John Nash and "A Beautiful Mind"

John Milnor

John F. Nash Jr. published his first paper, with his father, at age seventeen. His thesis, at age twenty-one, presented clear and elementary mathematical ideas which inaugurated a slow revolution in fields as diverse as Economics, Political Science, and Evolutionary Biology. During the following nine years, in an amazing surge of mathematical activity, he sought out and often solved the toughest and most important problems he could find in geometry and analysis. Then a mental breakdown led to thirty lost and painful years, punctuated by intermittent hospitalization, as well as occasional remission. However, in the past ten years a pronounced reawakening and return to mathematics has taken place. Meanwhile, the importance of Nash's work has been recognized by many honors: the von Neumann Prize, fellowship in the Econometric Society and the American Academy of Arts and Sciences, membership in the US National Academy of Sciences, culminating in a Nobel Prize.

"A Beautiful Mind"

Sylvia Nasar's biography, *A Beautiful Mind*,[1] tells this story in carefully documented detail, based on hundreds of interviews with friends, family, acquaintances and colleagues, as well as a study of available documents. Indeed, she is a highly talented interviewer, and in some cases seems to unearth material far beyond what one might expect. She gives detailed descriptions of the deliberations, not only for the 1958 Fields Medals where Nash had been one possible candidate, but even for the 1994 Nobel Prize in Economics: deliberations which were so explosive that they led to a radical restructuring of the prize and a complete change in the nominating committee. In general her sources are carefully identified, but in these particular cases they remain anonymous.

Although Nasar's training was in economics rather than mathematics, she is able to provide background, rough descriptions, and precise references for all of Nash's major work. Also, she gives a great deal of background description of the places and persons who played a role in his life. (Mathematical statements and proper names are sometimes a bit garbled, but the astute reader can usually figure out what is meant.) Thus we find fascinating information about the history of Carnegie Tech, Princeton, the Rand Corporation, MIT, the Institute for Advanced Study, and the Courant Institute, and also information about many well known and not so well known mathematical personalities. The discussion leads into many interesting byways: Her description of MIT is interwoven with a discussion of the McCarthy era, while her description of the Rand Corporation and of von Neumann leads to a discussion of the relation of Game Theory to Cold War politics. (Von Neumann, who advocated a preemptive strike against the Soviet Union, may have been the original model for Kubrik's "Dr. Strangelove".)

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[1] Sylvia Nasar, *A Beautiful Mind: A Biography of John Forbes Nash, Jr.*, Simon & Schuster, 1998, $25.00 hardcover, 459 pages, ISBN: 0684819066. (See also [Nas].)

Any discussion of Nasar's book must point out a central ethical dilemma: *This is an unauthorized biography, written without its subject's consent or cooperation.* Nash's mathematical activity was accompanied by a tangled personal life, which Nasar describes in great detail. This material is certainly of interest to a wide audience. (Oliver Sacks, quoted in the publisher's blurb, writes that the book is "*extraordinarily moving, remarkable for its sympathetic insights into both genius and schizophrenia*".) Inevitably however, the publication of such material involves a drastic violation of the privacy of its subject.

The book is dedicated to Alicia Nash, first his wife and later his steadfast companion, whose support through impossible difficulties has clearly played a major role in his recovery.

Nash's Scientific Work

Pure mathematicians tend to judge any work in the mathematical sciences on the basis of its mathematical depth, and the extent to which it introduces new mathematical ideas and methods, or solves long standing problems. Seen in this way, Nash's prize work is an ingenious but not surprising application of well known methods, while his subsequent mathematical work is far more rich and important. During the following years, he proved that every smooth compact manifold can be realized as a sheet of a real algebraic variety,[2] proved the highly anti-intuitive C^1-isometric embedding theorem, introduced powerful and radically new tools to prove the far more difficult C^∞-isometric embedding theorem in high dimensions, and made a strong start on fundamental existence, uniqueness, and continuity theorems for partial differential equations. (Compare [K1] and [M] for some further discussion of these results.)

However, when mathematics is applied to other branches of human knowledge, we must really ask a quite different question: *To what extent does the new work increase our understanding of the real world?* On this basis, Nash's thesis was nothing short of revolutionary. (Compare [N21], as well as [U].) The field of Game Theory was the creation of John von Neumann, and was written up in collaboration with Morgenstern. (One much earlier paper had been written by Zermelo.) The von Neumann-Morgenstern theory of zero-sum two-person games was extremely satisfactory, and certainly had application to warfare, as was amply noted by the military. However, it has few other applications. Their efforts to develop a theory of n-person or non-zero-sum games for use in economic theory were really not very useful. (Both Nash and the reviewer participated in one experimental study of n-person games [N10]. As far as I know, no such study has ever been able to detect much correlation between von Neumann-Morgenstern "solutions" and the real world.)

