Overview of selected topics in theoretical physics

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Part I

Classical Theory

Chapter 1

Introduction, notation, and preliminaries

1.1 Our guiding philosophy

This course is meant to be an introduction to the topics usually taught to undergraduate physics major. These are classical mechanics in both the Newtonian and Relativistic setting; classical electromagnetism; thermodynamics and statistical mechanics; and quantum mechanics. We have two audiences in mind: former physics majors who have seen the general content before and pure math majors with an interest is studying physical topics.

For the first audience, we present the physics in a way that emphasizes some of the overall mathematical structure. This approach can be of great pedagogical benefit by shining new light on old topics and preparing one for further study in field and string theory, to which many of the mathematical ideas we discuss apply.

For the second audience, the mathematical structure is there for psychological reasons, as well as pedagogical ones, softening the culture shock and yet introducing math that is interesting in its own right. We also present specific examples and solutions to get a hands-on feel for the physical ideas that they display.

In some cases, particularly when we cover quantum mechanics, some may find our mathematical approach to be vague and hand-waving at best. While this is somewhat regrettable, we will not apologize for it. One of the goals of this course is to offer the students a sense (perhaps even an intuition) for how physicists achieve progress, not in spite of eschewing mathematical rigor, but sometimes *because* of it.

1.2 Physical and mathematical preliminaries

Dimensional analysis and “naturalness”
Vector fields, differential forms, and calculus

Chapter 2

Classical mechanics

2.1 Newtonian mechanics on Euclidean space

2.1.1 Space, time and particle

Space We will generally refer to *space* meaning the 3-dimensional real vector space $\mathbb{R}^3 = \{\mathbf{x} = (x, y, z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\}$ with the *right-handed orientation* and the Euclidean inner product $\langle \cdot\,, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, to which we will refer as *dot product*. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$. We describe all events by coordinate expressions as that is the language most closely related to the realization of the event. We will also switch freely between various confusing but conventional notations to describe the coordinates. For example, it is common to write $\mathbf{x} \in \mathbb{R}^3$ variously as $\mathbf{x}^i$ for $i = 1, 2, 3$ or just x^i and also $\mathbf{x} = (x, y, z)$. Note that by convention $x^1 = x$ denotes the "x-coordinate", $x^2 = y$ denotes the "y-coordinate", and $x^3 = z$ denotes the "z-coordinate". In this language, $\mathbf{x} \cdot \mathbf{y} = \sum_{i,j=1}^{3} \delta_{ij} x^i y^j$ where δ_{ij} is the *Kronicker-delta*, equal to $+1$ when $i = j$ and 0 otherwise. We will use the *Einstein summation convention* meaning that when covariant and contravariant indices are repeated, a summation over the full range of the indices is implied, that is, for a vector x^i and covector p_i, $p_i x^i = \sum_{i=1}^{3} p_i x^i$.

For any vector $\mathbf{x}$ we define the unit vector $\hat{\mathbf{x}} = |\mathbf{x}|^{-1}\mathbf{x}$ where $|\mathbf{x}| \equiv \sqrt{\mathbf{x} \cdot \mathbf{x}} \equiv r$. The unit vectors in the x-, y-, and z-directions are denoted $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. The orientation on space defines a *cross-product* $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. The right-handed orientation is the one given by the *right-hand rule* $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = +\hat{\mathbf{z}}$. This can be expressed using the totally anti-symmetric tensor ϵ_{ijk} normalized to $\epsilon_{123} = +1$, that is $\epsilon_{ijk}(\hat{\mathbf{x}})^i(\hat{\mathbf{y}})^j(\hat{\mathbf{z}})^k = 1$, in terms of which for any two vectors $\mathbf{a}$, $\mathbf{b}$, $(\mathbf{a}\times\mathbf{b})^i = \delta^{ii'}\epsilon_{i'jk}a^j b^k$.

