows that $F(s)$ is in the domain $W^{2,q}$ of Δ and $\Delta F_\delta(s) \to \Delta F(s)$ in L^q, as $\delta \to 0$. This proves the formula in step 2.

Now fix a closed subinterval $[a, b]$ of $(0, T]$. By local Hölder continuity of $\tilde{f}$ there are positive constants K and β such that

$$\left\|\tilde{f}(s) - \tilde{f}(\sigma)\right\|_q \le K \left|s - \sigma\right|^\beta$$

for all $s, \sigma \in [a, b]$. Use the formulae in step 1 and step 2 to obtain that

$$\begin{aligned}
\left\|\Delta F_\delta(s) - \Delta F(s)\right\|_q &= \left\| \int_{s-\delta}^s \Delta e^{(s-\sigma)\Delta} \left(\tilde{f}(\sigma) - \tilde{f}(s)\right) d\sigma + \left(e^{\delta\Delta} - \mathbb{1}\right) \tilde{f}(s) \right\|_q \\
&\le KC' \int_{s-\delta}^s (s-\sigma)^{-1+\beta}\, d\sigma + \left\|\left(e^{\delta\Delta} - \mathbb{1}\right) \tilde{f}(s)\right\|_q \\
&\le \frac{KC'}{\beta} \delta^\beta + \left\|\left(e^{\delta\Delta} - \mathbb{1}\right) \tilde{f}(s)\right\|_q .
\end{aligned}$$

Here the first inequality follows by the calculation carried out earlier. Now the right hand side converges to zero, as $\delta \to 0$, uniformly in $s \in [a, b]$. This follows from the fact that the map $e^{\delta\Delta} - \mathbb{1} : L^q \to L^q$ is continuous by definition 9.4 and by (132) it converges pointwise to zero. Hence restricted to the compact set $\tilde{f}([a, b])$ we obtain uniform convergence.

Step 3. *Fix $\delta \in (0, T)$, then $F_\delta : (\delta, T] \to L^q$ is continuously differentiable and*

$$\frac{d}{ds} F_\delta(s) = \Delta F_\delta(s) + e^{\delta\Delta} \tilde{f}(s - \delta).$$

Fix $\delta < s < s + h \le T$. By definition of F_δ it follows that

$$\begin{aligned}
\frac{F_\delta(s+h) - F_\delta(s)}{h} &= \int_0^{s-\delta} \frac{e^{(\frac{\delta}{2}+h)\Delta} - e^{\frac{\delta}{2}\Delta}}{h} e^{(s-\frac{\delta}{2}-\sigma)\Delta} \tilde{f}(\sigma)\, d\sigma \\
&\quad + \frac{1}{h} \int_{s-\delta}^{s-\delta+h} e^{(s+h-\sigma)\Delta} \tilde{f}(\sigma)\, d\sigma.
\end{aligned}$$

Note that all exponents stay away from the singular value zero as they are bounded below by $\frac{\delta}{2}$. Hence the fraction inside the first integral converges to

$$\left.\frac{d}{dh}\right|_{h=\frac{\delta}{2}} e^{h\Delta} = \Delta e^{\frac{\delta}{2}\Delta},$$

as $h \to 0$, by (134) and the second integral to $e^{\delta\Delta}\tilde{f}(s-\delta)$. Using the first identity in (143) we obtain that the limit $h^{-1}(F_\delta(s+h) - F_\delta(s))$ exists in L^q, as $h \to 0$, and equals $\Delta F_\delta(s) + e^{\delta\Delta}\tilde{f}(s-\delta)$.

Step 4. *We prove part* (b) *of lemma 9.7.*

Fix a closed interval $[a,b] \subset (0,T]$ and suppose $0 < \delta < a$. Then $\frac{d}{ds}F_\delta$ converges to $\Delta F + \tilde{f}$ in L^q, as $\delta \to 0$, uniformly in $s \in [a,b]$. Here we used step 3, step 2, and the fact that the continuous map $e^{\delta\Delta} : L^q \to L^q$ converges pointwise, hence uniformly on the compact set $\tilde{f}([a/2,b])$, to the identity, as $\delta \to 0$. □

Lemma 9.8. *Fix a constant $q > 1$ and an integer $\ell > 0$. Then the following is true. If $h \in (0,1)$ and $x \in L^1(S^1) \cup W^{\ell,q}(S^1)$, then*

$$\left(e^{h\Delta} - \mathbb{1}\right) e^{s\Delta} x = \int_0^h \Delta e^{(\tau+s)\Delta} x \, d\tau$$

for every $s \in (0,1)$.

Proof. Assume $x \in W^{\ell,q}$. Then we know by remark 9.6 (d) that $\{e^{s\Delta}\}$ is a strongly continuous semigroup on $W^{\ell,q}$. Now fix $s \in (0,1)$. Then it follows by (140) that $e^{s\Delta}x \in W^{\ell+2,q} = \operatorname{dom}\Delta$. Hence $\Delta e^{(\tau+s)\Delta}x = \frac{d}{d\tau}e^{(\tau+s)\Delta}x$ by (136). Now the assertion follows by the fundamental theorem of calculus – provided the integral exists. To see this apply (140) with constant C' to get that

$$\left\|\Delta e^{(\tau+s)\Delta} x\right\|_{\ell,q} \le \left\|e^{(\tau+s)\Delta}\right\|_{\mathcal{L}(W^{\ell,q},W^{\ell+2,q})} \|x\|_{\ell,q} \le \frac{C'}{\tau+s}\|x\|_{\ell,q}. \tag{144}$$

Integrating the right hand side over $\tau \in (0,h)$ gives $C'\|x\|_{\ell,q}\ln(1+\frac{h}{s}) < \infty$. The case $x \in L^1$ follows by the same arguments using proposition 9.5 which applies to $L = \Delta$ and $Z = L^1$ by remark 9.6 (a) for $p = 1$. □

Lemma 9.9. *Fix constants $p \ge q > 1$ and integers $k \ge \ell \ge 1$. Suppose $\gamma \in W^{\ell,q}(S^1)$. Then the map $(0,1] \to W^{k,p}(S^1) : s \mapsto e^{s\Delta}\gamma$ is continuous.*

Proof. Fix $0 < s < s+h \le 1$, then

$$\begin{aligned}
\left\|e^{(s+h)\Delta}\gamma - e^{s\Delta}\gamma\right\|_{k,p} &\le \int_0^h \left\|\Delta e^{(\tau+s)\Delta}\gamma\right\|_{k,p} d\tau \\
&\le \int_0^h \left\|e^{(\tau+s)\Delta}\right\|_{\mathcal{L}(W^{\ell,q},W^{k+2,p})} \|\gamma\|_{\ell,q} \, d\tau \\
&= C\|\gamma\|_{\ell,q} \begin{cases} \frac{s^{-\kappa}-(s+h)^{-\kappa}}{\kappa} & ,\kappa > 0, \\ \ln\left(\frac{s+h}{s}\right) & ,\kappa = 0, \end{cases}
\end{aligned}$$

where $\kappa \in [0,\infty)$ is given by $\kappa = \frac{1}{2}(\frac{1}{q} - \frac{1}{p}) + \frac{1}{2}(k-\ell)$. In step one and four we applied lemma 9.8 and (140) with constant C, respectively. The right hand side of the estimate tends to zero as $h \to 0$. This proves continuity at s. □

Lemma 9.10. *Fix a perturbation $\mathcal{V}$ that satisfies* (V0–V1)*. Then the map*

$$f = \phi \circ \pi : X \supset \mathcal{U} \to Y$$

where ϕ and π are given by (125) and (127) is locally Lipschitz continuous.

