The Mathematical Work of Curt McMullen

John Milnor

McMullen has made important contributions to the study of Kleinian groups, hyperbolic 3-manifolds, and holomorphic dynamics. Indeed, following the lead of Dennis Sullivan, he clearly regards these three areas as different facets of one unified branch of mathematics. Following are descriptions of a few selected topics. I hope that these will illustrate the variety and depth of his work. However, by all means the reader should look at the original papers, since he is a master expositor. See especially his two books and his Berlin lecture [Mc10].

Solving the Quintic. His first work was on Smale's theory of purely iterative algorithms. By definition, these are numerical algorithms which can be carried out by iterating a single rational function, without allowing any "if $\cdots$ then $\cdots$" branching. In [Mc1] he showed that the roots of a polynomial of degree n can be computed by a generally convergent purely iterative algorithm if and only if $n \le 3$. With Peter Doyle [DMc] he showed that these roots can be computed by a tower of finitely many such algorithms if and only if $n \le 5$.

A Fat Julia Set. The Julia set J of a rational map f from the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself can be described roughly as the compact set consisting of all points $z \in \widehat{\mathbb{C}}$ such that the iterates of f, restricted to any neighborhood of z, behave chaotically. It is not known whether such a Julia set can have positive area, without being the entire Riemann sphere. However, McMullen [Mc2] produced very simple examples for the more general question in which we replace the rational map by a transcendental function, such as the map $z \mapsto \sin(z)$ of Figure 1. (A different and more complicated example was given in [EL].)

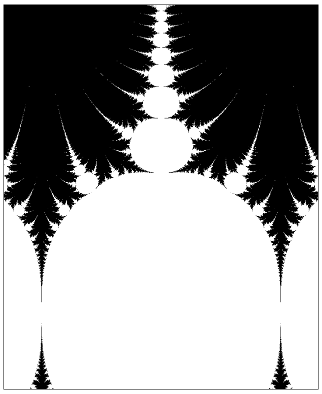

Figure 1. The Julia set for $z \mapsto \sin(z)$, shown in black, has positive area but no interior points.

The Kra Conjecture. To any Riemann surface X one can associate the Banach space $Q(X)$ consisting of all holomorphic quadratic differentials $\Phi = \phi(z)\,dz^2$ for which the norm $\|\Phi\| = \int |\phi|\,dz\,d\overline{z}$ is finite. Any covering map $f : X \to Y$ induces a push-forward operation $f_* : Q(X) \to Q(Y)$, where the image differential at a point y is obtained by summing over the points of $f^{-1}(y)$. This operation can never increase norms, $\|f_*(\Phi)\| \le \|\Phi\|$. In the special case of the universal covering

$f : \widetilde{Y} \to Y$ of a hyperbolic surface of finite area, Irwin Kra conjectured in 1972 that there is always some definite amount of cancellation between the different preimages of a point of Y, so that $\|f_*\|$ is strictly less than 1. This was proved by McMullen [Mc3, Mc5]. In fact McMullen considered a completely arbitrary covering map $f : X \to Y$, showing that $\|f_*\| < 1$ if and only if this covering is nonamenable.

Cusps in the boundary of Teichmüller space. Next a problem in Kleinian groups. In 1970, Lipman Bers compactified the Teichmüller space of complex structures on a hyperbolic Riemann surface of finite area by adding an ideal boundary consisting of algebraic limits of associated Kleinian groups. He conjectured that the "cusps", corresponding to ideal limits in which some simple closed curve has been pinched to a point, are everywhere dense in this boundary. (Compare Figure 2.) This was proved by McMullen [Mc4] in 1991, using a careful estimate for the change in the associated group representation $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ as some simple closed curve in the surface S shrinks to a point.

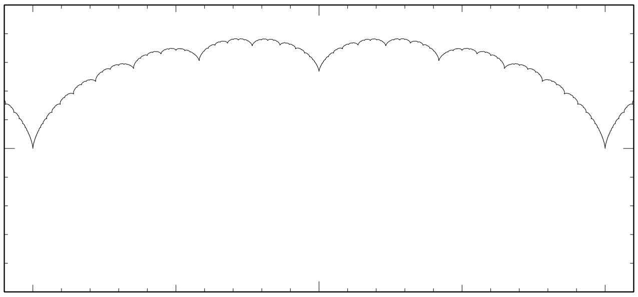

Figure 2 (courtesy of David Wright). Dense cusps in the boundary of Teichmüller space for a punctured torus, using the Maskit embedding. Teichmüller space is the region underneath this boundary curve.