Time and Particle In the Newtonian picture of nature, there is a universal clock defining time for all observers. We will denote this universal time by $t \in \mathbf{R}$. In general, particle motion is described, by definition, by a time-dependent vector $\mathbf{x}(t)$.[1] The *velocity* $\mathbf{v}$ of a particle is the derivative with respect to time of its position $\mathbf{v}(t) = \dot{\mathbf{x}}(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)$. Its acceleration $\mathbf{a}$ is the derivative of its velocity, or the second derivative of its position $a(t) = \ddot{\mathbf{x}}(t)$. We will often drop the argument of these physical quantities, leaving their time-dependence implicit. It is also common to denote the constant values of these quantities with a 'naught', e.g. $\mathbf{x}_0$ for constant position vector. We define the linear momentum $\mathbf{p}(t)$ of a particle as the product of its mass m and velocity $\dot{\mathbf{x}}$, $\mathbf{p} = m\dot{\mathbf{x}}$.

Symmetries The space symmetry group for Newtonian mechanics is given by the Euclidean group $\mathrm{SO}(3) \ltimes \mathbb{R}^3$ where the compact factor acts on the coordinates by rotations $x^i \mapsto \Lambda^i{}_j x^j : \Lambda^i{}_j \delta_{ik} \Lambda^k{}_l = \delta_{jl}$ and the non-compact factor acts by translations $x^i \mapsto x^i + a^i$. In Newtonian mechanics the time variable does not mix with the spacial coordinates. We therefore have a separate symmetry factor $\mathbb{R}$ of translations in time $t \mapsto t + c$. The physical interpretation of these space-time symmetries is that in writing equations, the origin and orientation of the coordinate system are conventions and in particular are not physical. That is, only the relative coordinates of space-time events are physical. In general, physical quantities are invariant under the space-time symmetry group. In practice we will always fix this ambiguity by specifying the coordinate system.

Note that when physical quantities are expressed in linear-algebraic language, the transformation laws are simple, that is, linear. When a physical formalism is expressed in the way, we say that the formalism is *covariant* – in this case with respect to the space symmetry group $\mathrm{SO}(3) \ltimes \mathbb{R}^3 \times \mathbb{R}$ – and that the space-time symmetry is *manifest*. It is always the case that a covariant formalism is expressed in terms of unphysical quantities because covariance means that the symmetries are manifest which means that they are realized linearly on the variables which, in turn, means that the variables are not invariant under the symmetries and hence not physical.

2.1.2 Newton's Laws

Newton I: An object of mass m in rectilinear motion $\mathbf{x}(t) = \mathbf{v}_0 t + \mathbf{x}_0$ will stay in rectilinear motion unless acted on by a force.

[1] This is the meaning of *particle* as opposed to an extended object for which we have to specify a distribution of positions as a function of time.

This statement defines the concept of *kinematics* or geodesic motion. It is equivalent to the statement that free particle trajectories satisfy the equation

$$\ddot{\mathbf{x}} = 0. \tag{2.1}$$

Which, in turn, is equivalent to the statement that, in the absence of force, momentum is conserved

$$\dot{\mathbf{p}} = 0. \tag{2.2}$$

Note that the equation is $\mathrm{SO}(3) \ltimes \mathbb{R}^3 \times \mathbb{R}$ covariant. In a local lagrangian system, the existence of a global symmetry implies, via Noether's theorem, the existence of a conserved current (c.f. section 2.1.5). Suffice it here to say that the current associated to the translations is the momentum $\mathbf{p}$. The kinematic equation (2.2) expresses that it is *conserved*, that is, constant in time. Similarly, there is a current associated to the rotational invariance – the *angular momentum* $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. Noting that $\mathbf{p} \parallel \dot{\mathbf{x}}$ and using the kinematical equation, we see that the angular momentum is conserved $\dot{\mathbf{L}} = 0$. Finally, the current associated to a shift in the time variable is $T = \frac{1}{2}m\dot{\mathbf{x}}^2 = \frac{1}{2m}\mathbf{p}^2$ and is called the (*kinetic*) *energy* (the normalization is conventional). Again, the kinematic equations imply that it is conserved.

Newton II: An object of mass m, when acted on by a force $\mathbf{F}$ will deviate from rectilinear motion with an acceleration $\mathbf{a} = \ddot{\mathbf{x}}$ according to the relation

$$\mathbf{F}(\mathbf{x}, t) = m\mathbf{a}(t) \tag{2.3}$$

or, equivalently,

$$\mathbf{F}(\mathbf{x}, t) = \dot{\mathbf{p}}(t). \tag{2.4}$$

This is the statement of *dynamics* or the deviation from geodesic motion due to an external influence. An equivalent way to express this is that the second law defines the *source* $\frac{1}{m}\mathbf{F}(\mathbf{x}, t)$ for the kinematic (read "source-less") equation $\ddot{\mathbf{x}} = 0$ or $\dot{\mathbf{p}}$ of the first law. In this sense, it defines what is meant by a force.

