Introduction to Part 1:

How these papers came to be written. (version of 12-7-95)

During the 1950's, I worked on an ongoing project of trying to understand one particularly simple class of manifolds, namely $2n$-dimensional manifolds which are $(n-1)$-connected. Although my intended paper on this subject was never finished, the project none the less led to the eight papers which follow.[1]

To understand how this came about, let me first describe the original plan. The homotopy theory of a closed manifold M^{2n} which is $(n-1)$-connected is relatively easy to describe. If the middle Betti number is β, then the manifold can be obtained (up to homotopy type) by attaching a $2n$-cell to a bouquet

$$W^n \;=\; S^n \vee \cdots \vee S^n$$

of β copies of the n-sphere. The attaching map can be described as an element of a homotopy group $\pi_{2n-1}(W^n)$ which is reasonably well understood. More explicitly, this group splits as the direct sum of β copies of the group $\pi_{2n-1}(S^n)$ together with $\beta(\beta-1)/2$ free cyclic groups, corresponding to Whitehead products of distinct generators of $\pi_n(W^n)$.

Better still, this space can be described in terms of cohomology theory. The cohomology groups $H^k = H^k(M^{2n};\mathbf{Z})$ are zero with the exception of H^0 and H^{2n} which are free cyclic and $H^n \cong H^n(W^n;\mathbf{Z})$ which is free abelian of rank β. The bilinear pairing $H^n \otimes H^n \to H^{2n}$ is either symmetric or skew according as n is even or odd, and has determinant ± 1 by Poincaré duality. Finally the "stable" attaching map can be described by a cohomology operation[2]

$$\psi \;:\; H^n \;\to\; H^{2n}(M^{2n};\Pi_{n-1}) \;\cong\; \Pi_{n-1}\ ,$$

where Π_{n-1} is the stable homotopy group $\pi_{k+n-1}(S^k)$ for $k > n$. This operation can be described as follows. Any element $\eta \in H^n \cong H^n(W^n;\mathbf{Z})$ corresponds to a homotopy class of maps $W^n \to S^n$. Composing with the attaching map in $\pi_{2n-1}(W^n)$, we obtain an element of $\pi_{2n-1}(S^n)$ which stabilizes to the required $\psi(\eta) \in \Pi_{n-1}$.

The problem arises when one tries to flesh out this homotopy picture by constructing actual manifolds realizing specified homotopy invariants. In the simplest case $\beta = 1$, we must attach a $2n$-cell to a single n-sphere in such a way as to obtain a manifold. It seems that the best chance of carrying out this construction is to first 'thicken' the n-sphere, replacing it by a tubular neighborhood in the hypothetical M^{2n}. In other words, we must form an n-disk bundle

$$D^n \;\hookrightarrow\; E^{2n} \;\twoheadrightarrow\; S^n\ .$$

[1] See also "*On simply connected 4-manifolds*", Sympos. Int. Top. Alg., UNAM, Mexico 1958, pp. 122-128; to be reprinted in Volume 4.

[2] More generally, if a space X has the property that all cohomology groups $H^i(X;A)$ with $p < i < q$ are zero, then there is an analogous operation

$$\psi : H^p(X;\mathbf{Z}) \to H^q(X;\Pi_{q-p-1})\ .$$

Compare §8 of "Groups of homotopy spheres", on p. xx.

In the hoped for situation, the boundary sphere bundle $\partial E^{2n} = \Sigma^{2n-1}$ will be homeomorphic to the $(2n-1)$-sphere. Hence we will be able to glue on a $2n$-disk by a boundary homeomorphism h so as to obtain a closed manifold $M^{2n} = E^{2n} \cup_h D^{2n}$ with the required homotopy type. Thus we are led to the following problem:

> *For which sphere bundles* $S^{n-1} \hookrightarrow \Sigma^{2n-1} \twoheadrightarrow S^n$ *is the total space* Σ^{2n-1} *homeomorphic to the sphere* S^{2n-1} *?*

Three basic examples had been discovered by Heinz Hopf, namely the fibrations[3]

$$S^1 \hookrightarrow S^3 \twoheadrightarrow S^2 , \quad S^3 \hookrightarrow S^7 \twoheadrightarrow S^4 , \quad \text{and} \quad S^7 \hookrightarrow S^{15} \twoheadrightarrow S^8 .$$

The corresponding $(n-1)$-connected $2n$-manifolds $E^{2n} \cup_h D^{2n}$ were respectively the complex projective plane, the quaternion projective plane, and the Cayley projective plane. For circle bundles over the 2-sphere, classified by elements of the homotopy group $\pi_1(\mathrm{SO}(2)) \cong \mathbf{Z}$, the Hopf fibration was the only possibility, up to sign. However, for 3-sphere bundles over S^4 , classified by elements of $\pi_3(\mathrm{SO}(4)) \cong \mathbf{Z} \oplus \mathbf{Z}$, there are infinitely many bundles which at least have the homotopy type of the 7-sphere. More precisely, such a bundle is classified by two elements of $H^4(S^4; \mathbf{Z}) \cong \mathbf{Z}$, namely the *Pontrjagin class* p_1 and the the *Euler class* e (denoted by $\bar{c}$ in the first paper), subject only to the relation $p_1 \equiv 2\mathrm{e} \pmod{4H^4}$. It is not difficult to check that the total space Σ^7 has the homotopy type of a 7-sphere if and only if the Euler class generates $H^4(S^4; \mathbf{Z})$. Thus we potentially have infinitely many distinct 3-connected 8-manifolds, twisted versions of the quaternion projective plane, which are distinguished by their Pontrjagin classes (and in some cases by homotopy invariants as well).

