[3.] Assume that α is a constant rank $2k+1$ one-form on M^N. Prove that there exists a basis $\{x_1, y_1, \ldots, x_k, y_k, z, \ldots\}$ with respect to which

$$\alpha = dz + \sum_{i=1}^{k} x^i dy^i.$$

Proof. Since α is rank $2k+1$, $\alpha \wedge (d\alpha)^k$ is nowhere zero, so $(d\alpha)^k$ is nowhere zero, but $(d\alpha)^{k+1}$ is identically zero, so $(d\alpha)$ is an exact (hence closed) two-form of constant rank k.

Then by Darboux's theorem, there exists a set of coordinates $\{x^1, y^1, \ldots, x^k, y^k, z^1, \ldots, z^{N-2k}\}$ with respect to which

$$d\alpha = \sum_{i=1}^{k} dx^i \wedge dy^i.$$

Then

$$d(\alpha - \sum_{i=1}^{k} x^i dy^i) = d\alpha - \sum_{i=1}^{k} dx^i \wedge dy^i = 0$$

So

$$\alpha - \sum_{i=1}^{k} x^i dy^i$$

is closed, hence locally exact, so that locally there exists a function z such that

$$\alpha - \sum_{i=1}^{k} x^i dy^i = dz.$$

Now

$$\alpha \wedge (d\alpha)^k = dz \wedge \bigwedge_{i=1}^{k} dx^i \wedge dy^i$$

because all the other terms have multiples of some dx^i or dy^i. Writing dz locally and omitting those terms which are functional multiples of dx^i or dy^i, this is

$$\left(\sum_{j=1}^{N-2k} \frac{\partial z}{\partial z^j} dz^j \right) \wedge \bigwedge_{i=1}^{k} dx^i \wedge dy^i.$$

Because α is of constant rank $2k+1$, this form is nowhere zero, so in particular,

$$\sum_{j=1}^{N-2k} \frac{\partial z}{\partial z^j} dz^j$$

is nowhere zero. This means that if the range of z (as a function $M \to \mathbb{R}^N\}$ is restricted to $\langle z^1, \ldots, z^{N-2k} \rangle$, its derivative is rank one, so it is a coordinate.

Then on $\langle x^1, y^1, \ldots, x^k, y^k, z^1, \ldots, z^{N-2k} \rangle$ it is still a coordinate, and in particular a coordinate independent of the set $\{x^1, y^1, \ldots, x^k, y^k\}$, as desired.