Proof. Given $\gamma \in \mathcal{U}$ we have to show that there are constants $\rho, L > 0$ such that

$$\|f(x) - f(y)\|_1 \leq L \|x - y\|_{1,2}$$

for all $x, y \in B := \{z \in X : \|z - \gamma\|_{1,2} \leq \rho\}$. Fix $\rho > 0$ sufficiently small such that $B \subset \mathcal{U}$. Recall that U is a fixed compact neighborhood of M in $\mathbb{R}^N$ used implicitely in the definition (127) of the projection π. Consider the constants

$$\kappa_1 := \max_{q \in U} \|d\,pr(q)\|_{\mathcal{L}(\mathbb{R}^N, T_{pr(q)}M)}, \quad \kappa_2 := \max_{q \in U} \left\|d^2\,pr(q)\right\|_{\mathcal{L}(\mathbb{R}^{N \times N}, T_{pr(q)}M)}$$

and

$$\beta_1 := \max_{q \in M} \|P(q)\|_{\mathcal{L}((T_qM)^{\times 2}, T_q^\perp M)}, \quad \beta_2 := \max_{q \in M} \|dP(q)\|_{\mathcal{L}((T_qM)^{\times 3}, T_q^\perp M)}.$$

Choose $x, y \in B$ and set $v = y - x$. Now the map $h : \mathcal{U} \to Y$, $x \mapsto (\operatorname{grad}\mathcal{V}) \circ \pi(x)$ is composed of two C^2 maps. Hence there exists $\tau \in (0,1)$ such that

$$\begin{aligned}
&\operatorname{grad}\mathcal{V}(\pi(x)) - \operatorname{grad}\mathcal{V}(\pi(x+v)) \\
&= \frac{d}{d\tau} \operatorname{grad}\mathcal{V}(\pi(x+\tau v)) \\
&= \nabla_\tau \operatorname{grad}\mathcal{V}(\pi(x+\tau v)) + P|_{\pi(x+\tau v)}\Big(\partial_\tau(\pi(x+\tau v)), \operatorname{grad}\mathcal{V}(\pi(x+\tau v))\Big)
\end{aligned}$$

pointwise at $t \in S^1$. Here we used that the covariant derivative in M equals the extrinsic derivative minus its normal component. From this we obtain that

$$\begin{aligned}
&\|\operatorname{grad}\mathcal{V}(\pi(x)) - \operatorname{grad}\mathcal{V}(\pi(x+v))\|_1 \\
&\leq \|\nabla_\tau \operatorname{grad}\mathcal{V}(\pi(x+\tau v))\|_1 + \beta_1 \|\partial_\tau(\pi(x+\tau v))\|_1 \|\operatorname{grad}\mathcal{V}(\pi(x+\tau v))\|_\infty \\
&\leq 2C_1 \|\partial_\tau(\pi(x+\tau v))\|_1 + \beta_1 C_0 \|\partial_\tau(\pi(x+\tau v))\|_1 \\
&= (2C_1 + \beta_1 C_0) \|d\pi(x+\tau v)\, v\|_1 \\
&\leq \kappa_1 (2C_1 + \beta_1 C_0) \|v\|_1 .
\end{aligned}$$

This estimate means that $\operatorname{grad}\mathcal{V}(\pi\cdot)$ is globally Lipschitz on $\mathcal{U}$ even with respect to the L^1 norm on the domain. To obtain the second inequality we used axioms (V0) and (V1) with constants $C_0 = C_0(\mathcal{V})$ and $C_1 = C_1(\mathcal{V})$, respectively. Now fix $t \in S^1$. By abuse of notation we denote the point $x(t)$ by x and $v(t)$ by v. Moreover, we abbreviate $\dot{x} = \partial_t x$. Since all maps involved are C^1 and take values in the ambient $\mathbb{R}^N$, there exists $\tau = \tau(t) \in (0,1)$ such that

$$
\begin{aligned}
&\Big| P|_{pr(x)}\Big(d\,pr|_x \dot x, d\,pr|_x \dot x\Big) - P|_{pr(x+v)}\Big(d\,pr|_{x+v}(\dot x+\dot v), d\,pr|_{x+v}(\dot x+\dot v)\Big)\Big| \\
&= \left| \frac{d}{d\tau} P|_{pr(x+\tau v)}\Big(d\,pr|_{x+\tau v}(\dot x+\tau\dot v), d\,pr|_{x+\tau v}(\dot x+\tau\dot v)\Big)\right| \\
&= \Big| dP|_{pr(x+\tau v)}\Big(d\,pr|_{x+\tau v} v, d\,pr|_{x+\tau v}(\dot x+\tau\dot v), d\,pr|_{x+\tau v}(\dot x+\tau\dot v)\Big) \\
&\quad + 2P|_{pr(x+\tau v)}\Big(d\,pr|_{x+\tau v}(\dot x+\tau\dot v), d^2\,pr|_{x+\tau v}(v, \dot x+\tau\dot v) + d\,pr|_{x+\tau v}\dot v\Big)\Big| \\
&\le 2(\beta_2\kappa_1^3 + 2\beta_1\kappa_1\kappa_2)\big(|\dot x|^2 + |\dot v|^2\big)\,|v| + 2\beta_1\kappa_1^2\big(|\dot x| + |\dot v|\big)\,|\dot v|\,.
\end{aligned}
$$

Now integrate this pointwise inequality over $t \in S^1$ to obtain that

$$
\begin{aligned}
&\left\| P(\pi x)\,(\partial_t(\pi x), \partial_t(\pi x)) - P(\pi y)\,(\partial_t(\pi y), \partial_t(\pi y))\right\|_1 \\
&\le 2(\beta_2\kappa_1^3 + 2\beta_1\kappa_1\kappa_2)\big(\|\dot x\|_2^2 + \|\dot v\|_2^2\big)\,\|v\|_\infty + 2\beta_1\kappa_1^2\big(\|\dot x\|_2 + \|\dot v\|_2\big)\,\|\dot v\|_2 \\
&\le \mu\,\|x-y\|_{1,2}\,.
\end{aligned}
$$

Here $\mu > 0$ depends on the constants κ_1, κ_2, β_1, β_2, ρ, the Sobolev constant associated to the embedding $W^{1,2}(S^1) \hookrightarrow L^\infty(S^1)$, and on $\|\gamma\|_{1,2}$. Note that

$$
\|\dot x\|_2 \le \|x\|_{1,2} \le \|\gamma\|_{1,2} + \rho, \quad \|\dot v\|_2 \le \|v\|_{1,2} = \|x-\gamma+\gamma-y\|_{1,2} \le 2\rho. \tag{145}
$$

We also used Hölder's inequality $\|fh\|_1 \le \|f\|_2\|h\|_2$. □

Lemma 9.11. *A map u defined on the interval $[0,T]$ is a solution of the Cauchy problem (128) if and only if it is a solution of the integral equation (129).*

Proof. "⇒" Suppose u is a solution of the Cauchy problem (128). In particular, this means by definition 9.1 that $\tilde f := f \circ u : (0,T] \to Y$ is locally Hölder continuous and bounded and u solves the linear inhomogeneous problem

$$
\frac{d}{ds}v(s) - \Delta v(s) = \tilde f(s), \qquad v(0) = \gamma \in Y.
$$

By [He81, thm. 3.2.2] this problem has a unique solution which is given by

$$
v(s) = e^{s\Delta}\gamma + \int_0^s e^{(s+\sigma)\Delta}\tilde f(\sigma)\,d\sigma.
$$

Now $v = u$ by uniqueness. Moreover, the first term in the sum is a continuous map $(0,T] \to X$ by lemma 9.9 with $r = \mu = 1$ and $p = q = 2$ and the second term is continuous even on $[0,T]$ by lemma 9.7 with $p = 2$.

"⇐" Now suppose u is a solution of the integral equation (129). Hence $u(s) = e^{s\Delta}\gamma + F(s)$ where F is given by (141) with $\tilde f := f \circ u : (0,T] \to Y$ being continuous and bounded by assumption. Then lemma 9.7 (a) asserts that $F(0) = 0$ and $F \in C^0([0,T], W^{1,2})$. Hence $F \in C^0([0,T], Y)$. Now Δ generates an analytic semigroup on Y by remark 9.6 a). Therefore the map

$(0,\infty) \to Y : s \mapsto e^{s\Delta}\gamma$ is continuously differentiable by (133). Furthermore it extends continuously to zero, taking on the value γ at zero, by the strong continuity property (132). To summarize we proved that the map $u : [0,T] \to Y$ is continuous and satisfies $u(0) = \gamma$. Also when restricted to $(0,T]$ the map u takes values in $\mathcal{U}$ by assumption.