It was natural to take a closer look at the structure of these hypothetical manifolds, using the Hirzebruch signature theorem.[4] For a smooth closed $4m$-manifold, this theorem expresses the signature (or "index") of the symmetric bilinear form $H^{2m} \otimes H^{2m} \to H^{4m} \cong \mathbf{Z}$ as a linear combination of the products of Pontrjagin classes which lie in dimension $4m$. For an 8-dimensional manifold, the formula reads

$$\text{signature} \;=\; \frac{7p_2 - p_1^2}{45}[M^8] .$$

In our case, the signature is ± 1 , and we can choose the the orientation so that it is $+1$. Hence the first Pontrjagin class must satisfy the congruence

$$p_1^2[M^8] + 45 \;\equiv\; 0 \qquad (\mathrm{mod}\ 7) .$$

[3] It is now known that such bundles can exist only in these particular dimensions. Compare "Some consequences of a theorem of Bott" on pages xx-xx. For background on the following remarks see Steenrod, "The Topology of Fibre Bundles", Princeton University Press 1951; as well as Milnor and Stasheff, "Characteristic Classes", Annals of Math. Studies **76**, Princeton University Press 1974.

[4] The existence of such a signature formula was proved by Thom, who worked out the 4 and 8 dimensional cases. At about the same time, Hirzebruch had conjectured the precise statement of the formula in all dimensions. Compare F. Hirzebruch, *The signature theorem: reminiscences and recreation*, pp. 3-31 of "Propects in Mathematics", Annals of Math. Studies **70**, Princeton U. Press 1971.

For any choice of p_1 *which does not satisfy this congruence, we have constructed a homotopy 7-sphere* Σ^7 *which cannot be smoothly homeomorphic to the standard 7-sphere.*

At this point, I believed that I had constructed a counter-example to the Poincaré conjecture in dimension 7. In other words, I assumed that Σ^7 could not even be continuously homeomorphic to the standard S^7. Fortunately however, before rushing into print with this claim, I did some experimentation, and discovered that this Σ^7 actually is homeomorphic to S^7. In fact, it can be obtained by pasting together the boundaries of two standard 7-disks under a boundary diffeomorphism. Hence, as an extra bonus, the proof also showed that there exists a non-standard diffeomorphism of the 6-dimensional sphere. These results are described in the paper "**On manifolds homeomorphic to the 7-sphere**", which follows.[5]

The next paper "**On the relationship between differentiable manifolds and combinatorial manifolds**", written in the same year but never published, carries the discussion a little further by tying it in with Henry Whitehead's theory of C^1-triangulation, and by citing Thom's proof that there exists a combinatorial 8-manifold, namely the $E^8 \cup_h D^8$ described above, which has no compatible differentiable structure, based on his theory of combinatorial Pontrjagin classes.[6] (A few years later, Michel Kervaire constructed a topological 10-manifold which cannot be given any differentiable structure at all, and Steve Smale constructed an analogous example in dimension twelve.[7] Still later, with Sergei Novikov's proof that rational Pontrjagin classes are actually topological invariants, it followed that the manifold $E^8 \cup_h D^8$ above has no differentiable structure at all.[8])

The paper "**Differentiable structures on spheres**", written three years later, carries out a similar argument based on hypothetical manifolds of the form $(S^p \vee S^q) \cup D^{p+q}$, where the two sub-spheres intersect transversally, with normal bundles described by elements of $\pi_{p-1}(\mathrm{SO}_q)$ and $\pi_{q-1}(\mathrm{SO}_p)$ respectively. This construction is much more robust, not being limited to dimensions 7 and 15. In fact, taking p and q divisible by 4, it actually yields non-standard structures on the $(p+q-1)$-sphere for every choice of $p \equiv q \equiv 0 \pmod 4$ with $p/2 < q < 2p$. I was able to verify this only for small values of p and q, but the corresponding assertion for p, $q \geq 16$ was proved some years later by Antonelli, Burghelea

[5] For an analogous construction in dimension 15, see N. Shimada, *Differentiable structures on the* 15*-sphere and Pontrjagin classes of certain manifolds*, Nagoya Math. J. **12** (1957) 59-69.

[6] For C^1-triangulations see Munkres, "Elementary Differential Toplogy", Princeton Univ. Press 1963. For combinatorial Pontrjagin classes compare the presentation in "Characteristic Classes", by Milnor and Stasheff.

[7] M. Kervaire, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv. **34** (1960) 257-270; and S. Smale, *The generalized Poincaré conjecture in higher dimensions*, Bull. AMS **66** (1960) 373-375 (compare footnote 16 below).

[8] S. P. Novikov, *Topological invariance of rational Pontrjagin classes*, Dokl. Akad. Nauk SSSR **163** (1965) 298-300 (Soviet Math. Dokl. **6** (1965) 921-923). For more precise information about the relationship between topological and combinatorial manifolds, see R. Kirby and L. Siebenmann, "Foundational Essays on Topological Manifolds, Smoothings, and Triangulations", Annals of Math. Studies **88**, Princeton Univ. Press 1977.

and Kahn, using an ingeneous diffeomorphism invariant due to Eells and Kuiper.[9]

These same non-standard spheres can be obtained by a bilinear pairing[10] which is described implicitly in this paper, and worked out more explicitly in the "Lectures on Differentiable Structures", on p. xx. Let me describe a smooth oriented manifold M^n as a *twisted sphere* with *twist* $f : S^{n-1} \to S^{n-1}$ if M^n can be obtained from two copies of the standard n-disk D^n by pasting boundaries together under the diffeomorphism f. Then the group Γ_n consisting of all diffeomorphism classes of twisted n-spheres can be characterized by the exact sequence

$$\pi_0\mathrm{Diff}^+(S^n) \;\to\; \pi_0\mathrm{Diff}^+(D^n) \;\to\; \pi_0\mathrm{Diff}^+(S^{n-1}) \;\to\; \Gamma_n \;\to\; 0$$

