MAT 205, Calculus III

Summary I for Final Exam

Study Similar Problems in the Problem Sets

1 Multiple Integral

1.1 Double Integral

The area of the region D in the xy-plane is given by

$$A = \int\int_D dA.$$

If f us continuous and $f(x, y) \geq 0$ on the region D, the volume of the solid under the surface $z = f(x, y)$ above the region D is given by

$$V = \int\int_D f(x, y) dA$$

1. Let D is the region bounded by $y = \sqrt{x}$, $y = 2-x$, and $y = 0$. Evaluate

$$\int_D 4x \, dA$$

   (See ex.1 in p890)

2. Find the volume under the plane $3x+y-z = 0$ above the elliptic region bounded by $4x^2 + 9y^2 \leq 36$, with $x \geq 0$ and $y \geq 0$. (See ex.3 in p892)

1.1.1 Change of Variables

Let T be a change of variable that maps a region D^* in uv-space onto a region D in xy-space, where

$$T: \quad x = x(u, v) \quad y = y(u, v).$$

Then

$$\int\int_D f(x, y) dx dy = \int\int_{D^*}\int f(u, v)|J(u, v)| du dv$$

where

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

1. Find the Jacobian for the change of variables from retangular to polar coordinates, namely

$$T: \quad x = r\cos\theta \quad y = r\sin\theta.$$

(Ex. I, p. 945)

$$J = r$$

2. Evaluate

$$\int\int_D (x^2 + y^2) dA$$

where D us tge region bounded by the x-axis, the line $y = x$, and the circle $x^2 + y^2 = 9$. (study Ex 1&2 in p.900)

3. Find the volume of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. (study Ex 3 in p.900-901)

4. Let D be the region in the xy-plane that is bounded by the coordinates axes and the line $x + y = 1$. Use the change of variable $u = x - y$, $v = x + y$ to compute the integral

$$\int\int_D \left(\frac{x-y}{x+y}\right)^5$$

5. Let D be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a, b \neq 0)$. Express the area of the region D as a double integral and find the area. (ex.3 in p.946)

6. Use the change of variable $x = ar\cos\theta$, $y = brsin\theta$ to evaluate

$$\int\int_S \exp\left(-\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dydx$$

where S is the quarter ellipse;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, \quad x \geq 0, y \geq 0$$

1.1.2 Surface Area

Assume that the function $f(x, y)$ has continuos partial derivatives f_x and f_y in a region R of the xy-plane. Then the portion of the surface $z = f(x, y)$ that lies over R has surface area

$$S = \int\int_R \sqrt{f_x^2 + f_y^2 + 1} dA$$

Study ex.1 in p.908.

2 Trifle Integrals

2.0.3 Trifle Integral over a general region

If S is a region in space that is bounded below by the surface $z = u(x,y)$ and above by $z = v(x,y)$ as (x,y) varies over the planar region A, then

$$\int\int\int_S f(x,y,z)\, dV = \int\int_A \int_{u(x,y)}^{v(x,y)} f(x,y,z) dz dA$$

The volume V of the solid region S is

$$V = \int\int\int_S dV$$

1. Set up but do not evaluate a triple integral for the volume of the solid S that is bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the plane $y + z = 2$. (see ex.2 in p.917)

2. Set up but do not evaluate a triple integral for the volume of the solid S that is bounded above by the paraboloid $z = 6 - x^2 - y^2 + z^2$ and below by the plane $z = 2x^2 + y^2$.

2.0.4 Change of Variables

; Let T be a change of variable that maps a region R^* in uvw-space onto a region R in xyz-space, where

$$T: \quad x = x(u,v,w) \quad y = y(u,v,w) \quad z = z(u,v,w) \quad .$$

Then

$$\int\int_R \int f(x,y,z) dx dy dz = \int\int_{R^*} \int f(u,v,w) |J(u,v,w)| du dv dw$$

where

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

An useful coordinate system to deal with problems with spherical symmetry is spherical coordinates, which lable a point P by a trifle (ρ, θ, ϕ), where (see the shaded box in p.939)

$\rho =$ the distance from the origin to the point P, $\quad \rho \geq 0$,
$\theta =$ the polar angle, $\quad 0 \leq \theta < 2\pi$,
$\phi =$ the angle measured down from the positive z-axies to the ray from the origin through the point P

Conversion formulars for spherical to rectangular is

$$\begin{aligned} x &= \rho \sin\phi \cos\theta, \\ y &= \rho \sin\phi \sin\theta, \\ z &= \rho \cos\phi, \end{aligned}$$

with the Jacobian

$$|J(\rho, \theta, \phi)| = \rho^2 \sin\phi$$

Conversly

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2 + z^2}, \\ \tan\theta &= \frac{y}{x}, \\ \phi &= \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \end{aligned}$$

etc.

1. Find the Jacobian of the change of variables given by

$$x = u\cos v, \qquad y = u\sin v, \qquad z = we^w$$

2. Let S is the solid sphre defined by $x^2 + y^2 + z^2 \leq 3$ evaluate

$$\int\int\int_S \frac{dxdydz}{\sqrt{x^2 + y^2 + z^2}}$$

   See Q.50-53 in p.943. I will give the Jacobian.