We prove the yet missing properties of u on the interval $(0,T]$. As mentioned above the map $s \mapsto e^{s\Delta}\gamma$ is continuously differentiable in Y. Moreover, this map takes values in $W^{2,1}$ and satisfies $\frac{d}{ds}e^{s\Delta}\gamma = \Delta e^{s\Delta}\gamma$ by (134). Assume we knew that $u : (0,T] \to \mathcal{U}$ was locally Hölder continuous. Then the composition $\tilde{f} = f \circ u : (0,T] \to Y$ is locally Hölder continuous, since $f : \mathcal{U} \to Y$ is locally Lipschitz by lemma 9.10. Now lemma 9.7 (b) with $q = 1$ applies. Consequently the sum $u(s) = e^{s\Delta}\gamma + F(s)$ takes values in $W^{2,1}$, is differentiable in Y and satisfies the differential equation (128).

It remains to prove that u is locally Hölder continuous on $(0,T]$ with respect to the $W^{1,2}$ norm. Set $K_1 := \sup_{(0,T]}\|\tilde{f}\|_1$ and fix $0 < s < s+h \leq T$. Then by the representation formula (129) we obtain that

$$
\begin{aligned}
u(s+h) - u(s) = &\int_0^s \left(e^{h\Delta} - \mathbb{1}\right) e^{(s-\sigma)\Delta}\tilde{f}(\sigma)\, d\sigma \\
&+ \int_s^{s+h} e^{(h+s-\sigma)\Delta}\tilde{f}(\sigma)\, d\sigma + \left(e^{h\Delta} - \mathbb{1}\right) e^{s\Delta}\gamma.
\end{aligned} \tag{146}
$$

Denote the sum of the three terms on the right hand side by $T_1 + T_2 + T_3$. By (138) with $p = 2$ and constant $C > 0$ it follows that

$$
\|T_2\|_{1,2} \leq \int_s^{s+h} \left\|e^{(h+s-\sigma)\Delta}\right\|_{\mathcal{L}(L^1,W^{1,2})}\left\|\tilde{f}(\sigma)\right\|_1 d\sigma \leq 4CK_1h^{1/4}.
$$

Now apply lemma 9.8 and (144) with $\ell = 1$ and $q = 2$ to obtain that

$$
\|T_3\|_{1,2} \leq \int_0^h \left\|\Delta e^{(\tau+s)\Delta}\gamma\right\|_{1,2} d\tau \leq C' \left\|\gamma\right\|_{1,2} \ln\left(1 + \frac{h}{s}\right). \tag{147}
$$

Note that $\ln(1 + h/s) \leq h/s \leq h/T_0$ whenever $s \in [T_0, T_1] \subset (0,T)$. Again by lemma 9.8 for $x = \tilde{f}(\sigma) \in L^1$ and (140) with constant C'' it follows that

$$
\begin{aligned}
\|T_1\|_{1,2} &\leq \int_0^s \int_0^h \left\|e^{(\tau+s-\sigma)\Delta}\tilde{f}(\sigma)\right\|_{3,2} d\tau\, d\sigma \\
&\leq \int_0^s \int_0^h \left\|e^{(\tau+s-\sigma)\Delta)}\right\|_{\mathcal{L}(L^1,W^{3,2})}\left\|\tilde{f}(\sigma)\right\|_1 d\tau\, d\sigma \\
&\leq C''K_1 \int_0^s \int_0^h (\tau + s - \sigma)^{-7/4}\, d\tau\, d\sigma \\
&= \frac{16}{3}C''K_1 \left(h^{1/4} + s^{1/4} - (h+s)^{1/4}\right) \leq \frac{16C''K_1}{3}\, h^{1/4}.
\end{aligned}
$$

□

Proof of theorem 9.3. The main idea to solve the integral equation (129) is to construct a complete metric space Z on which the map $\Psi := \Psi_\gamma$ defined in (129) acts as a strict contraction. We follow the exposition in [Ta96, ch. 15, sec. 1]. The construction is based on four facts, namely

$$\begin{aligned} e^{s\Delta} : X \to X & \text{ is a strongly continuous semigroup where } s \geq 0, \\ f : \mathcal{U} \to Y & \text{ is a locally Lipschitz continuous map,} \\ e^{s\Delta} : Y \to X & \text{ is a bounded linear operator for each } s > 0, \end{aligned} \tag{148}$$

and

$$\left\| e^{s\Delta} \right\|_{\mathcal{L}(Y,X)} \leq C s^{-3/4}, \qquad \forall s \in (0,1]. \tag{149}$$

For fact one see remark 9.6 (d) with $k = 1$ and $q = 2$, for fact two see lemma 9.10, and for facts three and four see remark 9.6 (b) with $p = 2$.

Since f is locally Lipschitz, there are positive constants ρ and L such that

$$\|f(x) - f(y)\|_1 \leq L \|x - y\|_{1,2} \tag{150}$$

for all x, y in the closed ball $B_\gamma \subset X$ of radius ρ and centered at γ. Choose ρ smaller if necessary to guarantee that $B_\gamma \subset \mathcal{U}$. Now pick $T \in (0,1]$ and consider the subset of the Banach space $C^0([0,T], X)$ given by

$$Z := \left\{ u \in C^0([0,T], X) : \|u(s) - \gamma\|_X \leq \rho \text{ for all } s \in [0,T] \right\}. \tag{151}$$

Observe that the elements of Z take values in the ball B_γ. Being closed Z is a complete metric space with respect to the sup norm. By (150) it follows that

$$\begin{aligned} \|f \circ u(s)\|_1 &\leq \|f \circ u(s) - f(\gamma)\|_1 + \|f(\gamma)\|_1 \\ &\leq L \|u(s) - \gamma\|_{1,2} + \|f(\gamma)\|_1 \\ &\leq L\rho + \|f(\gamma)\|_1 =: K_1 \end{aligned} \tag{152}$$

whenever $s \in [0,T]$ and $u \in Z$. By fact one in (148) and strong continuity (132) we can choose $T_1 = T_1(\gamma) \in (0,1]$ small enough such that

$$\left\| e^{s\Delta}\gamma - \gamma \right\|_{1,2} \leq \frac{\rho}{3}, \qquad \forall s \in [0, T_1].$$

Fix a positive constant $T_2 < \max\{T_1, \rho^4/(12CK_1)^4\}$. We prove that Ψ acts on Z whenever $T \leq T_2$. There are two conditions to be checked. To see the first condition observe that

$$\left\| \int_0^s e^{(s-\sigma)\Delta} f(u(\sigma))\, d\sigma \right\|_{1,2} \leq \int_0^s C(s-\sigma)^{-3/4} K_1 \, d\sigma = 4K_1 C s^{1/4} \leq \frac{\rho}{3}$$

for every $s \in [0,T]$. Here we used (138) with $p = 2$ and constant C and we also used estimate (152). Hence $\|\Psi u(s) - \gamma\|_{1,2} \leq \frac{2}{3}\rho$ whenever $s \in [0,T]$ by the last two estimates. The second condition is continuity of the map $\Psi u : [0,T] \to X$. Recall that $\Psi u(s) = e^{s\Delta}\gamma + F(s)$ where F is given by (141) with $\tilde{f}(s) := f \circ u(s)$.

Now $F : [0, T] \to X$ is continuous with $F(0) = 0$ by lemma 9.7 (a) with $p = 2$. The map $(0, T] \to X : s \mapsto e^{s\Delta}\gamma$ is continuous by lemma 9.9 with $q = p = 2$ and $r = \mu = 1$. By fact one in (148) and strong continuity (132) it extends continuously to zero. This proves that $\Psi u(0) = \gamma$ and that Ψ acts on Z.