of abelian groups, where $\pi_0\mathrm{Diff}^+(M)$ is the group of diffeotopy classes of orientation preserving diffeomorphisms of M. The required pairing $\pi_k\mathrm{SO}_\ell \otimes \pi_\ell\mathrm{SO}_\mathrm{k} \;\to\; \pi_0\mathrm{Diff}^+(\mathrm{S}^{\mathrm{k}+\ell})$, and hence

$$\pi_k\mathrm{SO}_\ell \otimes \pi_\ell\mathrm{SO}_\mathrm{k} \;\to\; \Gamma_{\mathrm{k}+\ell+1}\ ,$$

can be defined as follows. If $f : \mathbf{R}^k \to \mathrm{SO}_\ell$ and $g : \mathbf{R}^\ell \to \mathrm{SO}_\mathrm{k}$ map neighborhoods of infinity to the identity map, then the commutator of the two diffeomorphisms

$$(x, y) \;\mapsto\; (x\,,\, f(x)\cdot y)\,, \qquad (x, y) \;\mapsto (g(y)\cdot x\,,\, y)$$

of $\mathbf{R}^k \times \mathbf{R}^\ell$ is the identity outside of a compact set, and hence gives rise to the required diffeomorphism of the $(k+\ell)$-sphere. If $k = 4r-1$ and $\ell = 4s-1$, then this construction yields non-standard $(4r + 4s - 1)$-spheres whenever $r/2 < s < 2r$.

The expository lecture "**Sommes de variétés différentiables et structures différentiables des sphères**", presented at a conference in Lille, introduces this group Γ_n of oriented diffeomorphism classes of twisted spheres, and also the group Θ_n of smooth oriented manifolds having the homotopy type of S^n, up to the relation which was then called "J-equivalence" but is now known as "h-cobordism". Thus $\Gamma_n \to \Theta_n$, where the Γ_n are the groups one really wants to understand,[11] but the Θ_n are much easier to deal with.

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[9] See P. L. Antonelli, D. Burghelea, and P. J. Kahn, *The non-finite homotopy type of some diffeomorphism groups*, Topology **11** (1972); and J. Eells and N. Kuiper, *An invariant for certain smooth manifolds*, Annali di Math. **60** (1962) 93-110.

[10] Compare the two papers just cited; as well as P. J. Kahn, *Characteristic numbers and oriented homotopy type*, Topology **3** (1965) 81-95; A. Kosinski, *On the inertia group of π-manifolds*, Amer. J. Math. **89** (1967) 227-248; and T. Lawson, *Remarks on the pairings of Bredon, Milnor, and Milnor-Munkres-Novikov*, Indiana Univ. Math. J. **22** (1972/73) 833-843. For an application of related ideas to Riemannian geometry, see D. Gromoll, *Differenzierbare Strukturen und Metriken positiver Krümmung auf Sphären,* Math. Annalen **164** (1966) 353-371.

[11] In particular, these Γ_n appear as the coefficient groups of obstructions to smoothing. See J. Munkres, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Annals Math. **72** (1960) 521-554; and *Obstructions to imposing differentiable structures*, Ill. J. Math **8** (1964) 361-376; as well as M. Hirsch, *Obstruction theories for smoothing manifolds and mappings*, Bull. AMS **69** (1963) 352-356.

Although I talked with René Thom only a few times during the years when these papers were written, his influence was quite important. For example, his intuitive feeling for the structure of cobordism rings went far beyond his published work. During one particularly decisive conversation, he constructed an interesting example by a technique which I called *surgery*. (The same construction was introduced independently by Andrew Wallace,[12] who called it *spherical modification*. Both terms have been frequently used in the literature.) I developed this idea in the paper "**A procedure for killing homotopy groups of differentiable manifolds**". A fairly easy Morse theory argument shows that one manifold can be obtained from another by a sequence of spherical modifications if and only if the two belong to the same cobordism class. These ideas played a key role in further work on groups of homotopy spheres.[13]

The manuscript "**Differentiable manifolds which are homotopy spheres**", written in 1959, was never published since most of its results were absorbed into a larger paper "**Groups of homotopy spheres: I** ", written in collaboration with Michel Kervaire. These papers provided a much more systematic analysis of all possible homotopy spheres of any dimension (other than 3). In particular, Θ_n is finite for $n \neq 3$. The analysis is based on an exact sequence

$$0 \;\to\; bP_{n+1} \;\to\; \Theta_n \;\to\; \Pi_n/J\pi_n(\mathrm{SO}) \;.$$

Here bP_{n+1} (denoted by $\Theta_n(\partial\pi)$ in the 1959 manuscript) is the subgroup consisting of all homotopy n-spheres which bound parallelizable manifolds, and $J:\pi_n(\mathrm{SO}) \to \Pi_\mathrm{n}$ is the stable J-homomorphism. (Compare Part 3 of this volume.) The subgroup bP_{n+1} is trivial for n even and has at most two elements for $n \equiv 1 \pmod 4$. However, this group is quite large when $n \equiv 3 \pmod 4$. In fact bP_{4m} is cyclic of order

$$a_m\, 2^{2m-2}(2^{2m-1}-1)\ \mathrm{numerator}(B_m/m)\ ,$$

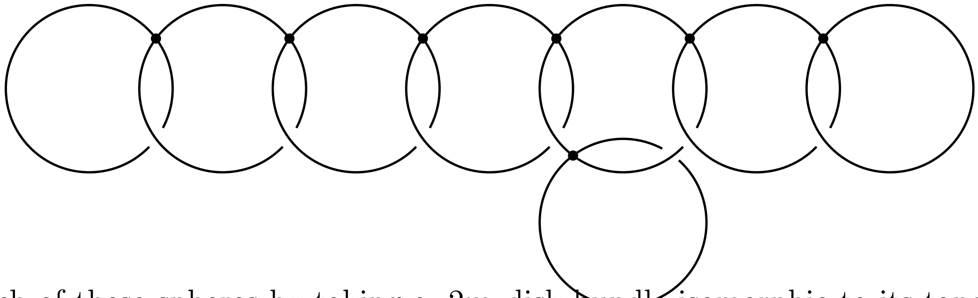
where B_m is the m-th Bernoulli number[14], and a_m is one or two according as m is even or odd. An explicit generator can be constructed as follows. (This form of the construction, based on the E_8-lattice, was suggested by Hirzebruch.[15] My original construction was somewhat more complicated.) Start with a $2m$-skeleton consisting of eight copies of the $2m$-sphere intersecting in seven points, indicated schematically as follows.

