MAT 561 - HW 1 Problem 2

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2 Assuming question 1.

(a) Let $\omega \in \Omega^2(M)$ be closed, non-degenerate. Claim: there exist local coordinates in a neighborhood of $m \in M$, (U, ϕ), $\phi = (x^1, \ldots, x^{2n})$ such that

$$\omega = dx^1 \wedge dx^2 + \ldots dx^{2n-1} \wedge dx^{2n}$$

Proof: For $m \in M$, pick a chart (U, ϕ), $\phi : U \to \mathbb{R}^{2n}$, and $\phi(m) = 0$. Consider $(\phi^{-1})^*\omega$.

It is a non-degenerate 2-form,[1] $\omega \in \Lambda^2(\mathbb{R}^{2n})^*$, so by problem 1a), $\exists\,(V, y),\ 0 \in V$ such that

$$(\phi^{-1})^*\omega\Big|_0 = dy^1 \wedge dy^2 + \ldots + dy^{2n-1} \wedge dy^{2n}\Big|_0$$

Further, $(\phi^{-1})^*\omega$ is also closed,[2] so, as a 2-form on $\mathbb{R}^{2n}$, by problem 1b),

$\exists\, f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, a diffeomorphism, defined on a neighborhood of 0, such that:

$$(f^* \circ (\phi^{-1})^*)\omega = dy^1 \wedge dy^2 + \ldots + dy^{2n-1} \wedge dy^{2n}$$

on the neighborhood. Then,

$$\begin{aligned} \omega &= (f^{-1} \circ \phi)^* dy^1 \wedge dy^2 + \ldots + dy^{2n-1} \wedge dy^{2n} \\ &= d(y^1 \circ f^{-1} \circ \phi) \wedge d(y^2 \circ f^{-1} \circ \phi) + \ldots + \\ &= +\, d(y^{2n-1} \circ f^{-1} \circ \phi) \wedge d(y^{2n} \circ f^{-1} \circ \phi) \end{aligned}$$

Let $x^i = y^i \circ f^{-1} \circ \phi$ and we have the coordinates desired:

$$\omega = dx^1 \wedge dx^2 + \ldots + dx^{2n-1} \wedge dx^{2n}. \quad \square$$

[1] This is because ω is non-degenerate and ϕ is a bijection, so their composition remains injective.

[2] Since pull-back commutes with external derivative.

(b) Suppose $\omega \in \Omega^2(M)$ is a *closed* form with constant rank $k \leq n$, i.e.:

$$\forall m \in M,\ \omega^k\Big|_m \neq 0,\ \text{and}\ \omega^{k+1}\Big|_m = 0$$

Claim: $\exists$ local coordinates such that

$$\omega = dx^1 \wedge dx^2 + \ldots + dx^{2k-1} \wedge dx^{2k}$$

Proof: by inducting on the dimension of M.
Base case: $\dim M = 2k$.
Then $\omega^k \neq 0 \Rightarrow \omega$ is a non-vanishing top form, so it is non-degenerate (and closed by assumption).
Thus, we can apply the results of 2a) to get local coordinates such that:

$$\omega = dx^1 \wedge dx^2 + \ldots + dx^{2k-1} \wedge dx^{2k}$$

Suppose that for $\dim M = 2k + i$ there exist local coordinates such that:

$$\omega = dx^1 \wedge dx^2 + \ldots + dx^{2k-1} \wedge dx^{2k}$$

Consider the case for $\dim M = 2k + i + 1$.
$\omega^{k+1} = 0 \quad \Rightarrow \quad \omega$ is degenerate,[3] so:

$$\exists X \in \mathfrak{X}(M) \quad \text{s.t.} \quad \iota_X \omega = 0$$

Further, we can choose local coordinates $\{z^1, \ldots, z^{2n-1}, y\}$ such that $X = \frac{\partial}{\partial y}$.
Then

$$\iota_X \omega = 0 \quad \Rightarrow \quad \omega = \sum_{i<j} a_{ij} dz^i \wedge dz^j$$

Want to show that a_{ij} is independent of y, so that ω will be defined on a dimension $2k + i$ submanifold and we can apply the inductive hypothesis.
ω closed $\quad \Rightarrow \quad d\omega = 0$, i.e.:

$$0 = \sum_{i<j} \left(\frac{\partial a_{ij}}{\partial y} dy \wedge dz^i \wedge dz^j + \text{stuff not involving } y \right)$$

The linear combination of basis vectors is 0 if and only if the coefficients are all 0, so we have that
$\forall i \forall j \ \frac{\partial a_{ij}}{\partial y} = 0$.
Thus the a_{ij}'s are independent of y, so that ω is a closed 2-form on a dimension $2n + i$ submanifold of M and we can apply the induction hypothesis to get that there are local coordinates such that:

$$\omega = dx^1 \wedge dx^2 + \ldots + dx^{2k-1} \wedge dx^{2k} \quad \square$$

[3] In the case $i = 0$, ω is degenerate, since it's a skew-symmetric form on an odd-dimensional manifold, so in local coordinates its matrix, ω_{ij} is skew-symmetric and the odd dimension gives $\omega_{ij} = -\omega_{ji}$, but $det(\omega_{ij}) = det(\omega_{ji})$, thus $det(\omega_{ij}) = 0$, hence it's a degenerate matrix. This holds for every point, so ω is degenerate as a form.