Nash, in his thesis, was the first to emphasize the distinction between *cooperative* games as studied by von Neumann and Morgenstern (roughly speaking these are games where the participants can sit around a smoke filled room and negotiate with each other), and the more fundamental *non-cooperative* games where there is no such negotiation. In

[2] Artin and Mazur used this work [N7] to prove the important result that every smooth self-map of a compact manifold can be approximated by one for which the number of periodic points of period p is less than some exponential function of p. For more than thirty years, no other proof was known. However, Kaloshin has recently given a much more elementary argument, based on the Weierstrass Approximation Theorem.

fact, the cooperative case can usually be reduced to the non-cooperative case by incorporating the possible forms of cooperation into the formal structure of the game. Nash made a start on the cooperative theory with his paper [N5] on the Bargaining Problem, to some extent conceived while he was still an undergraduate. (A related much earlier study is due to Zeuthen.) As one remark in this paper, Nash conjectured that every cooperative game should have a *value* which expresses "the utility to each player of the opportunity to engage in the game". Such a value was constructed by Shapley a few years later.

However the major contribution, which led to his Nobel Prize, was to the non-cooperative theory. Nash introduced the fundamental concept of *equilibrium point*: a collection of strategies by the various players, such that no one player can improve his outcome by changing only his own strategy. (Something very much like this concept had been introduced by Cournot, more than a hundred years earlier.) By a clever application of the Brouwer Fixed Point Theorem, he showed that at least one equilibrium point always exists. (For more detailed accounts, see [OR], [M].)

Over the years, the developments from Nash's seemingly simple idea have led to fundamental changes in Economics and Political Science. Nasar illustrates the dollars and cents impact of game theoretic ideas by describing "The Greatest Auction Ever" in 1994, when the US government sold off large portions of the electromagnetic spectrum to commercial users. A multiple round procedure was carefully designed by experts in the game theory of auctions to maximize both the payoff to the government and the utility of the purchased wavelengths to the respective buyers. The result was highly successful, bringing more that $10 billion to the government, while guaranteeing an efficient allocation of resources. By way of contrast, a similar auction in New Zealand, without such a careful game theoretic design, was a disaster in which the government realized only about 15% of its expected earnings, and the wavelengths were not efficiently distributed. (In one case, a New Zealand student bought a television station license for one dollar.)

One totally unexpected triumph of equilibrium theory has been its application to population genetics and evolutionary theory. Based on the pioneering work of Maynard Smith, game theoretic ideas are now applied to the competition between different species, or within a species. (Compare [MS], [HS], [W]. A more precise form of this theory, popularized by Dawkins [D1], holds that the competition is rather between individual genes.) There has also been an interesting reverse flow of ideas, from evolution back to economics. According to Binmore (in [W]):

> "*Despite Nash's remarks in his thesis about a possible evolutionary*[3] *interpretation of the idea of a Nash equilibrium, attention at the time was focused almost entirely on its interpretation as the only viable outcome of careful reasoning by ideally rational players.* ⋯ *Fortunately* ⋯ *Maynard Smith's book "Evolution and the Theory of Games" directed game theorists' attention away from their increasingly elaborate definitions of rationality. After all, insects can hardly be said to think at all, and so rationality cannot be*

[3] Here Binmore does not refer to biological evolution, but rather to a dynamic process in which repeated plays of a game converge to an equilibrium. Unfortunately, this discussion from Nash's thesis does not appear in his published work.

so crucial if game theory somehow manages to predict their behavior under appropriate conditions. Simultaneously the advent of experimental economics brought home the fact that human subjects are no great shakes at thinking either. When they find their way to an equilibrium of a game, they typically do so using trial-and-error methods."