Next we prove that Ψ is a strict contraction on $Z = Z(T)$ whenever $T \in (0, T_2]$ is sufficiently small. To see this fix $u, v \in Z$ and $s \in [0, T]$. Then by definition of Ψ in (129) and the estimates (149) and (150) it follows that

$$\begin{aligned}\left\|\Psi u(s) - \Psi v(s)\right\|_{1,2} &= \left\| \int_0^s e^{(s-\sigma)\Delta} \Big(f(u(\sigma)) - f(v(\sigma)) \Big)\, d\sigma \right\|_{1,2} \\ &\le \int_0^s C(s-\sigma)^{-3/4} \left\| f(u(\sigma)) - f(v(\sigma)) \right\|_1 d\sigma \\ &\le 4CLT^{1/4} \sup_\sigma \left\| u(\sigma) - v(\sigma) \right\|_{1,2}.\end{aligned}$$

Now choose $T < (8CL)^{-4}$ to obtain that $\|\Psi u - \Psi v\|_Z \le \frac{1}{2}\|u - v\|_Z$. By the Banach contraction principle Ψ has a unique fixed point u in Z; see e.g. [He81, section 1.2.6]. Now this fixed point of Ψ is a solution of the integral equation in the sense of definition 9.2 which by lemma 9.11 is a solution to the Cauchy problem (128) in the sense of definition 9.1. Uniqueness follows by uniqueness of the fixed point. □

9.1.2 Regularity

Theorem 9.12. *Fix a perturbation $\mathcal{V}$ that satisfies* (V0)–(V1) *and a loop $\gamma \in \Lambda M$. Then there is a constant $T = T(\gamma) > 0$ and a unique smooth solution*

$$u : (0, T] \times S^1 \to M, \qquad u(0, \cdot) = \gamma(\cdot),$$

of the heat equation (125) which is continuous on $[0, T] \times S^1$.

Proof. Set $f = \phi \circ \pi$ where ϕ and π are defined by (125) and (127), respectively. By theorem 9.3 there is a unique solution $u \in C^0([0, T], X)$ taking values in $\mathcal{U}$ of the integral equation (129) with $u(0) = \gamma$. The map $\tilde{f} := f \circ u : (0, T] \to Y$ is locally Hölder continuous and bounded by lemma 9.11 and definition 9.1. We denote this bound by

$$K_1 := \sup_{(0,T]} \|\tilde{f}\|_1.$$

For $s \in [0, T]$ define $F(s)$ by (141). In this notation the integral equation (129) becomes $u(s) = e^{s\Delta}\gamma + F(s)$. From now on $W^{k,p}$ abbreviates $W^{k,p}(S^1, \mathbb{R}^N)$ whenever it is convenient.

Step 1. *Fix $q \ge 2$. Then $u : (0, T] \to W^{1,q}(S^1)$ is locally Hölder continuous.*

Now if $0 < s < s + h \le T$ and $h \le s$, then by the representation formula (129) we obtain that

$$\begin{aligned}u(s+h) - u(s) = &\int_0^s \left(e^{h\Delta} - \mathbb{1}\right) e^{(s-\sigma)\Delta} \tilde{f}(\sigma)\, d\sigma \\ &+ \int_s^{s+h} e^{(h+s-\sigma)\Delta} \tilde{f}(\sigma)\, d\sigma + \left(e^{h\Delta} - \mathbb{1}\right) e^{s\Delta}\gamma.\end{aligned}$$

Denote the sum of the three terms on the right hand side by $T_1 + T_2 + T_3$. By (140) for $\ell = 0$ and $q = k = 1$ and with constant C it follows that

$$\|T_2\|_{1,q} \le \int_s^{s+h} \left\|e^{(h+s-\sigma)\Delta}\right\|_{\mathcal{L}(L^1,W^{1,q})} \left\|\tilde{f}(\sigma)\right\|_1 d\sigma \le 2qCK_1 h^{1/2q}.$$

To estimate T_3 for $q > 2$ we apply lemma 9.8 for $x = e^{\frac{s}{2}\Delta}\gamma \in W^{1,q}$ to get that

$$\begin{aligned}
\|T_3\|_{1,q} &\le \int_0^h \left\|\Delta e^{(\tau+\frac{s}{2})\Delta} e^{\frac{s}{2}\Delta}\gamma\right\|_{1,q} d\tau \\
&\le \int_0^h \left\|e^{(\tau+s)\Delta)}\right\|_{\mathcal{L}(W^{1,2},W^{3,q})} \left\|\gamma\right\|_{1,2} d\tau \\
&\le \frac{4qC'}{q-2} \|\gamma\|_{1,2} \left(s^{-\frac{q-2}{4q}} - (s+h)^{-\frac{q-2}{4q}}\right) \\
&\le \frac{4qC'}{q-2} \|\gamma\|_{1,2}\, \mu h.
\end{aligned}$$

Here step three is by (140) with constant C'. The last step is valid for all $s \in [T_0, T_1] \subset (0, T)$ and every sufficiently small $h > 0$. The constant μ depends on T_0, T_1, and $(q-2)/4q$. In the case $q = 2$ we obtained by (147) that $\|T_3\|_{1,2} \le c'\|\gamma\|_{1,2} {T_0}^{-1} h$ for $s \in [T_0, T_1]$. Next use (140) with constant C'' to obtain that

$$\begin{aligned}
\|T_1\|_{1,q} &\le \int_0^s \int_0^h \left\|e^{(\tau+s-\sigma)\Delta} \tilde{f}(\sigma)\right\|_{3,q} d\tau\, d\sigma \\
&\le \int_0^s \int_0^h \left\|e^{(\tau+s-\sigma)\Delta)}\right\|_{\mathcal{L}(L^1,W^{3,q})} \left\|\tilde{f}(\sigma)\right\|_1 d\tau\, d\sigma \\
&\le C''K_1 \int_0^s \int_0^h (\tau+s-\sigma)^{-2+1/2q}\, d\tau\, d\sigma \\
&= \frac{4q^2}{2q-1} C''K_1 \left(h^{1/2q} + s^{1/2q} - (h+s)^{1/2q}\right) \le \frac{4q^2 C'' K_1}{2q-1} h^{1/2q}.
\end{aligned}$$

In the last step we used that $s^{1/2q} \le (s+h)^{1/2q}$, because $h > 0$.

Step 2. *Fix $p \ge 1$. Then $f \circ u : (0, T] \to L^p(S^1)$ is locally Hölder continuous.*

The map $u : (0, T] \to W^{1,2p}$ is locally Hölder continuous by step 1 with $q = 2p$. Revisiting the proof of lemma 9.10 replacing $W^{1,2}$ by $W^{1,2p}$ and L^1 by L^p we observe that $f : W^{1,2p} \supset \mathcal{U}' \to L^p$ is locally Lipschitz. Here $\mathcal{U}'$ is a sufficiently small neighborhood of $W^{1,2p}(S^1, M)$ in $W^{1,2p}(S^1, \mathbb{R}^N)$ such that all elements of $\mathcal{U}'$ take values in the neighborhood U of M in $\mathbb{R}^N$ which was used to define π in (127). Hence the composition $f \circ u : (0, T] \to L^p$ is locally Hölder continuous.

Step 3. *Fix $p \ge 2$. Then $u \in C^1((0, T], L^p(S^1, \mathbb{R}^N))$.*

Recall that $u(s) = e^{s\Delta}\gamma + F(s)$. Now Δ generates the analytic semigroup $e^{s\Delta}$ on L^p by remark 9.6 (a). Hence the map $(0, \infty) \to L^p : s \mapsto e^{s\Delta}\gamma$ is real analytic by (133). To deal with the F part fix a constant $\delta \in (0, T)$. On the other hand, step 2 and lemma 9.7 (b) for $q = p$ and the map $\tilde{f} := f \circ u : (0, T] \to L^p$, which is bounded in L^1 by the constant K_1, show that $F:(0,T]\to L^p$ is continuously differentiable.