Thurston Geometrization. The still unproved Thurston Geometrization Conjecture asserts that every compact 3-manifold can be cut up along spheres and tori into pieces, each of which admits a simple geometric structure. Here eight possible geometries must be allowed. For six of these eight geometries, the problem is now well understood, but difficulties remain in the hyperbolic case, while the spherical case, including the classical Poincaré conjecture, is still intractable. For references, compare [Mi2].

Thurston outlined proofs that a 3-manifold admits a hyperbolic structure in two important special cases. First suppose that M is a *Haken manifold*, that is, suppose that M is S^2-irreducible and can be built up inductively from 3-balls by gluing together submanifolds of the boundary, taking care that no essential simple closed curve in this submanifold bounds a disk in the manifold. Thurston showed that M can be given a hyperbolic structure if and only if every $\mathbb{Z} \oplus \mathbb{Z}$ in its fundamental group comes from a boundary torus. McMullen [Mc6] used his work on the Kra conjecture to give a new and simpler proof of this theorem. (The details are still quite complicated.) The second case handled by Thurston concerned 3-manifolds which fiber over the circle. Again McMullen gave a new proof, which will be discussed below.

Renormalization. Let f be a smooth even map from the closed interval $I = [-1, 1]$ into itself with a nondegenerate critical point at the origin, and with no other critical points. We will say that f is *renormalizable* if there is an integer $n \geq 2$ so that the n-fold iterate $g = f^{\circ n}$ maps the subinterval $\{x\,;\,|x| \leq |g(0)|\}$ into itself with only one nondegenerate critical point. If we rescale by setting $\widehat{f}(x) = g(\alpha x)/\alpha$ where $\alpha = g(0)$, then $\widehat{f}$ will be a new map from the interval I into itself satisfying the original hypothesis. This $\widehat{f}$ is called the *renormalization* $\mathcal{R}_n(f)$. In 1978 Mitchell Feigenbaum, and independently Pierre Coullet and Charles Tresser, considered the special case $n = 2$, and studied maps f which are *infinitely renormalizable*, so that we can form a sequence of iterated renormalizations $f\,,\ \mathcal{R}_2 f\,,\ \mathcal{R}_2^{\circ 2} f\,,\ \ldots$, each mapping I to itself with one critical point. They observed empirically that this sequence of maps always seems to converge to a fixed smooth limit map. Their ideas, motivated by renormalization ideas from statistical mechanics, and by attempts to understand the onset of turbulence in fluid mechanics, now occupy a central role in one-dimensional dynamics, since the infinitely renormalizable maps are the most difficult ones to understand.

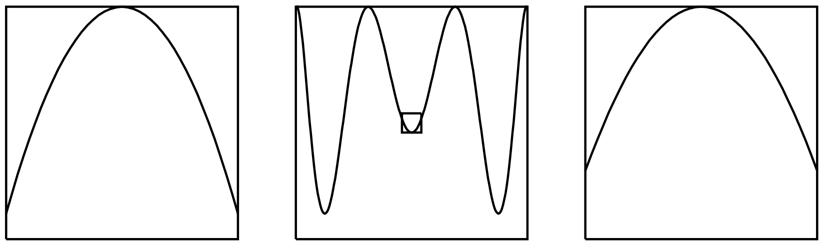

Figure 3. A quadratic map f, its iterate $f^{\circ 3}$, and its renormalization $\mathcal{R}_3(f)$. The right hand box in obtained from the small box in the middle by magnifying and rotating 180°.

This construction was extended to the complex case by Douady and Hubbard, using the idea of a *quadratic-like map*, that is, a proper holomorphic map $f : U \to V$ of degree two, where U and V are simply connected open sets in $\mathbb{C}$ and $\overline{U}$ is a compact subset of V. This has led to important work by mathematicians such as Dennis Sullivan, Curt McMullen, and Mikhail Lyubich. Most subsequent progress in understanding the real case has been based on complex methods. (One exception is Martens [Ma].) McMullen's book [Mc7] provided the first careful presentation of the foundations of renormalization. As one example, he was the first to notice the possibility of an aberrant form of "crossed renormalization" in the complex case, which does not fit into the usual pattern. He used his work on renormalization to obtain partial results on the generic hyperbolicity conjecture for real quadratic maps; that is, the conjecture that every such map can be approximated by one with an attracting periodic orbit. For example he showed that every component of the interior of the Mandelbrot set which meets the real axis is hyperbolic. The full conjecture was later proved by Lyubich [L1] and by Graczyk and Świątek [GS].

