

1 Ch. 14.1

Let $\vec{F} = u(x, y, z)\vec{i} + v(x, y, z)\vec{j} + w(x, y, z)\vec{k}$ a vector field on \mathbb{R}^3 . Know the definition of Operators

- $div \vec{F} = \nabla \cdot \vec{F} = \partial_x u + \partial_y v + \partial_z w$
- $curl \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix}$
- $div \nabla f = \nabla \cdot \nabla f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$

Fundamental Identities

$$\begin{aligned} \nabla \times (\nabla f) &= 0, \\ div(curl \vec{F}) &= 0. \end{aligned}$$

2 Ch. 14.2

2.1 Line integral over a vector field \vec{F}

Let C be a piecewise smooth curve with parametric representation

$$\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

for $a \leq t \leq b$ the line integral of \vec{F} along C is

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{R} &= \int_C (u dx + v dy + w dz) \\ &= \int_a^b \left(u(t) \frac{dx}{dt} + v(t) \frac{dy}{dt} + w(t) \frac{dz}{dt} \right) dt \end{aligned}$$

where $u(t)$ means $u(x(t), y(t), z(t))$, etc.

1. Study (ex 6) in p970 and the similar problems in the problem set 14.2

3 14.3. Independence of Path

3.1 Conservative Vector Fields

A vector field \vec{F} is said to be *conservative* in a region D if it can be represented in D as the gradient of a continuously differentiable function f , which is then called a *scalar potential* of \vec{F} ;

$$\vec{F} = \nabla f.$$

Note that the above condition automatically implies that $\text{curl } \vec{F} = 0$.
Important Definition.

- A region D is *connected* if any two points in D can be joined by a piecewise smooth curve that lies entirely within D .
- A *connected* region D is *simply connected* if every closed curve in D encloses only points that are also in D .

Important Theorem.

If \vec{F} and $\text{curl } \vec{F}$ are *continuous in the simply connected region* D , then \vec{F} is conservative in D if and only if $\text{curl } \vec{F} = 0$.

The conditions for the above theorem is crucial! If you use the above theorem you should state the conditions clearly.

Fundamental theorem of line integrals

If \vec{F} is *conservative* on the the region D and let f be a scalar potential for $\vec{F} = \nabla f$. Then, if C is *any piecewise* smooth curve lying entirely within D , with initial point P and terminal point Q , we have

$$\int_C \vec{F} \cdot d\vec{R} = f(Q) - f(P)$$

Application

1. For a given line integral find \vec{F}
2. Test if \vec{F} and $\text{curl } \vec{F}$ are *continuous in the simply connected region* D . If so and $\text{curl } \vec{F} = 0$, \vec{F} is conservative.
3. Now find the potential function.
4. Use the fundamental theorem of line integral.

Study (ex 2) in p977 and (ex4) in p979. *There will be a question, which requires exactly the above procedure. You should clearly state the conditions for the fundamental theorem of line integral.* It is important to know over what region in will be line integral of $\text{vec}F$ with $\text{curl } \text{vec}F = 0$.

For example consider problem 34 in p.983, which goes as follows

- over what region in the xy -plane will the line integral

$$\int_C \left((-yx^{-2} + x^{-1})dx + x^{-1}dy \right)$$

be independent of path?

First we note that

$$\vec{F} = u(x, y)\vec{i} + v(x, y)\vec{j}$$

where

$$\begin{aligned} u &= -yx^{-2} + x^{-1}, \\ v &= x^{-1} \end{aligned}$$

Then we find that

$$\text{curl } \vec{F} = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \vec{k} = 0.$$

Now we have to worry if \vec{F} is continuous in the simply connected region! We, however, see that \vec{F} is singular (not defined) for $x = 0$, which means the whole y -axes! Consequently the line integral is independent of path over the region $x < 0$ and $0 < x$.

