Mathematical Physics
September 13, 2006

Gabriel C. Drummond-Cole

September 13, 2006

Recall that for certain $\xi \in \mathscr{X}_\omega(\mathscr{M})$ (those for which the topological obstruction disappears) there exists a corresponding observable $\mathscr{O}_\xi \in C^\infty(\mathscr{M})$ such that $-d\mathscr{O}_\xi = \iota(\xi)_\omega$.

This describes the infinitessimal symmetry ξ via $\xi f = \{\mathscr{O}_\xi, f\}$ for any $f \in C^\infty(\mathscr{M})$.

We got as far as saying that there are a particular set of symmetries we're concerned with. There are the infinitessimal symmetries of time translation, $\zeta \in \mathscr{X}_\omega(\mathscr{M})$, and this has the observable $\mathscr{O}_\zeta$ where $-\mathscr{O}_\zeta$ is the energy or Hamiltonian. For a path x, $H(x) = \mathfrak{m}2|\dot{x}(t)|^2 + V(x(t))$. For $x \in \mathscr{M}$ this is independent of t.

[Is that obvious?]

Yes, I'll get to it in a second. That's where we left off last time. Any questions?

Before I finish off Hamiltonian dynamics, let me make some tangential but useful remarks about observables. Most of the observables we see in this class will be something like $\mathscr{O}_{(t,f)}$, defined for any time $t \in M^1$ and $f : X \to \mathbb{R}$. Then

$$\mathscr{O}_{(t,f)}(x) = f(x(t)).$$

Let me give another example, two examples.

1. If $X = \mathbb{E}^d$, then we can take $f = x^i$, and in this case $\mathscr{O}_{(t,f)}$ the x^i coordinate of the particle at time t.
2. The Hamiltonian, this is the energy of the particle at time t.

A jet of a function is essentially its Taylor series. The first type of observable depended on the 0-jet of the path; the Hamiltonian depends on the 1-jet. $\mathscr{O}_{(t,f)}$ is local in time, meaning it only depends on finitely many of these, only depends on a small neighborhood of a given time.

So what's the upshot? The structure on $\mathscr{M}$ is as follows. We have what is called a Hamiltonian system. That's the phase space with its symplectic structure and our function H, so the couple $(\mathscr{M}, H)$. So this is a symplectic manifold and a distinguished observable (energy) such that $-\xi_H$ is the infinitessimal time translation.

Let's look at the symmetries of this extended structure. Global symmetries are symplectomorphisms that preserve H. The infinitessimal symmetries are infintessimal symplectomorphisms $\xi \in \mathscr{X}(\mathscr{M})$ such that $\mathrm{Lie}(\xi)H = 0$. Now we're going to look at these symmetries in terms of observables.

Here "extended structure" should not be confused with the (closely related) "extended phase space" which is used in the construction of the Hamiltonian form of the variational principle. Extended phase space is just the Cartesian product of the original phase space (the cotangent bundle T^*X of the configuration space X) with the time $M^1 \cong \mathbb{R}$.

So if Q is an observable that corresponds to an infinitessimal symmetry, then we have the following relation: $\{H, Q\} = 0$. Now, for any observable, never mind that it's a symmetry of any system, time translation flow on phase space is induced by H. So we get that $\dot{\mathscr{O}} = \{H, \mathscr{O}\}$.

This equation deserves its own line:

$$\frac{\mathrm{d}\mathscr{O}}{\mathrm{d}t} = \{H, \mathscr{O}\} \ . \tag{1}$$

In the operator formalism of quantum mechanics this will be the equivalent of the Schrödinger equation.

Exercise: *Show that the classical evolution equation (1) is equivalent to Hamilton's equations.*

Recall that you already showed that Hamilton's equations (with the condition that the momentum is given in terms of the velocity by $p^i = m\dot{x}^i$ or, equivalently, that the Hamiltonian factorizes as $H = \frac{1}{2m}p^2 + V(x)$) are equivalent to Newton's second law. Therefore, once we fix the identification $TX \cong T^*X$, the evolution equation (1) is equivalent to Newton II.

Now we use the observable energy to tell us how things change with time. So now, thus, what we can conclude, assuming that the observable is a symmetry of the Hamiltonian system, for Q, if Q induces a symmetry of the Hamiltonian system, then we havea conservation law. We have that $\dot{Q} = \{H, Q\} = \mathrm{Lie}(-\xi_H)Q = \mathrm{Lie}(\xi_Q)H = 0$.

