MAT561 Homework 1

Problem 6

Part a

Let y^i be another basis of linear coordinates on V. Then, there exists $A = (A^i{}_j) \in GL(\mathbb{R}^n)$ such that $y^i = A^i{}_j x^j$. We have

$$\begin{aligned} dy^i &= A^i{}_j dx^j \\ \frac{\partial}{\partial y^i} &= A_i{}^j \frac{\partial}{\partial x^j} \end{aligned}$$

where $(A_i{}^j) = A^{-1}$.
Using above, we get:

$$\begin{aligned} \text{Hess} f &= \frac{\partial^2 f}{\partial y^i \partial y^j} dy^i \otimes dy^j \\ &= A_i{}^k \frac{\partial}{\partial x^k}\left(A_j{}^l \frac{\partial f}{\partial x^l}\right) A^i{}_m A^j{}_n dx^m \otimes dx^n \\ &= A_i{}^k A^i{}_m A_j{}^l A^j{}_n \frac{\partial}{\partial x^k}\left(\frac{\partial f}{\partial x^l}\right) dx^m \otimes dx^n \\ &= \delta^k{}_m \delta^l{}_n \frac{\partial}{\partial x^k}\left(\frac{\partial f}{\partial x^l}\right) dx^m \otimes dx^n \\ &= \frac{\partial^2 f}{\partial x^m \partial x^n} dx^m \otimes dx^n \end{aligned}$$

The fact that Hessian is symmetric follows from the commutativity of partial differentiations.

Part b

Let $f\colon V \to \mathbb{R}$. Then, we have

$$\begin{aligned} df\colon TV &\to T\mathbb{R} \\ (p, \dot{p}) &\mapsto (f(p), (df_p)(\dot{p})) \end{aligned}$$

So $\phi_f(p) = df_p = \frac{\partial f}{\partial x^i}(p) dx^i$. Now to find $d\phi_f$:

$$\begin{aligned} d\phi_f\colon TV &\to TV^* \simeq V^* \times V^* \\ (p, \dot{p}) &\mapsto (df_p, \frac{\partial^2 f}{\partial x^j \partial x^i}(p) \dot{p}^i dx^j) \end{aligned}$$

$d\phi_f = \text{Hess}f$ as a map from V to $V^* \otimes V^*$. So ϕ_f is a local homeomorphism at p if and only if the Hessian is non-singular at p.

Part c

From the computation in part b), we have:

$d\phi_{f,p} = H_{ij}(p)dx^i \otimes dx^j$, where H_{ij} denotes the Hessian of f.

By the chain rule, it follows that:

$$\begin{array}{rcl} d\phi_f^{-1} : TV^* & \to & TV \\ (\alpha, \dot{\alpha}) & \mapsto & \left(\phi_f^{-1}(\alpha), H^{ij}(\phi_f^{-1}(\alpha))\dot{\alpha}_i \dfrac{\partial}{\partial x^j}\right) \end{array}$$

Using this, we have:

$$\begin{array}{rcl} \mathcal{L}f(\alpha) & = & < id(\alpha), \phi_f^{-1}(\alpha) > -f \circ \phi_f^{-1}(\alpha) \\ \phi_{\mathcal{L}f}(\alpha) = d(\mathcal{L}f_\alpha) & = & < d(id)_\alpha, \phi_f^{-1}(\alpha) > + < id(\alpha), df_{\phi_f^{-1}}(\alpha) > -df_{\phi^{-1}(\alpha)} \circ d\phi_{f,\alpha}^{-1} \\ & = & < id, \phi_f^{-1}(\alpha) > +(\alpha - df_{\phi_f^{-1}(\alpha)})H^{-1}(\phi_f^{-1}(\alpha)) \\ & = & \phi_f^{-1}(\alpha) + (\alpha - df_{\phi_f^{-1}(\alpha)})H^{-1}(\phi_f^{-1}(\alpha)) \\ & = & \phi_f^{-1}(\alpha) + (\alpha - \phi_f(\phi_f^{-1}(\alpha)))H^{-1} = \phi_f^{-1}(\alpha) + (\alpha - \alpha)H^{-1} \\ \phi_{\mathcal{L}f}(\alpha) & = & \phi_f^{-1}(\alpha) \end{array}$$

where $H^{-1} = H^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$.

Part d

From the previous part, we have

$$\alpha = \phi_{\mathcal{L}f}^{-1}(x) = \phi_f(x)$$

for any x in V.

$$\begin{array}{rcl} \mathcal{L}\mathcal{L}f(x) = \mathcal{L}(\mathcal{L}f)(x) & = & < x, \phi_{\mathcal{L}f}^{-1}(x) > -\mathcal{L}f(\phi_{\mathcal{L}f}^{-1}(x)) \\ & = & < x, \alpha > -\mathcal{L}f(\alpha) \\ & = & < x, \alpha > -(< x, \alpha > -f \circ \phi_f^{-1}(\alpha)) \\ & = & f(\phi_f^{-1}(\phi_f(x))) = f(x) \end{array}$$

So it follows that $\mathcal{L}\mathcal{L} = id$.

Part e

One can generalize the notion of the Legendre transform to vector bundles in the following way:

Given $f : M \times V \to \mathbb{R}$, we define $\phi_f : M \times V \to M \times V^*$ by

$$\phi_f(m, v) = (m, d_V f_{(m,v)}),$$

where $d_V f$ is the "partial external differential" of f along V. (This is equivalent to fixing a point in M, and then taking the differential of f as a function on V.)

Since f is smooth on $M \times V$, so is ϕ_f. The local coordinates on the fibre are linear, so the "Hessian along V" is well-defined on each point of M, and it is precisely equal to $d_V \phi_f$.

ϕ_f is a local homeomorphism along the fibres, if for each point p on M, the Hessian along V in some local coordinates have none-zero determinant. Note that this doesn't necessarily give a local homeomorphism on the bundle.

Assuming that ϕ_f is a diffeomorphism along the fibre at each base point allows one to take the Legendre transform of f as before, only this time one has to mind the base point. This gives us a notion of Legendre transform on trivial vector bundles, and one can see that transforming a smooth function on $M \times V$ will yield a smooth function on $M \times V^*$ since it is the sum of compositions of smooth maps.

Finally, for a general vector bundle, one may perform Legendre transformation via the local trivializations.