MAT 561 Mathematical Physics II: HW1 **3/5/2007** **Andrew Stimpson**

Consider a thermodynamic system whose only configurational variable is volume V so that the equilibrium submanifold M is 2-dimensional. **NOTE:** For any of these results to make sense, we must assume that $Q \neq 0$.

(a) Let Q denote the heat 1-form and let p denote pressure. Explain how we can interpret the ratio of two forms as a function

$$f = \frac{Q \wedge dp}{Q \wedge dV}.$$

Proof: Because 2-forms are top forms on a 2-manifold, in local coordinates (y^1, y^2) we can write

$$Q \wedge dp = f_p(y)\, dy^1 \wedge dy^2 \quad \text{and} \quad Q \wedge dV = f_V(y)\, dy^1 \wedge dy^2.$$

Because V is a configurational variuable, $Q \wedge dV$ is nowhere-vanishing, so we can provisionally define $f \equiv f_p/f_V$. This definition is coordinate-free because in any other coordinates, the top form corresponding to $dy^1 \wedge dy^2$ will be related to $dy^1 \wedge dy^2$ by a multiplication by a non-vanishing function[1], which will cancel out in the quotient. ■

(b) Fix a point $x \in M$ and any adiabatic vector $\xi \in T_xM$. Show that

$$f(x) = \frac{dp(\xi)}{dV(\xi)} \quad \text{which justifies the expression} \quad f = \left(\frac{dp}{dV}\right)_{\text{adiabatic}}$$

where the right hand side is just the ratio of 1-forms evaluated on some adiabatic vector.

Proof: Pick $\eta \in T_xM$ such that $Q(\eta) \neq 0$. Part (a) showed that

$$Q \wedge dp = f\, Q \wedge dV.$$

So at the point x,

$$Q(\eta)\, dp(\xi) = (Q \wedge dp)(\eta, \xi) = f(x)(Q \wedge dV)(\eta, \xi) = f(x)\, Q(\eta)\, dV(\xi)$$

because $Q(\xi) = 0$. So because $Q(\eta) \neq 0$, $dp(\xi) = f(x)\, dV(\xi)$. Note that since f is independent of the choice of ξ, the ratio $\frac{dp(\xi)}{dV(\xi)}$ is also independent of ξ. ■

(c) Let T denote temperature. Based on the discussion above, state what is meant by

$$\frac{dT \wedge dp}{dT \wedge dV} = \left(\frac{dp}{dV}\right)_{\text{isothermal}}.$$

Statement: To calculate, $\frac{dp}{dV}$ along infinitesimal isothermal directions, we can just take the ratio of dp and dV evaluated on isothermal tangent vectors. But just as in part (b), this can be calculated by taking the ratio of the two 2-forms $dT \wedge dp$ and $dT \wedge dV$ because an isothermal tangent vector is in the kernel of dT by definition.

[1] the determinant of the Jacobian of the transition function

(d) Since the differentials of V, T and p, T are linearly independent, we may write

$$Q = \Lambda_V dV + C_V dT \quad \text{or} \quad Q = \Lambda_p dp + C_p dT$$

where the Λs and Cs are functions on M. Show that

$$\left(\frac{dp}{dV}\right)_{\text{adiabatic}} = \gamma \left(\frac{dp}{dV}\right)_{\text{isothermal}}$$

where $\gamma = C_p/C_V$.

Proof:

$$\left(\frac{dp}{dV}\right)_{\text{adiabatic}} = \frac{Q \wedge dp}{Q \wedge dV} = \frac{(\Lambda_p dp + C_p dT) \wedge dp}{(\Lambda_V dV + C_V dT) \wedge dV} = \frac{C_p \; dT \wedge dp}{C_V \; dT \wedge dV} = \frac{C_p}{C_V}\left(\frac{dp}{dV}\right)_{\text{isothermal}}$$

■

(e) An *ideal gas* is one that, in equillibrium, obeys the constraints

$$pV = nT \quad \text{and} \quad \gamma = \text{constant}$$

where n are the moles of gas. Use part (d) to show that the adiabatic curves for an ideal gas are given by

$$pV^\gamma = \text{constant.}$$

Proof: Using the coordinates (V, T) for M, for an isothermal vector τ,

$$dp(\tau) = \frac{\partial p}{\partial V} dV(\tau) + \frac{\partial p}{\partial T} dT(\tau) = \frac{\partial p}{\partial V} dV(\tau).$$

Because pressure has the formula $p(V, T) = \frac{nT}{V}$,

$$\left(\frac{dp}{dV}\right)_{\text{isothermal}} = \frac{\partial p}{\partial V} = -\frac{nT}{V^2} \quad \Longrightarrow \quad \left(\frac{dp}{dV}\right)_{\text{adiabatic}} = -\gamma \frac{nT}{V^2}$$

Let ξ be an adiabatic vector in $T_x M$.

$$\begin{aligned} d(pV^\gamma)(\xi) &= V^\gamma dp(\xi) + p\gamma V^{\gamma-1} \, dV(\xi) = \left(-V^\gamma \gamma \frac{nT}{V^2} + p\gamma V^{\gamma-1}\right) dV(\xi) \\ &= \left(-V^\gamma \gamma \frac{nT}{V^2} + \frac{nT}{V}\gamma V^{\gamma-1}\right) dV(\xi) = 0. \end{aligned}$$

So infinitesimal adiabatic vectors are in the kernel of $d(pV^\gamma)$. Thus as long as $Q \neq 0$, Q must be proportional to $d(pV^\gamma)$, which means that adiabatic curves must be level sets of pV^γ. ■