So look at $\dot{H}$. This is $\{H, H\}$ which is zero. So H is conserved. Such observables, here's more jargon, are called, and this is why I used Q, are called conserved charges.

So here's the big idea, big enough to put in a box. Symmetries imply conservation laws.

Exercise 1 *Compute these conserved charges. The physical situation is the free particle in Euclidean space. We have the huge symmetry group, which is the isometries of $\mathbb{E}^d$.*

Compute the conserved charges for translations and for rotations. These will be momentum and angular momentum.

The term linear momentum *is sometimes used to distinguish these two types of momenta.*

Okay, let's talk about Lagrangian mechanics. For particles we have solutions to Newton's second law, $\mathscr{M} \subset \mathscr{P} = \mathrm{Map}(M^1, X)$. The idea of Lagrangian mechanics is to describe $\mathscr{M}$ as the critical submanifold of a function $S : \mathscr{P} \to \mathbb{R}$.

This function S is called the action, and $\mathscr{M}$ would be paths x such that $\delta S(x) = d_{\mathscr{P}} S(x) = 0$. So δ is the exterior derivative on $\mathscr{P}$.

This is the variational principle: we want the action to be stationary with respect to variations $\delta x^i(t)$ (which form a basis of $H^*(\mathscr{P})$) in the path $x^i(t)$. This philosophy can be motivated in various ways with various degrees of rigor. One such (rigorless) way is the following. In Newtonian mechanics, particle motion tends to minimize the potential energy; if a ball is sitting on an inclined plane it will roll to the bottom. The action principle is the precise embodiment of this intuition.

[Is this why physicists want a path integral?]

That's for quantum mechanics.

The Path Integral and the Principle of Least Action: A preview *Quantum theory introduces a fundamental unit of action $\hbar$ called Planck's constant. The path integral Z is the probability amplitude for a particle at position $\mathbf{x}_i$ at time t_i to be found at a $\mathbf{x}_f$ at a later time t_f. It is given (schematically) by*

$$Z = \int_{\mathbf{x}(t_i)=\mathbf{x}_i}^{\mathbf{x}(t_f)=\mathbf{x}_f} [\mathrm{d}\mathbf{x}(t)] \exp\left(\frac{i}{\hbar} S[\mathbf{x}(t)]\right) \qquad (2)$$

The boundary conditions on the path are indicated by the "limits of integration" and $[\mathrm{d}\mathbf{x}(t)]$ is a "measure" on the space of paths $\mathscr{P}$. This formula expresses the fact that the probability of finding a particle at $\mathbf{x}_f$ at time t_f given that it was at $\mathbf{x}_i$ at time t_i is given by a sum over all paths (a.k.a. "histories") with these boundary conditions weighted by a unimodular complex number whose phase is the action ($\hbar = 1$ in natural units). Now consider the classical limit $\hbar \to 0$. *When the action is away from its stationary point, any small deviation in the path causes wild fluctuations in the exponential with "frequency" $\frac{1}{\hbar} \to \infty$. The claim is that these fluctuations average to 0 so that the path integral has, in the classical limit, support only on those paths for which the action is extremal, that is $\delta S = 0$. The principle of least action therefore follows naturally from the quantum principle of "sum over histories".*

So these equations, call these x paths, they satisfy what are called Euler-Lagrange equations. We'll eventually see that these are just Newton's second law. Let me just continue with the philosophical baloney. This sort of variational principle is also found in geometry, where it used to obtain nice PDEs, like the harmonic PDE.

The terms "Newton's second law", "Euler-Lagrange equation" and "Hamilton's equations" are all examples of "equations of motion". The phrase "equation of motion" or EOM *is used interchangeably (and non-commitally) with any of these.*

The Lagrangian approach gives us back our phase space, but it gives us a lot more than that. The symplectic form was borrowed and depended on a time t. In the Lagrangian approach, we'll get, the information embedded in this Lagrangian mechanics, which are the Euler Lagrange equations and the submanifold $\mathscr{M}$, but also a family of one-forms on $\mathscr{M}$ parameterized by time. Finally, these one-forms will give us the symplectic structure naturally, and that won't depend on t.

I've kept you guys ten minutes long, I apologize. But in this sense, physicists equate "theory" with a particular Lagrangian, which has all of this information in it.

[I thought it was the action?]

That's the integral of the Lagrangian, which I think is more basic.