[12] A.H. Wallace, *Modifications and cobounding manifolds*, Canad. J. Math. **12** (1960) 503-528.

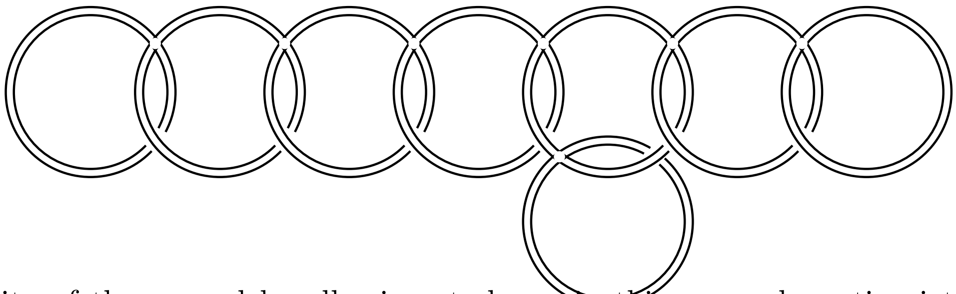
[13] For further developments, see for example, C. T. C. Wall, "Surgery on Compact Manifolds", Academic Press 1971, and W. Browder, "Surgery on Simply-Connected Manifolds", Springer 1972.

[14] These numbers grow rapidly as $m \to \infty$, as one sees from the identity $$B_m \;=\; 2\left(1+2^{-2m}+3^{-2m}+\cdots\right)(2m)!/(2\pi)^{2m}\ .$$

[15] Compare the discussion on pages 673 and 801 of Hirzebruch's "Gesammelte Abhandlungen, I", Springer 1987.



Now thicken each of these spheres by taking a $2m$-disk bundle isomorphic to its tangent disk bundle, with Euler number $+2$. The spheres are to cross each other transversally at each intersection point:



(The non-triviality of the normal bundles is not shown in this very schematic picture.) The result will be a parallelizable $4m$-dimensional manifold Q^{4m} having a homotopy sphere as boundary,[16] provided that $m > 1$. It is shown that this boundary ∂Q^{4m} is a generator for the finite cyclic group consisting of all homotopy spheres of dimension $4m-1$ which bound parallelizable manifolds.

The more precise results of these papers seem to be achieved at the cost of replacing the groups Γ_n by the cruder groups Θ_n. In other words, topological spheres were replaced by homotopy spheres, and the relation of diffeomorphism by the apparently weaker relation of h-cobordism. However, just at this point, Smale[17] proved the h-Cobordism Theorem, showing, except in low dimensions,[18] that differentiable homotopy spheres are

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[16] As noted below, the boundary is actually a topological sphere, so that we can attach a topological disk by a boundary homeomorphism h to obtain a topological manifold $Q^{4m} \cup_h D^{4m}$. I am indebted to Moe Hirsch for the following observation. If $m \equiv 3 \pmod 4$ so that $\pi_{2m}(B_{\mathrm{SO}}) = 0$, then it is not hard to show that the resulting manifold does not even have the homotopy type of any differentiable manifold.

[17] S. Smale, *On the structure of manifolds*, Amer. J. Math. **84** (1962), 387-399. (Compare Milnor, Siebenmann, and Sondow "Lectures on the h-Cobordism Theorem", Princeton Mathematical Notes, Princeton Univ. Press 1965.)

[18] The most striking demonstration that low dimensions can really be different came many years later with Donaldson's construction of uncountably many differentiable (and even combinatorial) structures on Euclidean 4-space. No such examples can exist in other dimensions. See S. K. Donaldson, *The geometry of 4-manifolds*, Proc. Int. Cong. Math. Berkeley 1986, AMS 1987, pp. 43-54; as well as J. Stallings, *The piecewise-linear structure of Euclidean space*, Proc. Cam. Phil. Soc. **58** (1962) 481-488; and E. Moise, *Affine structures in 3-manifolds, V, The triangulation theorem and the Hauptvermutung*, Annals Math. **56** (1952) 96-114.

always "twisted" spheres, and that simply-connected h-cobordant manifolds are actually diffeomorphic. Thus, after the fact, we see that $\Gamma_n \cong \Theta_n$, except possibly in the case $n = 3$. In fact the group Γ_n is zero[19] for $n < 7$, and the next few groups are given as follows.

$n =$	7	8	9	10	11	12	13	14	15
$\Gamma_n \cong$	$\mathbf{Z}/28$	$\mathbf{Z}/2$	(order 8)	$\mathbf{Z}/6$	$\mathbf{Z}/992$	0	$\mathbf{Z}/3$	$\mathbf{Z}/2$	(order 16256)

The projected "Groups of homotopy spheres: II " was never completed, although a very small part of it found its way into the expository paper "**Differential topology**", which was published a few years later.

Meanwhile, the original project of publishing a paper on $2n$-manifolds which are $(n-1)$-connected, and a closely related project of studying "Spaces with a gap in cohomology" got lost in the shuffle. They were finally abandoned in 1962 when Wall published a beautiful exposition of the subject which made my attempts unnecessary.[20]

[19] The statement that $\Gamma_3 = 0$ follows from Smale, *Diffeomorphisms of* S^2 , Proc. Am. Math. Soc. **10** (1959) 621-626, while the statement that $\Gamma_4 = 0$ is due to J. Cerf, *Sur les difféomorphismes de la sphère de dimension trois (* $\Gamma_4 = 0$ *)*, Lect. Notes Math. **53**, Springer 1968. See also A. Hatcher, *A proof of the Smale conjecture*, Annals Math. **117** (1983) 553-607.