An important point to note is that the second law is *linear* in the force. This implies that we have the

Principle of superposition: If there are 2 forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the same particle, the effective force $\mathbf{F}_{\text{total}}$ the particle experiences is the vector sum of the individual forces $\mathbf{F}_{\text{total}} = \mathbf{F}_1 + \mathbf{F}_2$. In particular, two opposing forces of equal magnitude and opposite direction applied to the same particle produce no net dynamics.

A second important point is that the second law can be interpreted as defining the mass of an object to be the ratio of a stimulus $|\mathbf{F}|$ to the response $|\mathbf{a}|$ in its motion by $m = \frac{|\mathbf{F}|}{|\mathbf{a}|}$. In this sense, we see that m refers to an *inertial mass*, that is, a property describing its resistance to a change in motion.

Newton III: An object, when acted on by some agent with a force $\mathbf{F}_{\text{action}}$ will exert a force $\mathbf{F}_{\text{reaction}}$ on the agent of equal magnitude and opposite direction, *id est*,

$$\mathbf{F}_{\text{action}} = -\mathbf{F}_{\text{reaction}}. \tag{2.5}$$

This is a statement of linear momentum conservation during a collision. Intuitively, when pressing on an object with some force, the object presses back (otherwise, we wouldn't be able to feel it). The third law is the statement that the reaction force is of precisely the same magnitude as the applied force.[2]

Newton's law of universal gravitation Consider two objects, one of mass m_1 and the other of mass m_2. They will exert a gravitational force on one another given by

$$\mathbf{F}_{\text{gravitation}} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}} \tag{2.6}$$

where $G \equiv \frac{1}{4\pi\kappa} \approx 6.67259 \times 10^{-11}\mathrm{Nm}^2\mathrm{kg}^{-2}$ is the least precisely known fundamental constant of nature.

This formula is fundamentally different from the second law. Firstly, it introduces a constant G which is claimed to be fundamental in the sense that it is the same number no matter what material form the masses take.

Secondly and related to this, the masses $m_{1,2}$ entering it could be called *gravitational masses* since they describe a property of an object we are calling gravitation and should probably have been called gravitational charge. A priori, this is a different type of mass than the inertial mass entering the dynamical second law. Therefore, Newton's law of universal gravitation is making the bold assertion that gravitational mass and inertial mass are equivalent.

Finally, we note that setting $\mathbf{a}_2 = -\frac{Gm_2}{r^2}\hat{\mathbf{r}}$ to be the acceleration due to the gravity of the mass m_2 at a distance r from its position, we find the form $\mathbf{F}_{\text{gravitation}} = m_1\mathbf{a}_2$. Taking $m = m_{♁}$ to be the mass of the earth and $r = r_{♁}$ its radius, we find the famous acceleration due to gravity $g = |\mathbf{a}_{♁}| \approx 9.8ms^{-2}$.

[2] This law causes some confusion when used in conjunction with the second to the effect that if the object pushes back with exactly the same force, the forces should cancel and there should be no resulting dynamics. Indeed, there is no relative dynamics between the hand and the object, rather, the object will accelerate relative to the ground against which we are also pushing when we try to accelerate the object.

2.1.3 Potentials

There are various drawbacks to the vector space formulation of Newtonian mechanics, not the least of which is that all defining equations are vector equations. In most cases of physical interest, drastic simplifications are made possible by switching to a description in terms of energy. Suppose the force is *holonomic* $\nabla \times \mathbf{F} = 0$. Then we can define the *potential energy function* $U(\mathbf{x}, t)$ s.t. $\mathbf{F} = -\nabla U$. The sign comes from the observation that a force acts so as to decrease the potential energy. The total energy $E = T + U$ is the sum of the kinetic and potential energy. Just as the kinetic energy was conserved in the absence of external forces, the total energy of a system is conserved when the force is the gradient of the potential energy and the latter does not depend explicitly on time: $\dot{E} = m\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} + \nabla U \cdot \dot{\mathbf{x}} + \frac{\partial U}{\partial t} = 0$ by Newton II. This is the famous *principle of the conservation of energy*. It is very powerful because it is, in the cases in which it applies, equivalent to the second law but it is a scalar equation, making it much easier to use.