Step 4. *Fix constants $p\geq 2$ and $\delta\in(0,T)$. Then the map $[\delta,T]\times S^1\to\mathbb{R}^N:(s,t)\mapsto u(s,t)$ is of class $\mathcal{W}^{1,p}$.*

By definition of the space $\mathcal{W}^{1,p}$ we need to show that u, $\Delta u:=\partial_t\partial_t u$, and $\partial_s u$ are in $L^p([\delta,T]\times S^1,\mathbb{R}^N)$. By step 2 and step 3 we know that $f\circ u$ and $\frac{d}{ds}u$ are in $C^0((0,T],L^p)$. On the other hand, by theorem 9.3 we have $\Delta u(s)\in L^1$ and

$$\frac{d}{ds}u(s)=\Delta u(s)+f\circ u(s) \tag{153}$$

for every $s\in(0,T]$. Hence Δu is in $C^0((0,T],L^p)$ as well. Now every element of $C^0((0,T],L^p)$ restricts to an element of $L^p([\delta,T],L^p)=L^p([\delta,T]\times S^1,\mathbb{R}^N)$.

Step 5. *The map u defined on $[0,T]\times S^1$ takes values in M. It is continuous on $[0,T]\times S^1$, smooth on $(0,T]\times S^1$, and satisfies (126).*

Recall that prior to (127) we identified a compact neighborhood U of M in $\mathbb{R}^N$ with a neighborhood of the zero section of the normal bundle $pr:\nu_M\to M$ of M in $\mathbb{R}^N$. Moreover, every element of the neighborhood $\mathcal{U}$ of the space of $W^{1,2}$ loops ΛM takes values in U. Since u takes values in $\mathcal{U}$ by theorem 9.3, we identify $u(s,t)\in U$ with the pair $(v(s,t),\eta(s,t))$ where η is the field of normal vectors corresponding to u and $v(s,t):=pr\circ u(s,t)\in M$ are the corresponding base points. On the normal bundle fix the Riemannian metric provided by the ambient Euclidean space $\mathbb{R}^N$ and the associated Levi Civita connection ∇'. Then (153) translates into the pair of equations

$$\begin{pmatrix}\partial_s v\\ {\nabla_s}'\eta\end{pmatrix}=\begin{pmatrix}\partial_t\partial_t v-P(v)\left(\partial_t v,\partial_t v\right)+\operatorname{grad}\mathcal{V}(v)\\ {\nabla_t}'{\nabla_t}'\eta\end{pmatrix}. \tag{154}$$

Now the section η of the normal bundle satisfies $\eta(0,\cdot)=0$, since $u(0)=\gamma\in\Lambda M$. Moreover, by (154) and integration by parts we obtain that

$$\frac{d}{ds}\left\|\eta(s)\right\|_2^2=2\left\langle{\nabla_s}'\eta(s),\eta(s)\right\rangle=2\left\langle{\nabla_t}'{\nabla_t}'\eta(s),\eta(s)\right\rangle=-2\left\|{\nabla_t}'\eta(s)\right\|_2^2\leq 0$$

for every $s\in(0,T]$. Here we used that the section $\eta(s)$, its s derivative, and the first two t derivatives are L^p integrable over S^1 whenever $p\geq 2$; see proof of step 4. Of course, the inequality above proves that $\eta=0$. But this means that $u=v$. Hence u satisfies the heat equation by the first component in (154) and by step 4 it is in $\mathcal{W}^{1,p}([\delta,T]\times S^1,M)$ for $p\geq 2$. Thus u is smooth on $(\delta,T]\times S^1$ by theorem 4.2 and this is true for all $\delta\in(0,T)$. □

9.1.3 Global existence and asymptotic behavior

Theorem 9.13 (Global forward existence)**.** *Fix a perturbation $\mathcal{V}$ that satisfies* (V0)–(V3)*, a time $T>0$, and an initial loop $\gamma\in\Lambda M$. Then the following is true. Every solution u of the Cauchy problem (126) on $[0,T)$ with $u(0)=\gamma$ extends to a smooth solution on $(0,\infty)$.*

Proof. Assume by contradiction and without loss of generality (rename if necessary) that $[0,T)$ is the maximal interval of existence. This means u does not extend to a solution on $[0,T')$ for $T'>T$. By theorem 9.12 we may assume that $u \in C^0([0,T),\Lambda M)\cap C^\infty((0,T),\mathcal{L}M)$. The idea is to prove in two steps that u extends to time T continuously in ΛM. Then by theorem 9.3 we solve the integral equation (129) for the initial value $u(T)$ to get a solution on $[T,T']$ for some $T'>T$. Concatenation then provides a solution on $[0,T']$ and by lemma 9.11 every solution of the integral equation solves the heat equation (125). This contradicts maximality of $[0,T)$.

STEP 1. *There is a constant K_1 such that $\|\phi(u(s))\|_1 \le K_1$ for every $s\in[0,T)$.*
Since the action functional $\mathcal{S}_\mathcal{V}$ is decreasing along solutions, it follows that

$$\|\partial_t u(s)\|_2^2 = 2\mathcal{S}_\mathcal{V}(u(s)) + 2\mathcal{V}(u(s)) \le 2\mathcal{S}_\mathcal{V}(u(0)) + 2C_0 \le \|\partial_t\gamma\|_2^2 + 4C_0$$

for every $s\in[0,T)$. The first and the last step are by definition of $\mathcal{S}_\mathcal{V}$. We used axiom (V0) with constant $C_0=C_0(\mathcal{V})>0$ in the second and the last step. Now we obtain for ϕ defined by (125) the estimate

$$\begin{aligned}\|\phi(u(s))\|_1 &\le \|P\|_\infty \|\partial_t u(s)\|_2^2 + \|\operatorname{grad}\mathcal{V}(u(s))\|_\infty \\ &\le \|P\|_\infty \left(\|\partial_t\gamma\|_2^2 + 4C_0\right) + C_0 =: K_1\end{aligned}$$

for every $s\in[0,T)$. Here we used the second estimate of axiom (V0).

STEP 2. *The limit $\lim_{s\to T} u(s)$ exists in $W^{1,2}(S^1)$.*
Recall that $f := \phi\circ\pi$ where π is defined by (127). Note that $\pi\circ u = u$, because u takes values in ΛM. Hence by step 1 the map $\tilde f := f\circ u = \phi\circ u : [0,T)\to L^1$ is bounded from above by the constant K_1. Now fix a constant $p>2$, choose $\max\{0,T-1\}<s<\sigma<T$, and set $h:=\sigma-s$. By lemma 9.11 each solution of the Cauchy problem solves the integral equation (129). Recall that the difference $u(\sigma)-u(s) = u(s+h)-u(s) = T_1+T_2+T_3$ is given by (146) and that we already have $W^{1,2}$ estimates for the terms T_j, namely

$$\|T_1\|_{1,2} \le \frac{16C''K_1}{3}(\sigma-s)^{\frac14}, \qquad \|T_2\|_{1,2} \le 4CK_1(\sigma-s)^{\frac14},$$

and

$$\|T_3\|_{1,2} \le C'\|\gamma\|_{1,2}\ln\left(\frac{\sigma}{s}\right).$$

This shows that $\|u(\sigma)-u(s)\|_{1,2}$ converges to zero as $s<\sigma$ both converge to T. Hence the sequence is Cauchy and therefore the desired limit exists. □

Theorem 9.14 (Asymptotic forward limit)**.** *Fix $\gamma\in\Lambda M$. If all critical points of $\mathcal{S}_\mathcal{V}$ of action less than $\mathcal{S}_\mathcal{V}(\gamma)$ are nondegenerate, then the solution u in theorem 9.13 converges to one of them in $C^2(S^1)$ as $s\to\infty$.*

Proof. Observe that the solution u provided by theorem 9.13 is smooth on $[\delta,\infty)$ for each $\delta>0$. Now apply theorem 1.8 (F) to the shifted solution $\tilde u(\cdot) = u(\cdot+\delta)$ which is smooth on $[0,\infty)$. □

9.1.4 Differentiable dependence on initial value

Theorem 9.15. *Fix a perturbation $\mathcal{V}$ that satisfies* (V0)–(V1). *For any initial loop $\gamma \in \Lambda M$ consider the solution $u_\gamma : [0,T] \to \mathcal{U} \subset X$ with $u_\gamma(0) = \gamma$ of the Cauchy problem (128) provided by theorem 9.3. Then*

$$\varphi_s : \mathcal{U} \supset \Lambda M \to \mathcal{U} \subset X : \quad \gamma \mapsto u_\gamma(s)$$

is a continuously differentiable map for each time $s \in [0,T]$.