[20] C.T.C. Wall, *Classification of* $(n-1)$*-connected* $2n$*-manifolds*, Annals of Math. **75** (1962), 163-189. See also his papers *The action of* Γ_{2n} *on* $(n-1)$*-connected* $2n$*-manifolds*, Proc. AMS **13** (1962), 943-944, and *On simply-connected 4-manifolds*, J. Lond. Math. Soc. **39** (1964) 141-149.

Introduction to Part 2: Expository Lectures.

I am very happy that the lecture notes on "**Differential Topology**" by Jim Munkres are herewith finally in print. These notes, based on lectures at Princeton in the Fall of 1958, describe some of the foundations of the field, including Whitney embedding, Thom transversality, and tangent bundles of smooth manifolds, together with an outline of non-orientable cobordism theory.

The notes "**Smooth Manifolds with Boundary**", based on lectures in Mexico City in 1960, provide a bit more detail on some fundamental constructions, using only C^1-smooth maps in order to simplify the proofs. They also include an outline of how these constructions lead to exotic structures on spheres.

The notes on "**Differentiable Structures**", from lectures at Princeton in the Spring of 1961, cover further fundamental material, including the diffeotopy extension problem, connected sums, and tubular neighborhoods.

"**Hassler Whitney, an appreciation**" is adapted from remarks at a memorial gathering in Princeton.

Introduction to Part 3: Relations with Algebraic Topology.

To any real vector bundle $\mathbf{R}^m \hookrightarrow E \twoheadrightarrow B$ there are associated characteristic classes

$$w_i \in H^i(B;\mathbf{Z}/2) \qquad \text{and} \qquad p_j \in H^{4j}(B;\mathbf{Z})$$

in the base space. Wu Wen-Tsün, in his study of the cohomology of Grassmann manifolds, had described relations between these two kinds of classes. In particular, for vector bundles over the sphere S^{4k}, it follows from his work that the Stiefel-Whitney class w_{4k} is zero if and only if the Pontrjagin class p_k is divisible by 4. On the other hand Raoul Bott, in his study of the homotopy groups of classical groups, showed that the Pontrjagin class of such a bundle over S^{4k} is always divisible by $(2k-1)!$. (A sharper statement of his result will be given shortly.) The next two papers "**On the parallelizability of the spheres**" (written with Bott), and "**Some consequences of a theorem of Bott**", combine these two statements to show that there exists a bundle over S^n with $w_n \neq 0$ if and only if[1] n equals 1, 2, 4, or 8. As easy consequences, it follows that there exists a division algebra over the real numbers only in these dimensions, and that the sphere S^{n-1} is parallelizable only for these values of n.

The next paper, "**On the Whitehead homomorphism** J" describes a relationship between differential topology, algebraic topology, and number theory which I found quite surprising. In particular, it provides a lower bound for the size of the stable homotopy group $\Pi_{4k-1} = \pi_{4k-1+q}(S^q)$, with q large, in terms of the denominator[2] of the Bernoulli number B_k. This relationship is based on the J-homomorphism

$$J \;:\; \pi_p(\mathrm{SO_q}) \;\rightarrow \pi_{\mathrm{p+q}}(\mathrm{S^q})$$

of Heinz Hopf and George Whitehead, which provides a fundamental connection between homotopy groups of rotation groups and homotopy groups of spheres, and is also intimately related to the study of differentiable manifolds. In particular, the J-homomorphism generalizes to the Pontrjagin-Thom construction which assigns to each embedded $M^p \subset S^{p+q}$ with framed normal bundle a map $f : S^{p+q} \rightarrow S^q$ which carries each normal q-disk in a tubular neighborhood onto S^q, with M^p mapping to a base point x_0 and with the complement of the tubular neighborhood mapping to the antipodal point $-x_0$. Taking M^p to be the standard $S^p \subset S^{p+q}$ with an arbitrary normal framing, we recover the J-homomorphism. In particular, taking q large, we obtain the stable J-homomorphism $J : \pi_p(\mathrm{SO}) \rightarrow \Pi_{\mathrm{p}}$.

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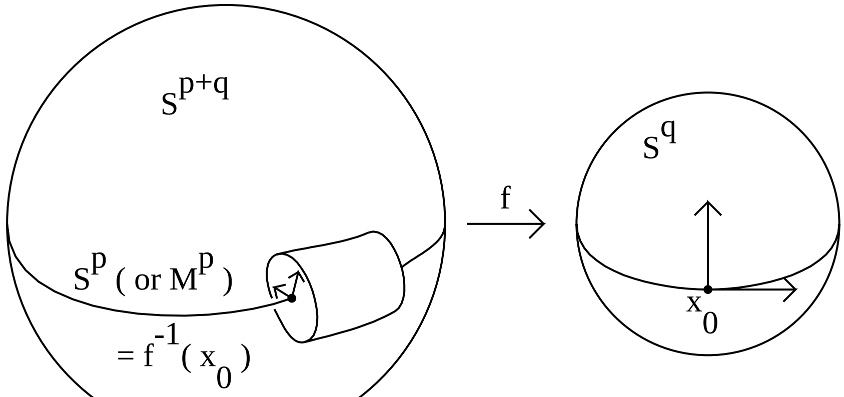
[1] This result would also follow from J. F. Adams, *On the nonexistence of elements of Hopf invariant one*, Bull. AMS **64** (1958) 279-282.

[2] Similarly, the size of the group of homotopy spheres Θ_{4k-1} can be estimated in terms of the numerator of B_k. Compare the introduction to Part I above. For number theoretic properties of Bernoulli numbers, see for example Appendix B of "Characteristic Classes", by Milnor and Stasheff.



