

Problem 1. Let V be a finite-dimensional vector space and let $(,) : V \times V \rightarrow V$ be a skew-symmetric bilinear pairing on V . Then V has a basis of the form

$$\{x_i, y_i\}_{i=1}^m, \{z_i\}_{i=1}^n \text{ such that } \begin{cases} (x_i, y_i) = 1 \\ (x_i, y_j) = 0 \\ (z_i, \alpha) = 0 \quad \forall \alpha \in V \end{cases}$$

If $(,)$ is identically zero, then any basis for V is of the above form, with $n = \dim(V)$ and $m = 0$. If not, then $(x, y) \neq 0$ for some $x, y \in V$. Furthermore $x \neq y$ by skew-symmetry, so after rescaling if necessary we have $(x, y) = 1$.

Now we can write $V = \text{span}\{x, y\} \oplus V'$, where $V' = \{\alpha \in V \mid (\alpha, x) = (\alpha, y) = 0\}$, and apply the same argument recursively to V' . Since V is finite dimensional, the recursion terminates and we arrive at a basis of the desired form.

Corollary 1. If $(,)$ is nondegenerate, then the dimension of V is even.

Problem 2. If ω_1 and ω_0 are symplectic forms on \mathbb{R}^{2n} and $\omega_1 = \omega_0$ at 0, then there exists a local diffeomorphism f defined in some neighborhood of 0 so that $f^*\omega_1 = \omega_0$.

We will solve this problem using time-dependent vector-fields. Given a time dependent vector field $\xi : \mathbb{R} \times M \rightarrow TM$, let $\phi : \mathbb{R} \times M \rightarrow M$ be the solution to the initial value problem

$$\begin{cases} \xi_s(\phi_s(x)) = d\phi_{(s,x)}\left(\frac{\partial}{\partial t}\right) \\ \phi_0(x) = x \end{cases}$$

Lemma 1. For any time-dependent form α_t , time-dependent vector field ξ_t and one parameter family of diffeomorphisms ϕ_t generated by ξ_t , we have

$$\frac{\partial}{\partial t}(\phi_t^* \alpha_t) = \frac{\partial}{\partial t} \alpha_t + \text{Lie}_{\xi_t} \alpha_t$$

Proof: First define $\bar{\phi} : \mathbb{R} \times M \rightarrow \mathbb{R} \times M$ by extending ϕ by the identity, so that $\bar{\phi}(s, x) = (s, \phi_s(x))$. Then $\phi_t^* \alpha_t = \bar{\phi}^* \alpha_t$.

First consider the case where α_t is a time-dependent function. At a point $(s, x) \in \mathbb{R} \times M$, we have

$$\begin{aligned} \frac{\partial}{\partial t}(\phi_t^* \alpha_t)_{(s,x)} &= \frac{\partial}{\partial t}(\alpha \circ \bar{\phi}_t)_{(s,x)} \\ &= d\alpha_{(s,\phi_s(x))} \circ d\bar{\phi}_{(s,x)}\left(\frac{\partial}{\partial t}\right) \\ &= d\alpha_{(s,\phi_s(x))} \circ (d(\text{ID}_{\mathbb{R}}) + d\phi)_{(s,x)}\left(\frac{\partial}{\partial t}\right) \\ &= d\alpha_{(s,\phi_s(x))}\left(\frac{\partial}{\partial t} + \xi_s(\phi_s(x))\right) \\ &= \frac{\partial}{\partial t} \alpha_{(s,\phi_s(x))} + \xi_s \alpha(\phi_s(x)) \\ &= \phi_t^* \left(\frac{\partial}{\partial t} \alpha_t + \text{Lie}_{\xi_t} \alpha_t\right)(s, x) \end{aligned}$$

Next consider the case where $\alpha_t = df_t$ is an exact 1-form. Then

$$\begin{aligned} \frac{\partial}{\partial t}(\phi_t^* df_t)_{(s,x)} &= d\left(\frac{\partial}{\partial t}\phi_t^* f_t\right) \\ &= d\left(\phi_t^*\left(\frac{\partial}{\partial t}f_t + \text{Lie}_{\xi_t}f_t\right)\right) \\ &= \left(\phi_t^*\left(\frac{\partial}{\partial t}df_t + \text{Lie}_{\xi_t}df_t\right)\right) \end{aligned}$$

because d commutes with $\frac{\partial}{\partial t}$ and Lie_{ξ_t} .

Now, every form α_t can be written locally as a wedge product of functions and exact 1-forms. Since pullback commutes with wedge and $\frac{\partial}{\partial t}$ is a derivation of the wedge product, the lemma follows. ■

We proceed with the proof of Problem 2. Define:

$$\omega_t = t\omega_1 + (1-t)\omega_0$$

Observe that ω_t is a linear combination of closed forms, and thus is closed and locally exact. Then there exists a 1-form β such that:

$$d\beta = \omega_1 - \omega_0$$

At the point 0, $\omega_1 = \omega_0$ and thus $\omega_t = t\omega_1 + (1-t)\omega_0 = \omega_0$. Therefore ω_t is nondegenerate at 0. Nondegeneracy is an open condition, so ω_t is nondegenerate in a neighborhood of 0 as well. Thus it is possible to define a time-dependent vector field ξ_t by the relation:

$$\iota_{\xi_t}\omega_t = -\beta$$

Let ϕ_t be the one parameter family generated by ξ_t . Then by our Lemma,

$$\begin{aligned} \frac{\partial}{\partial t}(\phi_t^*\omega_t) &= \phi_t^*\left(\frac{\partial}{\partial t}\omega_t + \text{Lie}_{\xi_t}\omega_t\right) \\ &= \phi_t^*(\omega_1 - \omega_0 + d(\iota_{\xi_t}\omega_t) + \iota_{\xi_t}d\omega_t) \\ &= \phi_t^*(d\beta - d\beta + 0) \\ &= 0 \end{aligned}$$

Thus $\phi_1^*\omega_1 = \phi_0^*\omega_0$. But ϕ_0^* is the identity, so $\phi_1^*\omega_1 = \omega_0$. ϕ_1 is the desired diffeomorphism.