Proof. It is known that the degree of smoothness of the map φ_s coincides with the degree of smoothness of the perturbation f in (128); see e.g. [He81, thm. 3.4.4]. Hence it remains to prove that f is of class C^1 on $\mathcal{U}$ and this is the content of lemma 9.16 below.

However, since [He81, thm. 3.4.4] is stated in a slightly different situation, we briefly recall the main steps of the proof in our setting. Fix $\gamma \in \Lambda M$ and positive constants ρ and L such that the Lipschitz estimate (150) for f holds on the closed ball $B^\rho(\gamma) \subset \mathcal{U} \subset X$ of radius ρ about γ. Suppose $T \in (0,1]$ and recall that the complete metric space Z is given by

$$Z = Z_\gamma(T,\rho) := \left\{ u \in C^0([0,T],X) : \|u(s) - \gamma\|_{1,2} \le \rho \text{ for all } s \in [0,T] \right\}.$$

By μ we denote the constant in (139) for $p = 2$ and set $\delta := \rho/6\mu$. Then

$$\left\|e^{s\Delta}(x-\gamma)\right\|_{1,2} \le \left\|e^{s\Delta}(x-\gamma)\right\|_2 + \left\|e^{s\Delta}\partial_t(x-\gamma)\right\|_2 \le 2\mu \|x-\gamma\|_{1,2} \le \frac{\rho}{3}$$

for all $s > 0$ and $x \in B^\delta(\gamma) =: \mathcal{B}$. For $x \in \mathcal{B}$ and $u \in Z$ define $\Psi_x u$ by (129). Then for every sufficiently small $T > 0$ the map

$$\Psi : \mathcal{B} \times Z \to Z, \quad (x,u) \mapsto \Psi_x u$$

is a *uniform contraction on Z*, namely

$$\|\Psi_x u - \Psi_x v\|_Z \le \frac{1}{2} \|u - v\|_Z$$

for all $x \in \mathcal{B}$ and $u,v \in Z$. To see this choose $x \in \mathcal{B}$ and $u \in Z$. The main point is to prove that $\Psi_x u$ lies in Z. It follows as in the proof of theorem 9.3 that $\Psi_x u = e^{\cdot\Delta}x + F \in C^0([0,T],X)$ and $(\Psi_x u)(0) = x$. Now

$$\begin{aligned}
\|(\Psi_x u)(s) - \gamma\|_{1,2} &= \left\| e^{s\Delta}x - \gamma + \int_0^s e^{(s-\sigma)\Delta} f(u(\sigma))\, d\sigma \right\|_{1,2} \\
&\le \left\|e^{s\Delta}(x-\gamma)\right\|_{1,2} + \left\|e^{s\Delta}\gamma - \gamma\right\|_{1,2} + 4K_1 C s^{1/4} \\
&\le \frac{\rho}{3} + \frac{\rho}{3} + \frac{\rho}{3}
\end{aligned}$$

for every $s \in (0,T_2]$. See the proof of theorem 9.3 for the constants T_2, K_1, and C and the last two of the three terms estimated in the final step. The estimate

$\|\Psi_x u - \Psi_x v\|_Z \le \frac{1}{2}\|u-v\|_Z$ is independent of x and follows exactly as in the proof of theorem 9.3 whenever $T < (8CL)^{-4}$.

Next we prove that the uniform contraction Ψ of the ball $Z \subset C^0([0,T],X)$ of radius ρ about the constant in s map γ is continuously differentiable. To see this observe first (see [He81, lemma 3.4.3]) that the map $Z \to C^0([0,T],Y) : u \mapsto f \circ u =: \tilde f$ is of class C^1, because $f : \mathcal{U} \to Y = L^1(S^1,\mathbb{R}^N)$ is of class C^1 by lemma 9.16. Secondly, being linear and by (138) for $p=2$ the map

$$C^0([0,T],Y) \to C^0([0,T],X), \quad \tilde f \mapsto \int_0^{\cdot} e^{(\cdot-\sigma)\Delta}\tilde f(\sigma)\, d\sigma$$

is smooth. Thirdly, again by linearity and by remark 9.6 (d) for $k=1$ and $q=2$ the map $\mathcal{B} \to C^0([0,T],X) : x \mapsto e^{\cdot\Delta}x$ is smooth. Since Ψ is equal to the sum of map three with the composition of maps one and two, it follows that $\Psi(x,u)$ is smooth in x and of class C^1 in u. Hence by [He81, sec. 1.2.6, second theorem] the map $\mathcal{B} \to Z$ which assigns to x the unique fixed point u_x of $\Psi_x : Z \to Z$ is of class C^1. Observe that u_x solves the Cauchy problem (128) with $u_x(0) = x$ by lemma 9.11 and it is C^∞ smooth on $(0,T]$ actually taking values in $\mathcal{L}M$ and solving the heat equation (125) by theorem 9.12. Hence for $s \in [0,T]$ the composition $\mathcal{B} \to Z \to \mathcal{U} \subset X : x \mapsto u_x \mapsto u_x(s)$ is of class C^1. For $\sigma > T$ compose this map with the smooth map $u_x(s) \mapsto u_x(\sigma)$ using that u_x actually extends smoothly to $(0,\infty)$ by theorem 9.13. □

The following lemma is used in the proof of theorem 9.15.

Lemma 9.16. *Fix a perturbation $\mathcal{V}$ that satisfies* (V0)–(V1). *Then the map*

$$df = d(\phi \circ \pi) : X \supset \mathcal{U} \to \mathcal{L}(X,Y)$$

is continuous. Here ϕ and π are given by (125) and (127), respectively.

Proof. Fix $\gamma \in \mathcal{U}$ and a sufficiently small constant $\rho > 0$ such that the ball $B := \{z \in X : \|z-\gamma\|_{1,2} \le \rho\}$ is contained in $\mathcal{U}$. Given $x \in B$ we need to show that

$$\|df(x) - df(y)\|_{\mathcal{L}(X,Y)} := \sup_{\|\xi\|_{1,2}\le 1} \|df(x)\xi - df(y)\xi\|_1 \longrightarrow 0$$

whenever $y \in B$ converges to x in the $W^{1,2}$ topology. To prove this we will use the constants κ_j and β_j defined in the proof of lemma 9.10 for $j=1,2$ and set

$$\kappa_3 := \max_{q\in U}\left\|d^3\, pr(q)\right\|_{\mathcal{L}((\mathbb{R}^N)^{\times 3},T_{pr(q)}M)}, \quad \beta_3 := \max_{q\in M}\left\|d^2 P(q)\right\|_{\mathcal{L}((T_qM)^{\times 3},T_q^\perp M)}.$$

Now choose $x, y \in B$, set $v = y - x$, and pick $\xi \in T_xX = W^{1,2}(S^1,\mathbb{R}^N)$. Fix $t \in S^1$ and set

$$h(v(t)) = P|_{pr(x(t)+v(t))}\Big(d\,pr|_{x(t)+v(t)}(\dot x(t)+\dot v(t)), d\,pr|_{x(t)+v(t)}(\dot x(t)+\dot v(t))\Big)$$

where $\dot x = \partial_t x$. By abuse of notation, but for simplicity, we abbreviate from now on the projection pr by π and the point $x(t)$ by x and similarly for v and