Consider a homotopy class in the kernel of this J-homomorphism. Then application of the Pontrjagin-Thom construction shows that S^p, with its normal framing, is the boundary of a normally framed submanifold Q^{p+1} in the disk D^{p+q+1}. Gluing a standard $(p+1)$-disk onto the sphere $S^p = \partial Q^{p+1}$, we obtain a closed differentiable manifold M^{p+1}.

Now suppose that $p+1 = 4k$. Since Q^{4k} is parallelizable, it follows that all of the Pontrjagin classes p_i of M^{4k} are zero for $i < k$. Hence the signature formula reduces to the simple form

$$\text{signature} \;=\; s_k\, p_k[M^{4k}]\ .$$

Here the coefficient s_k is a rational number which can be computed explicitly as a multiple of the k-th Bernoulli number,

$$s_k \;=\; \frac{2^{2k}(2^{2k-1}-1)}{(2k)!} B_k\ ,$$

for example $s_1 = 1/3$, $s_2 = 7/45$, $s_3 = 2/945$. Since the signature is an integer,[3] it follows that the Pontrjagin number $p_k[M^{4k}]$ is always divisible by the denominator of the rational number s_k. For example, taking $k=2$, the number $p_2[M^8]$ must be divisible by $\text{denom}(s_2) = 45$.

The structure of the stable homotopy groups $\pi_{n-1}(\mathrm{SO}) \cong \pi_n(\mathrm{B_{SO}})$ has been worked out by Bott. In particular, for $n = 4k$, the group $\pi_{4k-1}(\mathrm{SO})$ is free cyclic. Furthermore, there is a canonical embedding which I will denote by

$$p_{k*}\ :\ \pi_{4k-1}(\mathrm{SO}) \;\to\; \mathbf{Z}\ ,$$

defined as follows. To each $\eta \in \pi_{4k-1}(\mathrm{SO})$ there is associated an SO-bundle ξ_η over the sphere S^{4k}, with Pontrjagin class $p_k(\xi_\eta) \in H^{4k}(S^{4k};\mathbf{Z}) \cong \mathbf{Z}$. Now set $p_{k*}(\eta)$ equal to the corresponding integer $p_k(\xi_\eta)[S^{4k}]$. Bott showed that the image $p_{k*}(\pi_{4k-1}(\mathrm{SO})) \subset \mathbf{Z}$ is generated by $a_k(2k-1)!$, where a_k is either 1 or 2 according as k is even or odd. (See his description on p. xx.)

If η belongs to the kernel of the J-homomorphism, then it is not hard to show that $p_{*k}(\eta)$ can be identified with the Pontrjagin number $p_k[M^{4k}]$ of the manifold M^{4k}

[3] In fact the signature is an integer divisible by 8, since Q^{4k} is parallelizable. However, this extra information is not very useful (except when $k \le 2$) because of the large factor 2^{2k} in the formula for s_k.

constructed above, up to sign. (See Lemma 1 of the paper.) Hence, in this case, the product $s_k\, p_{k*}(\eta)$ must be an integer.

Since the stable homotopy groups of spheres are finite, by a theorem of Serre, the image $J\pi_{4k-1}(\mathrm{SO}) \subset \Pi_{4\mathrm{k}-1}$ must be a finite cyclic group. Let j_k be its order. Then a generator η_0 of the kernel of the J-homomorphism will map to the integer

$$p_{k*}(\eta_0) \;=\; j_k a_k (2k-1)! \;.$$

Hence it follows from the argument above that the product $s_k j_k a_k(2k-1)!$ must be an integer. *In other words, the order* j_k *of the group* $J\pi_{4k-1}(\mathrm{SO})$ *must be a multiple of the denominator of the rational number*

$$s_k a_k(2k-1)! \;=\; 2^{2k} a_k (2^{2k-1}-1) B_k/2k \;.$$

Using classical results about Bernoulli numbers, we see that the factor of $2^{2k-1}-1$ never causes any cancellation, so this is just the odd primary part of the denominator of B_k/k. Furthermore, an odd prime power p^a divides this denominator if and only if k is a multiple of $p^{a-1}(p-1)/2$. As examples, for $k = 1, 2, \ldots, 6$ this denominator takes the values

$$3\,, \qquad 3\cdot 5\,, \qquad 3^2\cdot 7\,, \qquad 3\cdot 5\,, \qquad 3\cdot 11\,, \qquad 3^2\cdot 5\cdot 7\cdot 13$$

respectively.

This estimate is sharpened to include the prime 2 in the paper "**Bernoulli numbers, homotopy groups, and a theorem of Rohlin**", written with Kervaire, which shows that the order of the cyclic group $J\pi_{4k-1}(\mathrm{SO})$ is at least equal to the denominator of the rational number $B_k/4k$. As examples, for $k = 1, 2, \ldots, 6$ the ratio $B_k/4k$ takes the values

$$\frac{1}{2^3\cdot 3}\,, \quad \frac{1}{2^4\cdot 3\cdot 5}\,, \quad \frac{1}{2^3\cdot 3^2\cdot 7}\,, \quad \frac{1}{2^5\cdot 3\cdot 5}\,, \quad \frac{1}{2^3\cdot 3\cdot 11}\,, \quad \frac{691}{2^4\cdot 3^2\cdot 5\cdot 7\cdot 13}$$

respectively. The proof is based on work of Borel and Hirzebruch,[4] who define a rational number $\hat{A}(M^{4k})$ (the $\hat{A}$-*genus*) for any smooth closed $4k$-manifold, and show that $\hat{A}(M^{4k})$ is an integer whenever $w_2(M^{4k}) = 0$. In fact, if k is odd then Hirzebruch shows that $\hat{A}(M^{4k})$ is an even integer. Like the signature, $\hat{A}(M^{4k})$ is expressible as a rational linear combination of Pontrjagin numbers, in fact

$$\hat{A}(M^{4k}) \;=\; \frac{-B_k}{2\,(2k)!}\, p_k[M^{4k}] \;+\; (\text{terms in } \; p_1\,, \ldots, p_{k-1}) \;.$$

The rest of the proof is completely analogous to the one outlined above, simply making use of $\hat{A}(M^{4k})$ in place of the signature.