2.1.4 Hamiltonian

Very closely related to the energy formulation of Newtonian mechanics is the *Hamiltonian formalism*. In this formulation, the fundamental variables are the position x^i and the momentum p_i vs. the position and the velocity (c.f. section 2.1.5). A physical trajectory is a graph in the *phase space* $\{x^i, p_i\}_{i=1}^3$.[3] Note that the momentum is treated as a 1-form in this formulation (which, as we will soon learn, is the proper interpretation of this quantity). The dynamics is encoded in the *Hamiltonian* $H(x, p)$ which, when evaluated on a point in the phase space, is equal to the energy E introduced in section 2.1.3. In particular, it is the sum of the kinetic energy function $T(p)$ which we take to be a function only of the momentum (usually $T = \frac{1}{2m}p^2$)[4] and the potential energy function $U(x)$ with we take to depend only on the position. We can now easily show that the definition of momentum and the second law imply

Hamilton's Equations

$$\dot{x}^i = \frac{\partial H}{\partial p_i}$$

[3] In general, the space parameterized by x may be any $\mathcal{C}^2$ 3-manifold M. Then the phase space is defined to be the co-tangent bundle T^*M. From this point of view, it is easy to see the symplectic structure.

[4] On a more general space the kinetic energy function will depend on x through the metric: $T = \frac{1}{2m}g^{ij}(x)p_i p_j$.

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} \tag{2.7}$$

The form of these equations[5] displays an important aspect of the phase space, namely, its *symplectic structure*: The phase space comes equipped with its Poincaré 1-form $p_i \mathrm{d}x^i$ and therefore the symplectic 2-form $\mathrm{d}p_i \wedge \mathrm{d}x^i$. This statement is often implicit in a discussion of Hamilton's equations in which one considers transformations of the variables (x, p) which preserve the 'form' of Hamilton's equations. These *canonical transformations* are the symplectomophisms – smooth transformations on the phase space coordinates which preserve the symplectic structure.

From the Poincaré 1-form and a phase space trajectory γ (a path in phase space) we can construct the *action* (*functional*)

$$S[\gamma] = \int_\gamma p_i \mathrm{d}x^i. \tag{2.8}$$

A useful generalization of the phase space includes the time coordinate as an additional variable. This 7-dimensional space is called the *extended phase space*. Similarly to the action functional (2.8) on the un-extended phase space, from the Poincaré 1-form and the Hamiltonian function we can construct the action functional[6]

$$S[\gamma] = \int_\gamma \left[p_i \mathrm{d}x^i - H(x, p)\mathrm{d}t \right]. \tag{2.9}$$

It is important to remember that $(x^i(t), p_i(t))$ are functions of the time parameter t. As such, we are allowed to "vary" them. That is, we consider an infinitesimal deformation of the trajectory $\gamma \to \gamma' = \gamma + \delta\gamma$. The *variational* or *functional derivative* of the action functional is defined to be the linear part of $S[\gamma']$, that is

$$\frac{\delta S}{\delta \gamma} \equiv \lim_{\delta\gamma \to 0} S[\gamma + \delta\gamma]. \tag{2.10}$$

This notation δ for ∂ for the functional derivative is customary in the calculus of variations.

The path has two linearly independent variations in the x-direction and the p-direction. It is therefore possible to define the partial variations in these directions. The following notation is customary (and, hopefully, self-explanatory)

$$\delta S = \delta x^i \frac{\delta S}{\delta x^i} + \delta p_i \frac{\delta S}{\delta p_i}. \tag{2.11}$$

[5]Note that the Hamilton equations (2.7) are coupled ordinary differential equations of the *first* order which contain the same information as Newton's second law which is second order.

[6]Note that this is (the negative of) an integrated Legendre transformation of $H(x, p)$.

The action is called *stationary* when $\delta S = 0$. Since the x- and p-variations are independent, stationary action implies

$$\begin{aligned} 0 &= \frac{\delta S}{\delta x^i} = -\dot{p}_i - \frac{\partial H}{\partial x^i} \\ 0 &= \frac{\delta S}{\delta p_i} = \dot{x}^i - \frac{\partial H}{\partial p_i} \end{aligned} \tag{2.12}$$

and we recover Hamilton's equations (2.7).[7] This is the *principle of stationary action*; the physical trajectories in phase space are those which extremize (usually minimize) the action.