ξ. In addition, we set $q_\tau = x(t) + \tau v(t)$. Since all maps involved are C^1 and take values in the ambient $\mathbb{R}^N$, there exists $\tau = \tau(t) > 0$ such that

$$
\begin{aligned}
&|dh(0)\xi - dh(v)\xi| = \left|\frac{d}{d\tau} dh(\tau v)\xi\right| \\
&= \left|\frac{d}{d\tau}\frac{d}{d\rho}\Big|_0 h(\tau v + \rho\xi)\right| \\
&= \left|\frac{d}{d\tau}\frac{d}{d\rho}\Big|_0 P|_{\pi(q_\tau+\rho\xi)}\Big(d\pi|_{q_\tau+\rho\xi}(\dot x + \tau\dot v + \rho\dot\xi), d\pi|_{q_\tau+\rho\xi}(\dot x + \tau\dot v + \rho\dot\xi)\Big)\right| \\
&= \Bigg|\frac{d}{d\tau} dP|_{\pi(q_\tau)}\Big(d\pi|_{q_\tau}\xi, d\pi|_{q_\tau}(\dot x + \tau\dot v), d\pi|_{q_\tau}(\dot x + \tau\dot v)\Big) \\
&\quad + 2\frac{d}{d\tau} P|_{\pi(q_\tau)}\Big(d^2\pi|_{q_\tau}(\xi, \dot x + \tau\dot v) + d\pi|_{q_\tau}\dot\xi, d\pi|_{q_\tau}(\dot x + \tau\dot v)\Big)\Bigg| \\
&= \Bigg|d^2P|_{\pi(q_\tau)}\Big(d\pi|_{q_\tau}v, d\pi|_{q_\tau}\xi, d\pi|_{q_\tau}(\dot x + \tau\dot v), d\pi|_{q_\tau}(\dot x + \tau\dot v)\Big) \\
&\quad + dP|_{\pi(q_\tau)}\Big(d^2\pi|_{q_\tau}(v,\xi), d\pi|_{q_\tau}(\dot x + \tau\dot v), d\pi|_{q_\tau}(\dot x + \tau\dot v)\Big) \\
&\quad + 2dP|_{\pi(q_\tau)}\Big(d\pi|_{q_\tau}\xi, d^2\pi|_{q_\tau}(v, \dot x + \tau\dot v) + d\pi|_{q_\tau}\dot v, d\pi|_{q_\tau}(\dot x + \tau\dot v)\Big) \\
&\quad + 2dP|_{\pi(q_\tau)}\Big(d\pi|_{q_\tau}v, d^2\pi|_{q_\tau}(\xi, \dot x + \tau\dot v) + d\pi|_{q_\tau}\dot\xi, d\pi|_{q_\tau}(\dot x + \tau\dot v)\Big) \\
&\quad + 2P|_{\pi(q_\tau)}\Big(d^3\pi|_{q_\tau}(v,\xi,\dot x + \tau\dot v) + d^2\pi|_{q_\tau}(\xi,\dot v) + d^2\pi|_{q_\tau}(v,\dot\xi), d\pi|_{q_\tau}(\dot x + \tau\dot v)\Big) \\
&\quad + 2P|_{\pi(q_\tau)}\Big(d^2\pi|_{q_\tau}(\xi, \dot x + \tau\dot v) + d\pi|_{q_\tau}\dot\xi, d^2\pi|_{q_\tau}(v, \dot x + \tau\dot v) + d\pi|_{q_\tau}\dot v\Big)\Bigg| \\
&\le 2\left(\kappa_1^3\beta_3 + 5\kappa_1^2\kappa_2\beta_2 + 2\kappa_1\kappa_3\beta_1 + 2\kappa_1^2\beta_1\right)\Big(|\dot x|^2 + |\dot v|^2\Big)|v|\cdot|\xi| \\
&\quad + 2\kappa_1^3\beta_2\left(|\dot x| + |\dot v|\right)|v|\cdot|\xi| + 2\kappa_1^2\beta_1|\dot v|\cdot\left|\dot\xi\right| \\
&\quad + 2\kappa_1\left(\kappa_1^2\beta_2 + 2\kappa_2\beta_1\right)\left(|\dot x| + |\dot v|\right)|\dot v|\cdot|\xi| \\
&\quad + 4\kappa_1\kappa_2\beta_1\left(|\dot x| + |\dot v|\right)|v|\cdot\left|\dot\xi\right|.
\end{aligned}
$$

Now integrate this pointwise inequality over $t \in S^1$ to obtain that

$$\|dh(0)\xi - dh(v)\xi\|_1 \le \mu\,\|v\|_{1,2}\,\|\xi\|_{1,2}\,.$$

Here $\mu > 0$ depends on the constants ρ, κ_j, β_j, $j = 1, 2, 3$, the Sobolev constant associated to the embedding $W^{1,2}(S^1) \hookrightarrow L^\infty(S^1)$, and on $\|\gamma\|_{1,2}$. Recall from (145) that

$$\|\dot x\|_2 \le \|\gamma\|_{1,2} + \rho, \qquad \|\dot v\|_2 \le 2\rho.$$

We also used Hölder's inequality $\|fg\|_1 \le \|f\|_2\|g\|_2$. The estimate for the $\mathcal{V}$ part follows similarly using axioms (V0)–(V3). $\square$

9.2 Morse homology and singular homology

Let $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ be a perturbation that satisfies (V0)–(V3). Assume a is a regular value of $\mathcal{S}_\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ and $\mathcal{S}_\mathcal{V}$ is *Morse–Smale* below level a. Fix a choice ν of orientations of the unstable manifolds of all $x \in \mathcal{P}^a(\mathcal{V})$. Our goal in this section is to propose a strategy of how to prove theorem 1.14. In other words, we aim to calculate singular homology of the sublevel set $\mathcal{L}^a M$ in terms of the Morse complex

$$\mathrm{CM}(\mathcal{L}^a M, \mathcal{S}_\mathcal{V}, \nu) := (\mathrm{CM}^a_*(\mathcal{V}, \nu), \partial(\mathcal{V}, \nu))$$

defined in section 8.2. Recall that its chain groups are generated by the critical points x of $\mathcal{S}_\mathcal{V} : \mathcal{L}^a M \to \mathbb{R}$, together with the information about the chosen orientations, and the Morse boundary operator counts heat flow trajectories between critical points of Morse index difference one with appropriate signs. These signs are determined by the chosen orientations of the unstable manifolds.

A standard idea is to consider an intermediate chain complex which, on one hand, is isomorphic to the Morse complex and, on the other hand, whose homology is known to represent singular homology. A well known candidate is the cellular complex associated to a suitable filtration; see [M65] in the case of a finite dimensional manifold. For a Banach manifold with a flow generated by a C^1 tangent vector field a suitable filtration has been constructed by Abbondandolo and Majer in [AM06] where they also provide full details of the natural isomorphism between Morse and singular homology. While our heat flow situation does not quite match the assumptions in [AM06] the structure of all proofs still carries over. Concerning details it remains to replace some of their tools with those provided in the present text.

To meet the assumptions in [AM06] we choose the Hilbert manifold $\Lambda^a M$ of $W^{1,2}$ loops in M of action less or equal to a. Note that the L^2 gradient $\nabla_t \partial_t + \operatorname{grad} \mathcal{V}$ of $\mathcal{S}_\mathcal{V}$ is surely not a tangent vector field of ΛM as it is not even defined everywhere. Increasing regularity of the loops by choosing $W^{k,2}$ loops, $k \geq 2$, does not help either, because the L^2 gradient looses regularity. (When evaluated on any $W^{k,2}$ loop the resulting vector field along that loop is only of class $W^{k-2,2}$, whereas $W^{k,2}$ is required for tangent vector fields.) On the other hand, the L^2 gradient is a tangent vector field of $\mathcal{L}M$, the set of smooth loops, but $\mathcal{L}M$ is not a Banach manifold. However, we proved in section 9.1 that the negative L^2 gradient generates at least a C^1 *semi*flow

$$\varphi : (0, \infty) \times \Lambda M \to \Lambda M$$

which extends continuously to zero. In this case, while the large scale structure of proof still carries over from the case of a genuine flow (see [AM06] and [M69, S90] in finite dimensions) the arguments to prove major steps do not. In what follows we recall the major steps to construct the desired natural isomorphism and comment on how to prove them in our semiflow situation. From now on all statements are with respect to the manifold $\Lambda^a M$.