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[4] See A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces, III*, Amer. J. Math. **82** (1960) 491-504; and F. Hirzebruch, *A Riemann-Roch theorem for differentiable manifolds*, Séminaire Bourbaki 1958/59, no. **177**, Fév. 1959, or Hirzebruch, "Topological Methods in Algebraic Geometry" (3rd ed.), Springer 1966. (Compare Problem 7 of Hirzebruch, *Some problems on differentiable and complex manifolds*, Annals Math. **60** (1954) 213-236.)

It turns out, after the fact, that the lower bound obtained in this way is optimal. The much more difficult problem of finding an upper bound for the order of $J\pi_{4k-1}(\mathrm{SO})$ was solved a few years later by Frank Adams,[5] who showed that the order of $J\pi_{4k-1}(\mathrm{SO})$ is precisely equal to the denominator of $B_k/4k$. (Equivalently, the order of $J\pi_{4k-1}(\mathrm{SO})$ can be expressed as a product $\prod p^a$ where p ranges over all primes such that $2k$ is a multiple of $p-1$, and where p^{a-1} is the highest power of p dividing $4k$.)

[5] J. F. Adams, *On the groups* $J(X)$ – IV, Topology **5** (1966) 21-71.

Introduction to Part 4: Cobordism, and to the Appendix.

The research announcement "**On the cobordism ring Ω^***" describes some of the structure of the cobordism ring for oriented manifolds. Details are provided in the paper, "**On the cobordism ring Ω^* and a complex analogue, Part I**", which combines the techniques introduced by Thom with work of Frank Adams to show that the oriented cobordism ring (nowadays denoted by Ω_*) has no odd torsion. It also shows that the analogous ring Ω_*^{U} for manifolds with a complex structure on the stable normal bundle has no torsion at all. These results were also obtained by Averbuh and Novikov in the Soviet Union.[1]

The projected Part II of this paper was never written. In fact I am chagrined to discover that I have never published any details about some of the announced results which were intended to appear in it. I am very grateful to Thom for his permission to reprint his Bourbaki lecture "**Travaux de Milnor sur le cobordisme**", which gives a better account of this work than anything which I have published. In particular, it sketches a proof that the complex cobordism ring Ω_*^{U} is a polynomial ring of the form $\mathbf{Z}[Y^2, Y^4, Y^6, \ldots]$, where the Y^{2k} are certain complex manifolds. Furthermore, the oriented cobordism ring Ω_* modulo 2-torsion is a polynomial ring of the form $\mathbf{Z}[Y^4, Y^8, Y^{12}, \ldots]$, using only those Y^{2k} with dimension divisible by 4.

The full structure of the oriented cobordism ring Ω_* was worked out shortly afterwards by Terry Wall.[2]

The expository paper "**A survey of cobordism**" begins with an outline of the results described above, and ends with a very tentative exploration of more general cobordism theories, for example for topological or piecewise linear manifolds. However, its main emphasis is on cobordism for smooth Riemannian manifolds with an "X-structure" on the stable tangent bundle, where X is some space on which the infinite orthogonal group $\mathrm{O} = \varinjlim \mathrm{O}_n$ acts. Let us define an *X-structure* on M as a section of the X-bundle associated with the tangent bundle of M. For example, if $X = \mathrm{O}/G$ where G is a subgroup of O, then the X-structures on M correspond uniquely to reductions of the structural group of the stable tangent bundle to this subgroup G. Thus an O/SO-structure is just an orientation, while an O/U-structure is a stable almost complex structure, and an $\mathrm{O}/\{1\}$-structure is a stable framing of M. For each choice of X, there is an associated cobordism theory, which could be denoted by $\mathcal{N}_*^G$, or better by Ω_*^G when $G \subset \mathrm{SO}$. (The notations which are actually used in the paper, $H_*(\mathcal{M})$ for the cobordism groups associated with some class $\mathcal{M}$ of compact manifolds, or $\mathcal{N}_*(X)$ for the cobordism groups for smooth manifolds with X-structure, seem rather confusing, and are not recommended.)

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[1] See B. G. Averbuh, *Algebraic structure of intrinsic homology groups*, Dokl. Akad Nauk SSSR **125** (1959) 11-14, and S. P. Novikov, *Some problems in the topology of manifolds connected with the theory of Thom spaces*, Dokl. Akad. Nauk SSSR **132** (1960) 1031-1034 (Sov. Math Dokl. **1** 717-720), as well as *Homotopy properties of Thom complexes*, Mat. Sb. **57** (1962) 407-442.

[2] C.T.C. Wall, *Determination of the cobordism ring*, Annals Math. **2** (1960) 292-311.

Remark. The examples which are cited above all have the important property that an X-structure on the stable tangent bundle determines an X-structure on the stable normal bundle up to homotopy, and conversely.[3] However, examples such as $X = \mathrm{O}/\mathrm{O}_n$ show that a general X-structure will not have this property.

This paper also attempts to define the concept of "spin structure" on an oriented manifold within the same framework. However, the construction which it describes is wrong (see the erratum).

A correct but quite different definition for the concept of spin structure, suggested by Moe Hirsch, is described in the paper "**Spin structures on manifolds**", which also contains some very preliminary computations of the spin cobordism ring Ω_*^{Spin}.

The next paper, "**On the Stiefel-Whitney numbers of complex manifolds and spin manifolds**", describes the image of the homomorphism

$$\Omega_*^{\mathrm{U}} \to \Omega_* \to \mathcal{N}_*$$

from complex cobordism to oriented or non-oriented cobordism, and gives partial results on a corresponding question for spin cobordism.