This point of view has many advantages. Firstly, it generalizes the intuitive idea that physical processes are such that they minimize the energy. Secondly, a modification of this formalism (c.f. section 2.1.5) us a very powerful tool to solve complicated concrete problems in analytical dynamics especially dynamical systems defined in terms of constrained degrees of freedom. Finally, the principle of stationary action will fit seamlessly into the description of quantum mechanical systems c.f. chapter **??**. There we will see that quantum mechanical corrections to classical mechanics have the interpretation of deviations $\delta\gamma$ of the phase space trajectories.

2.1.5 Lagrangian

A complementary formulation of Newtonian mechanics is the Lagrangian formulation. The Lagrangian formulation is a "Legendre transform of the Hamiltonian formulation". Indeed, the space replacing the phase space of Hamiltonian mechanics is the space parameterized by positions q^i and velocities $\dot{q}^i$.[8] The *Lagrangian function* $L(q^i, \dot{q}^i)$ is the Legendre transform of the Hamiltonian $H(x^i, p_i)$

$$L(q^i, \dot{q}^i) = p_i \dot{q}^i - H(x, p). \tag{2.13}$$

Plugging in the form $H = T + U$ and substituting $p_i = m\dot{q}_i$ we find that $L = T - U$. By the definition of the action (2.9), the Lagrangian function is the unintegrated action density

$$S[\gamma] = \int_\gamma L(q, \dot{q}) \tag{2.14}$$

[7] In the first equation, we have integrated the time derivative by parts. This is legal since the surface term is proportional to δx which vanishes when evaluated at the endpoints of the path (recall that we vary the path but keep the endpoints fixed).

[8] For a general space M the Lagrangian formalism is defined on the tangent bundle TM. It is conventional in this context to denote the positions by q^i instead of x^i.

where γ is re-interpreted as a section of the tangent bundle.[9] The stationary phase principle in this case implies the *Euler-Lagrange equation*

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} = 0. \tag{2.15}$$

The advantages of the Lagrangian formalism over the Hamiltonian one include the use of the stationary action principle to solve complicated problems in anaytical dynamics and the possibility to easily manifest Lorentz invariance in relativistic theory (c.f. **??**).

Noether's theorem Consider a time-independent infinitesimal transformation of coordinates $q^i \mapsto q'^i \approx q^i + \epsilon^i$ under which the action is invariant $S \mapsto S$, that is, a *symmetry* of the theory. Now promote the parameter $\epsilon^i \to \epsilon^i(t)$ to a function. The resulting change in the action must be proportional to $\dot{\epsilon}$ since, when ϵ is constant, the transformation is a symmetry. Given this, there must be a function $J_i(q, \dot{q}, t)$ such that $\delta S = \int \dot{\epsilon}^i J_i = \epsilon^i J_i| - \int \epsilon^i \dot{J}_i$. The first term is the "surface term" $\epsilon J|_{t_i}^{t_f}$ – the difference of the quantity ϵJ at the final time t_f and the initial time t_i. The second term vanishes by the equation of motion. (Prove it!) When ϵ is constant, we see that J is conserved $J(t_f) = J(t_i)$. Such conserved functions arising from symmetries of the theory are called *Noether currents*. In this case the symmetry is a translation and the current is $J_i = \partial L/\partial \dot{q}^i$, which is the definition of the momentum. In the absence of external forces, this is indeed conserved.

With an eye to the future we will refer to a time-independent symmetry as a *global symmetry*. Noether's theorem is the statement that for every global symmetry of the action, there is a conserved current and *vice versa*.

2.1.6 Examples

Gravitational potentials and solvable systems.
Potential theory and the need for fields.

[9] Usually this whole story is reversed: One defines the Lagrangian function as the difference between the potential and kinetic energy functions and develops the Lagrangian formalism and stationary action principle. Subsequently the Legendre transformation to the Hamiltonian function is performed. It is then proven that the resulting Hamiltonian is independent of $\dot{q}$ and the phase space picture is developed.

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Bibliography

2.2 Physics

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2.3 Math

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