Cellular filtration

The construction of a suitable cellular filtration requires a different idea than the one in [AM06] to choose open neighborhoods of the critical points and let them flow in forward time. In our case the union of these sets over positive time would not be an open set. The problem is that the heat flow φ_t in general does not map open sets to open sets. (Each $W^{1,2}$ loop becomes smooth after any positive time.) However, the heat flow is continuous and therefore *preimages* of open sets are open.

The idea is to generalize the notion of Conley index pair associated to a flow invariant set S. Here S is simply a critical point. Assume for each critical point $x \in \mathcal{P}^a(\mathcal{V})$ we have a pair of open subsets $L \subset N$ of $\Lambda^a M$ such that $x \in N \setminus \mathrm{cl} L$, no other critical point is contained in the closure of N, and

$$\begin{aligned} \gamma \in L,\ \varphi_{[0,t]}(\gamma) \subset N \quad &\Rightarrow \quad \varphi_t(\gamma) \in L, \\ \gamma \in N \setminus L \quad &\Rightarrow \quad \exists t > 0 : \varphi_{[0,t]}(\gamma) \subset N. \end{aligned}$$

The first condition says that L is **positively invariant in N** and the second says that every flow line which leaves N goes through L first. Hence L is called the **exit set** of N. Below we denote (N, L) by (N_x, L_x). We say a set is **positively invariant** if it is invariant under the forward semiflow φ. Assume further that

$$\mathrm{cl} N_x \cap (\varphi_t)^{-1} (\mathrm{cl} N_y) = \emptyset, \quad \forall t > 0, \tag{155}$$

for all pairs of critical points $x \neq y$ with $\mathrm{ind}_{\mathcal{V}}(x) \leq \mathrm{ind}_{\mathcal{V}}(y)$. This condition guarantees that the Morse index strictly decreases whenever there is a trajectory from N_x to N_y. The proof of the corresponding construction in [AM06, prop. 2.6] uses the Morse–Smale and the Palais–Smale condition which are both satisfied in our case.

If the Morse index of x is zero, it is natural to take a (strict) sublevel set with respect to a value $c + \varepsilon$ slightly larger than $c = \mathcal{S}_{\mathcal{V}}(x)$. Since x is a nondegenerate local minimum, one can choose $\varepsilon > 0$ sufficiently small such that the connected component containing x is positively invariant and contains no other critical points. This connected component is N_x and we set $L_x := \emptyset$. For $T > 1$ sufficiently large consider the open positively invariant set given by

$$F_0 := \bigcup_{\substack{x \in \mathcal{P}^a(V) \\ \mathrm{ind}_V(x) = 0}} (\varphi_T)^{-1} N_x.$$

We tacitly intersect all sets with $\Lambda^a M$.

Assume further that all points in the exit sets of the critical points of Morse index one enter F_0 in uniform time. By choosing T larger, if necessary, we assume this time is T. Note that each individual such point enters F_0 in finite time by theorem 9.14. Now define

$$F_1 := F_0 \cup \bigcup_{\substack{x \in \mathcal{P}^a(V) \\ \mathrm{ind}_V(x) = 1}} (\varphi_T)^{-1} N_x.$$

This set is open by continuity of the forward flow and it is positively invariant, because each point enters in finite time either the positively invariant set F_0 or one of the sets N_x. Now the only way to leave N_x is through the exit set L_x. But all these points end up after time T in F_0. Let $N = N(a)$ be the maximal Morse index among the critical points below action level a. Then for $k = 1, \ldots, N$ and by choosing $T > 1$ again larger if necessary define

$$F_k := F_{k-1} \cup \bigcup_{\substack{x \in \mathcal{P}^a(V) \\ \mathrm{ind}_V(x) = k}} (\varphi_T)^{-1} N_x.$$

Set $F_k = \emptyset$ whenever $k < 0$ or $k > N$. Singular homology H_* is understood to have integer coefficients. The next steps are then to prove that

$$\mathrm{H}_*(\Lambda^a M) \simeq \mathrm{H}_*(F_N)$$

and that

$$\mathcal{F} := (F_k)_{k \in \mathbb{Z}}$$

is a **cellular filtration of F_N**. By definition this means that $\mathcal{F}$ is a sequence of subsets of the topological space F_N such that

(i) $F_k \subset F_{k+1}$ for every $k \in \mathbb{Z}$;

(ii) every singular simplex in F_N is a simplex in F_k for some k;

(iii) relative singular homology $\mathrm{H}_\ell(F_k, F_{k-1})$ vanishes whenever $\ell \neq k$.

Whereas (i) is by construction of the F_k, condition (ii) follows since each F_k is open. The main idea to prove (iii) is to write F_k as union of F_{k-1} and a set

$$U_k := \bigcup_{\substack{x \in \mathcal{P}^a(V) \\ \mathrm{ind}_V(x) = k}} U(x)$$

where the open sets $U(x)$ have the property that they are pairwise disjoint as a consequence of (155) and the pair $(U(x), U(x) \cap F_{k-1})$ is homotopy equivalent to a k-dimensional disk modulo its boundary. Think of $U(x)$ as the unstable manifold of x suitably thickened. Then use excision to conclude that

$$\begin{aligned} \mathrm{H}_*(F_k, F_{k-1}) &\simeq \mathrm{H}_*(U_k, U_k \cap F_{k-1}) \\ &\simeq \bigoplus_{\substack{x \in \mathcal{P}^a(V) \\ \mathrm{ind}_V(x) = k}} \mathrm{H}_*(U(x), U(x) \cap F_{k-1}). \end{aligned}$$

This implies (iii). More precisely, it follows that

$$\mathrm{H}_k(F_k, F_{k-1}) \simeq \bigoplus_{\substack{x \in \mathcal{P}^a(V) \\ \mathrm{ind}_V(x) = k}} \mathbb{Z}\, x.$$

Cellular filtration and singular homology

The cellular complex $\mathrm{C}\mathcal{F} := \left(\mathrm{C}_*\mathcal{F}, \partial_*^{triple}\right)$ consists of the chain groups

$$\mathrm{C}_k\mathcal{F} := \begin{cases} \mathrm{H}_k(F_k, F_{k-1}) & , k \in \{0, 1, \ldots, N\}, \\ \{0\} & , \text{otherwise}, \end{cases}$$

and the boundary operator

$$\partial_k^{triple} : \mathrm{C}_k\mathcal{F} \to \mathrm{C}_{k-1}\mathcal{F}$$

associated to the triple (F_k, F_{k-1}, F_{k-2}). More precisely, it is the composition

$$\mathrm{H}_k(F_k, F_{k-1}) \to \mathrm{H}_{k-1}(F_{k-1}) \to \mathrm{H}_{k-1}(F_{k-1}, F_{k-2})$$

of the boundary homomorphism of the pair (F_k, F_{k-1}) and the homomorphism induced by inclusion. It is well known that the homology of the cellular complex associated to a filtration of a topological space is naturally isomorphic to singular homology of the space itself; see e.g. [D80, sec. V.1] or [M65]. This means that the homology of the cellular complex $\mathrm{C}\mathcal{F}$ is isomorphic to singular homology of F_N. Hence we obtain that

$$\mathrm{H}_k\left((\mathrm{C}_*\mathcal{F}, \partial_*^{triple})\right) \simeq \mathrm{H}_k(F_N) \simeq \mathrm{H}_k(\Lambda^a M), \qquad k \in \mathbb{Z}.$$

Cellular filtration and Morse homology

The final step is to construct an isomorphism

$$\Theta_k = \Theta_k(\mathcal{V}, v, a) : \mathrm{CM}_k^a(\mathcal{V}, v) \to \mathrm{C}_k\mathcal{F} := \mathrm{H}_k(F_k, F_{k-1})$$

as in [AM06, thm. 2.8] which is induced by orientation preserving embeddings of the canonically oriented closed unit ball $D^k \subset \mathbb{R}^k$ into the unstable manifolds of the critical points of index k and prove that Θ commutes with the two boundary operators. This concludes our sketch of proof of theorem 1.14. Full details will be provided in a forthcoming paper.

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