- evaluate the line integral above if C is defined by

$$\vec{R}(t) = (\cos^3 t)\vec{i} + (\sin 3t)\vec{j}$$

for $0 \leq t \leq \frac{\pi}{3}$.

the curve C looks nasty, which doesn't really matter. There are only two possibilities (i) C cross the y -axes. (ii) C does not cross the y -axes. To see what is the case we examine if $\cos^3 t = 0$ for $0 \leq t \leq \frac{\pi}{3}$. As $\cos 0 = 1$ and $\cos t$ decrease and becomes 0 for $t = \frac{\pi}{2}$, we conclude that $\cos^3 t \neq 0$ for $0 \leq t \leq \frac{\pi}{3}$. Since C does not contains singularity and lying entirely in the (good) region $0 < x$ and $\text{curl}\vec{F} = 0$ the line integral is path independent.

Then we can use the fundamental theorem of line integral. Let's find the potential f of $\vec{F} = \nabla f = f_x\vec{i} + f_y\vec{j}$ From $f_x = u$ and $f_y = v$

$$f(x, y) = \int v dy = \int (x^{-1}) dy = x^{-1}y + c(x)$$

Now use

$$f_x = -yx^{-2} + x^{-1} = -yx^{-2} + \frac{\partial c(x)}{\partial x}$$

Thus $c(x) = \log x$. We have

$$f = x^{-1}y + \log x$$

Now $P = (\cos^3 0, \sin 0)$ and $Q = (\cos^3 \frac{\pi}{3}, \sin \pi)$ The line integral is then

$$\int_C \vec{F} \cdot d\vec{R} = f(Q) - f(P)$$

4 14.4. Green's Theorem

What is *Jordan Curve* and its application in the definition of *simply connected region*?

What is *positively oriented piecewise oriented boundary curve* of a simply connected region D ?

Green's Theorem for Simply Connected Region

Let D be a simply connected region with a positively oriented piecewise-smooth boundary C . Then if the vector field $\vec{F}(x, y) = M(x, y)\vec{i} + N(x, y)\vec{j}$ is continuously differentiable on D , we have

$$\int_C (Mdx + Ndy) = \int \int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Read the proof in p986.

Green's Theorem for Doubly Connected Region

Let R be a doubly-connected region (one hole) in the plane, with outer boundary C_1 oriented *counter-clockwise* and boundary C_2 of the hole oriented *clockwise*. If the boundary curves and $\vec{F}(x, y) = M(x, y)\vec{i} + N(x, y)\vec{j}$ satisfy the hypotheses of Green's theorem then

$$\int \int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_{C_1} (Mdx + Ndy) + \int_{C_2} (Mdx + Ndy)$$

Assuming that we know the Green's theorem of simply connected region, sketch the essential idea (see p.989 figure) of the proof of the above theorem. Study Ex 4 in p989 and the similar problem in the problem sets. Remember that I emphasized the importance of this concept so many times.

5 14.5 Surface Integrals

Let S be a surface defined by $z = f(x, y)$ and R_{xy} its projection on the xy -plane. If f , f_x and f_y are continuous in R_{xy} and g is continuous on S , then the *surface integral* of g over S is

$$\int \int_S g(x, y, z) dS = \int \int_{R_{xy}} \int g(x, y, f(x, y)) \sqrt{f_x^2 + f_y^2 + 1} dA_{xy}.$$

Solve problem 19 in p.1001 as a simple exercise but why now? do it using stokes theorem

6 14.6. Stokes Theorem

What is the compatible orientation of a closed path C on the surface S with the orientation on S ?

Stoke's theorem

Let S be an oriented surface with unit normal vector \vec{N} , and assume that S is bounded by a closed, piecewise smooth curve C whose orientation is compatible with that of S . If \vec{F} is a vector field that is continuously differentiable on S , then

$$\int_C \vec{F} \cdot d\vec{R} = \int \int_S (\text{curl} \vec{F} \cdot \vec{N}) dS$$

There are two crucial examples (ex 2 in p1005) and ex 3 in p1006. The actual question will be roughly the same level of difficulty. Study the similar problems in the problem sets. Assume that S' has the same boundary as S with the same orientation and can be continuously deformed to each other (or enclosing simply connected region) Then

$$\int \int_S (\text{curl} \vec{F} \cdot \vec{N}) dS = \int \int_{S'} (\text{curl} \vec{F} \cdot \vec{N}') dS'$$

Finally read *Evaluation of Line Integrals* part in p.1020.