

- (a) Start with a one-form $\alpha \in \Omega^1(\mathbb{R}^{2k})$ of constant rank $2k$. Show there exist local coordinates x^1, \dots, x^{2k} such that

$$\alpha \wedge (d\alpha)^{k-1} = g dx^2 \wedge dx^3 \wedge \dots \wedge dx^{2k}$$

for some positive g . Define a function f so that $f^k = g$ and define a one-form σ so that $\alpha = f\sigma$. Show that σ has rank $2k - 1$.

Proof. By definition of a rank $2k$ one-form, $(d\alpha)^k$ is nonvanishing, and $\alpha \wedge (d\alpha)^k \equiv 0$. (In this case, this second property is trivial since $(d\alpha)^k$ is a top form.) We can assume $\alpha \wedge (d\alpha)^{k-1}$ does not vanish locally, since its derivative is $(d\alpha)^k$. Since $\alpha \wedge (d\alpha)^{k-1}$ is a nonvanishing $2k - 1$ -form, we can find a locally-defined nonvanishing vector field ξ such that $i(\xi)[\alpha \wedge (d\alpha)^{k-1}] \equiv 0$. Therefore, around any point, we can find local coordinates $t, x^2, x^3, \dots, x^{2k}$ such that $\xi = \frac{\partial}{\partial t}$ in this coordinate system.

Hence, $i(\xi)[\alpha \wedge (d\alpha)^{k-1}] = i(\xi)[dx^2 \wedge \dots \wedge dx^{2k}] = 0$. Since $\alpha \wedge (d\alpha)^{k-1}$ is nonvanishing, at each point p there is a unique one dimensional subspace of the tangent space at p where $\xi(p)$ must reside. Ergo, $\alpha \wedge (d\alpha)^{k-1}$ and $dx^2 \wedge \dots \wedge dx^{2k}$ are linearly dependent. So $\alpha \wedge (d\alpha)^{k-1} = g dx^2 \wedge \dots \wedge dx^{2k}$ for some continuous, nonvanishing g . If g is negative, we can switch x^2 to $-x^2$ to make g positive.

Let f be such that $f^k = g$ and let $\sigma = \frac{1}{f}\alpha$.

$$d\sigma = -\frac{1}{f^2}df \wedge \alpha + \frac{1}{f}d\alpha$$

Since $\alpha \wedge \alpha = 0$, we can quickly calculate:

$$\begin{aligned} (d\sigma)^{k-1} &= -(k-1)\frac{1}{f^k}df \wedge \alpha \wedge (d\alpha)^{k-2} + \frac{1}{f^{k-1}}(d\alpha)^{k-1} \\ \sigma \wedge (d\sigma)^{k-1} &= \frac{1}{f^k}\alpha \wedge (d\alpha)^{k-1} = \frac{1}{g}(g dx^2 \wedge \dots \wedge dx^{2k}) = dx^2 \wedge \dots \wedge dx^{2k} \\ (d\sigma)^k &= d(\sigma \wedge (d\sigma)^{k-1}) = d(dx^2 \wedge \dots \wedge dx^{2k}) = 0 \end{aligned}$$

Hence, $\sigma \wedge (d\sigma)^{k-1}$ vanishes nowhere and $(d\sigma)^k$ is identically 0; i.e. σ has constant rank $2k - 1$. □

- (b) Prove that the above still holds when $n \geq k$ and $\alpha \in \Omega^1(\mathbb{R}^{2n})$ still has rank $2k$.

Proof. We know that $(d\alpha)^k$ is nowhere vanishing, and $\alpha \wedge (d\alpha)^k \equiv 0$. Hence, $(d\alpha)^{k+1} = d(\alpha \wedge (d\alpha)^k) \equiv 0$. Thus, $d\alpha$ is a 2-form of constant rank k . Therefore, by Darboux's Theorem (Problem 2), there exist local coordinates x^1, x^2, \dots, x^{2n} such that $d\alpha = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \dots + dx^{2k-1} \wedge dx^{2k}$.

Then $(d\alpha)^k = k!(dx^1 \wedge dx^2 \wedge \dots \wedge dx^{2k-1} \wedge dx^{2k})$, and since α has rank $2k$, $\alpha \wedge (d\alpha)^k = 0$. Expressed in this coordinate system, $\alpha = \sum_{i=1}^{2n} a^i dx^i$ for some functions a^i . $\alpha \wedge (d\alpha)^k = 0$ implies that $a^i \equiv 0$ for $i > 2k$. So

$$\alpha = \sum_{i=1}^{2k} a^i(x^1, \dots, x^{2n}) dx^i.$$

$d\alpha = \sum_{i=1}^{2k} \sum_{j=1}^{2j} \frac{\partial a^i}{\partial x^j} dx^j \wedge dx^i$. Therefore, for any $1 \leq i \leq 2k$ and for any $j > 2k$, $\frac{\partial a^i}{\partial x^j} = 0$, since the only term of the sum that carries a factor of $dx^j \wedge dx^i$ is $\frac{\partial a^i}{\partial x^j} dx^j \wedge dx^i$, and this term does not appear at all in $d\alpha$ by choice of coordinates. Hence, each a^i is independent of x^j for $j > 2k$. Thus,

$$\alpha = \sum_{i=1}^{2k} a^i(x^1, \dots, x^{2k}) dx^i.$$

is completely independent of x^{2k+1}, \dots, x^{2n} . Therefore, we can restrict to the $2k$ -dimensional submanifold generated by $\{x^1, \dots, x^{2k}\}$, and apply part (a). □

- (c) Use part (b) to show that if $\alpha \in \Omega^1(M)$ has constant rank $2k$ then we can find a positive function f and a one-form σ of constant rank $2k - 1$ such that

$$\alpha = f\sigma.$$

Use this fact and question (3) to find coordinates that verify the claim.

Proof. Part (b) allows us to define σ and f locally as follows: Given a coordinate chart $x : U \rightarrow \mathbb{R}^{2n}$, $x_*\alpha$ is a one form of constant rank $2k$ on \mathbb{R}^{2n} . Thus, there exist a positive function f' and a rank $2k - 1$ 1-form σ' on \mathbb{R}^{2n} such that $f'\sigma' = x_*\alpha$. Let $f = x^*f'$ and $\sigma = x^*\sigma'$.

By question 3, there are local coordinates $z^1, z^2, \dots, z^n, y^1, y^2, \dots, y^n$ such that $\sigma = dy^1 + \sum_{i=2}^k z^i dy^i$. Then $\alpha = f dy^1 + \sum_{i=2}^k f z^i dy^i$. Define $x^1 = f$ and $x^i = f z^i$ for $2 \leq i \leq k$. $(d\alpha)^k = dx^1 \wedge dy^1 \wedge \dots \wedge dx^k \wedge dy^k \neq 0$, so $x^1, \dots, x^k, y^1, \dots, y^k$ are linearly independent. Hence, the x^i, y^i are part of a coordinate system such that $\alpha = \sum_{i=1}^k x^i dy^i$. □