In all of the applications, one very important corollary must be emphasized: *Although equilibrium theory, as developed by Nash and his successors, seems to provide the best known description of what is likely to happen in a competitive situation, an equilibrium is not necessarily a good outcome for anyone.* In contrast to the classical economic theory of Adam Smith where free competition leads to best possible results, and in contrast to classical Darwinian theory, where natural selection always leads to improvement in the species,[4] the actual dynamics of unregulated competition can be disastrous. We all know that political conflict between nations can lead to an arms race which is bad for everyone concerned, and in extreme cases can lead to totally unnecessary war. Similarly, in evolutionary theory, an arms race within a species or between competing species over geological periods of time can be extremely detrimental.[5] Indeed, it seems perfectly conceivable that natural selection may sometimes lead to a dead end, and eventual extinction. Here is an only mildly exaggerated example which goes back to Darwin. (Compare [D3], [D4].) Suppose that an amorous peahen will always choose the peacock with the most splendid tail. This must lead to an evolutionary arms race, during which the tails get progressively larger until the males become so clumsy that they cannot escape from predators.

Similar comments apply to economic theory. In this case, one hopes that carefully chosen government regulation can modulate the negative effects of unbridled competition and lead to a better outcome for all concerned. However, the question as to just who will do the careful choosing is of course a matter of politics, and leads to an even more complicated problem for equilibrium theory.

References

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[D1] R. Dawkins, "The Selfish Gene", Oxford U. Press 1976.

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[4] According to Darwin, "*As natural selection works solely by and for the good of each being, all corporeal and mental endowments will tend to progress towards perfection.*" However, he also expressed a contrary view. Compare the discussion of "*why progress does not rule the history of life*" in Gould [G1]. Unfortunately, there has been much unnecessary misunderstanding and bad feeling between those like Maynard Smith who work with theoretical models for evolution and those like Gould [G2] who emphasize that the real world is more complicated than any model.

[5] Compare [DK], [D2]. When interpreting the phrase "arms race" for an evolutionary contest, remember that reproductive success is more important than battle prowess. The best evolutionary strategy is often to "make love not war"!

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[Zer] E. Zermelo, *Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels*, Proc. 5th International Congress of Mathematicians, v.**2**, 501-504, Cambridge U. Press 1913.

[Zeu] F. Zeuthen, "Problems of Monopoly and Economic Welfare," Routledge, London 1930.

[6] Historical correction: I claimed in [M] (quoted by Nasar, p. 68) that Nash's ideas on desingularizing algebraic varieties date back to the early 1950s. In fact the correct dates are 1963-64. (Nasar, Chapter 42.) Nash's construction was probably first published by Nobile [No] in 1975.

Sidebar

Publications by John Nash

[N1] J. F. Nash Jr. (with J. F. Nash Sr.), *Sag and tension calculations for wire spans using catenary formulas*, Electrical Engineering (1945).

[N2] J. F. Nash Jr., *Equilibrium points in n-person games*, Proc. Nat. Acad. Sci. USA **36** (1950) 48-49. (Also in [K2].)

[N3] ——, *Non-cooperative games*, Thesis, Princeton University, May 1950.

[N4] —— (with L. S. Shapley), *A simple three-person poker game*, pp. 105-116 of "Contributions to the Theory of Games", Annals of Math. Studies **24**, Princeton U. Press 1950.

[N5] ——, *The bargaining problem*, Econometrica **18** (1950) 155-162. (Also in [K2].)

[N6] ——, *Non-cooperative games*, Annals Math. **54** (1951) 286-295. (Also in [K2].)

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[N8] —— (with J. P. Mayberry and M. Shubik), *A comparison of treatments of a duopoly situation*, Econometrica **21** (1953) 141-154.

[N9] ——, *Two-person cooperative games*, Econometrica **21** (1953) 128-140.

[N10] —— (with C. Kalisch, J. Milnor and E. Nering), *Some experimental n-person games*, pp. 301-327 of "Decision Processes", Thrall, Coombs and Davis editors, Wiley 1954.

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[N17] ——, *Le problème de Cauchy pour les équations différentielles d'un fluide général*, Bull. Soc. Math. France **90** (1962) 487-497.

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[N19] ——, *Autobiographical essay*, Les Prix Nobel 1994, Stockholm: Norsteds Tryckeri, 1995.

[N20] ——, *Arc structure of singularities*, Duke J. Math. **81** (1995) 31-38. (Written in 1966.)

[N21] —— (with H. Kuhn, J. Harsanyi, R. Selten, J. Weibull, E. van Damme, P. Hammerstein), *The work of John F. Nash Jr. in Game Theory*, Duke J. Math **81** (1995) 1-29. (Also in J. Econom. Theory **69** (1996) 153–185.)

In addition, there were also a number of Rand Corporation memoranda written by Nash, on diverse subjects such as machine memories and parallel control (Nasar pp. 403, 407, 411, 436), as well as an unpublished lecture at the World Congress of Psychiatry in Madrid in 1996.