The last paper, "**Remarks concerning spin manifolds**", contains two unrelated results. After a review of spin structures, section 2 uses the Eells-Kuiper invariant to construct a non-standard smooth involution of the 7-sphere.[4] The last section uses a construction of Atiyah in topological K-theory (or more explicitly KO-theory) to study the relationship between spin cobordism and exotic spheres. If $D^{8k} \hookrightarrow E \twoheadrightarrow B$ is a 8k-disk bundle with spin structure, then there is a canonical KO*-orientation class*, belonging to the group $KO(E, \dot{E}) \cong \widetilde{KO}(E/\dot{E})$, where $\dot{E}$ is the boundary sphere-bundle. Applying this construction to the universal Spin_{8k}-bundle, where $8k > n$ so that

$$\pi_{8k+n}(E/\dot{E}) \cong \Omega_n^{\mathrm{Spin}} ,$$

we see that this orientation class gives rise to a homomorphism

$$\Omega_n^{\mathrm{Spin}} \to \widetilde{KO}(S^{8k+n}) \cong \widetilde{KO}(S^n) .$$

Now specialize to the case $n \equiv 1$ or $n \equiv 2 \pmod 8$, so that $\widetilde{KO}(S^n)$ is cyclic of order 2, and consider the corresponding group of homotopy spheres Θ_n. For $9 \le n \le 18$ it is shown that the composition

$$\Theta_n \to \Omega_n^{\mathrm{Spin}} \to \widetilde{KO}(S^n) \cong \mathbf{Z}/2$$

is non-zero. If we assume the results of Adams cited at the end of the previous introduction, then the proof extends immediately[5] to all higher values of n which are congruent to 1

[3] Compare R. K. Lashof, *Poincaré duality and cobordism*, Trans. AMS **109** (1963) 257-277.

[4] For other free actions of finite groups on spheres, see Volume 2 of these Collected Papers.

[5] Compare B. Lawson and M.-L. Michelsohn, "Spin Geometry", Princeton Univ. Press 1989, p. 93.

or 2 mod 8.

The full structure of the spin cobordism ring was worked out shortly afterwards by Anderson, Brown and Peterson.[6] Their work depends on a concept of KO characteristic classes which generalizes the construction described above.

[6] D.W. Anderson, E.H. Brown and F.P. Peterson, *The structure of the spin cobordism ring*, Annals Math. **86** (1967) 271-298. Compare the exposition in R. Stong, "Notes on Cobordism Theory", Princeton Univ. Press 1968.

Erratum for "A survey of cobordism".

In order to define the concept of "spin structure" on a smooth oriented Riemannian manifold, we must lift the structural group of its tangent bundle from the rotation group to the spin group. More generally, suppose that we have a smooth homomorphism $h : G \to H$ between Lie groups (or well behaved limits of Lie groups) and some principal H-bundle $H \hookrightarrow P \twoheadrightarrow M$. If h embeds G as a subgroup of H , then a reduction of the structural group to G can be identified with a cross section of the associated H/G-bundle. In the case of a general homomorphism this quotient space H/G no longer makes sense. However, we can form a "homotopy quotient" $H/\!\!/G$, as follows. Let us thicken H by taking the cartesian product with a contractible space E_G on which G operates freely, and define $H/\!\!/G$ to be the quotient

$$(H \times E_G)/G \ ,$$

using the diagonal action $(A, e) \cdot g = (A \cdot h(g)\, ,\, e \cdot g)$. Now starting with any principal H-bundle $H \hookrightarrow P \twoheadrightarrow M$ we can form the associated $X = H/\!\!/G$-bundle

$$X \ \hookrightarrow \ P \times_H X = (P \times E_G)/G \ \twoheadrightarrow \ M.$$

There is a canonical G-bundle $G \hookrightarrow P \times E_G \twoheadrightarrow (P \times E_G)/G$ over the total space of this bundle, so any section $s : M \to (P \times E_G)/G$ induces a G-bundle over the original base space M .

(In fact this space $X = H/\!\!/G$ fits into a one-sided infinite sequence

$$\cdots \ \to \Omega(H) \ \to \ \Omega(X) \ \to \ G \ \to \ H \ \to \ X \ \to \ B_G \ \to \ B_H \ ,$$

where any two successive maps form the inclusion and projection of a fibration, up to homotopy type. Any H-bundle is classified by a map into B_H , and we can lift to a map into B_G if and only if there exists a section of the associated $X = H/\!\!/G$-bundle.)

In particular, taking H to be the infinite rotation group SO and taking G to be the infinite Spin group, defined as the non-trivial central extension

$$1 \ \to \ \{\pm 1\} \ \to \ \mathrm{Spin} \ \to \ \mathrm{SO} \ \to \ 1 \ ,$$

it is not hard to check that $X = \mathrm{SO}/\!\!/\mathrm{Spin}$ has the homotopy type of the infinite real projective space RP^∞ , and that the fundamental group $\pi_1(\mathrm{SO})$ maps isomorphically to $\pi_1(X)$. Unfortunately however, the paper identifies X with RP^∞ , using the usual action of SO on RP^∞ . This is just wrong, since $\pi_1(\mathrm{SO})$ maps trivially to $\pi_1(RP^\infty)$.

(An interesting variant of this example is provided by the group Spin^c , defined as the non-trivial central extension

$$1 \ \to \ S^1 \ \to \ \mathrm{Spin}^c \ \to \ \mathrm{SO} \ \to \ 1 \ .$$

In this case, the homotopy quotient $\mathrm{SO}/\!\!/\mathrm{Spin}^{\mathrm{c}}$ has the homotopy type of the infinite complex projective space CP^∞ . Again the map $\mathrm{SO} \to \mathrm{SO}/\!\!/\mathrm{Spin}^{\mathrm{c}}$ is non-trivial, as can be checked using cohomology.)