McMullen's second book [Mc8] developed renormalization theory further and tied it up with Mostow rigidity, and also with Thurston geometrization. (See also [Mc10].) He introduced the concept of a "deep point" in a fractal subset $X \subset \mathbb{C}$. By definition, p is *deep* if there are positive constants ϵ and c so that the distance from an arbitrary point q to X is at most $c|p-q|^{1+\epsilon}$. Taking $p = 0$ for convenience, we can understand this concept by zooming in on the origin so as to magnify the set X by some large constant λ. Replacing X by the magnified copy λX, we can replace the constant c by c/λ^ϵ, which tends to zero as $\lambda \to \infty$. In other words, these magnified images will fill out the complex plane more and more densely, with gaps which become smaller and smaller, as λ becomes large. (Compare [Mi1].)

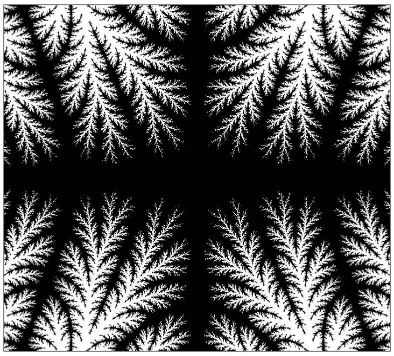

Figure 4. The critical point (at the center of symmetry) is a deep point of the Julia set for the Feigenbaum infinitely renormalizable map $z \mapsto 1 - az^2$, where $a = 1.401155189\cdots$. This Julia set has no interior points.

Now consider a hyperbolic 3-manifold which may have infinite volume. The *convex core* of such a manifold M can be described as the smallest geodesically convex subset which is a strong deformation retract of M. Assuming both upper and lower bounds for the injectivity radius at points of the convex core, McMullen's inflexibility theorem asserts that two such manifolds M and M' which are "pseudo-isometric" must actually be related by a diffeomorphism which becomes exponentially close to an isometry as we penetrate deeper into the convex core. Closely related is the statement that actions of $\pi_1(M)$ and $\pi_1(M')$ on the 2-sphere at infinity for hyperbolic 3-space are quasiconformally conjugate, and that this quasiconformal conjugacy is actually conformal at every deep point of the limit set for this action.

As an application, he gave a new proof of the second Thurston geometrization theorem. To any surface diffeomorphism $\psi : S \to S$ we can associate the *mapping torus* T_ψ, that is the quotient of $S \times \mathbb{R}$ under the $\mathbb{Z}$ action which is generated by $(x, t) \mapsto (\psi(x), t+1)$. If S has genus two or more and ψ is pseudo-Anosov, then Thurston showed that T_ψ is a hyperbolic 3-manifold. McMullen proved this by using his inflexibility result to construct a hyperbolic structure on $S \times \mathbb{R}$ which is invariant under the given $\mathbb{Z}$ action.

Next he applied these ideas to renormalization. One basic result is a rigidity theorem for bi-infinite "towers" of renormalizations. We can think of such a tower as a bi-infinite sequence $(\cdots, q_{-1}, q_0, q_1, q_2, \cdots)$ of quadratic-like maps $q_j : U_j \to V_j$, where each q_{j+1} is a renormalization $\mathcal{R}_{n_j}(q_j)$. If the renormalization periods n_j are bounded, and if the annuli $V_j \smallsetminus \overline{U}_j$ have modulus bounded away from zero, then he showed that the entire tower is uniquely determined up to a suitable isomorphism relation by its quasiconformal conjugacy class.

Consider an infinitely renormalizable real quadratic map f, with periodic combinatorics. Using the complex theory, McMullen showed that the successive renormalizations converge exponentially fast to a map which is periodic under renormal-

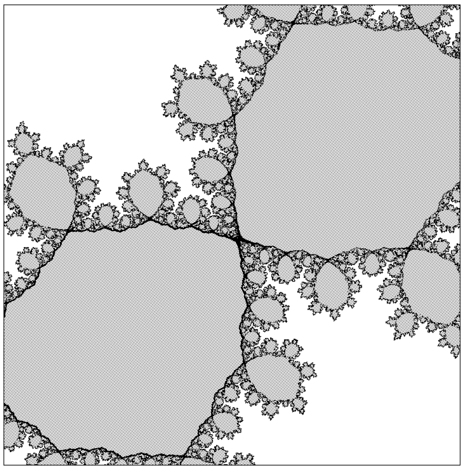

Figure 5. Filled Julia set associated with the golden mean Siegel disk, with rotation number $\rho = 1/(1+1/(1+1/(1+\cdots)))$. The Siegel disk is the large region to the lower left.

ization. Closely related is the statement that the critical point is a deep point for the Julia set of f. (Figure 4.)

Now consider a quadratic map $f(z) = z^2 + c$ which has a Siegel disk of rotation number ρ. That is, choose the constant c so that the derivative of f at one of its two fixed points is equal to $e^{2\pi i\rho}$, where $\rho \in \mathbb{R} \smallsetminus \mathbb{Q}$ satisfies a suitable Diophantine condition. If the continued fraction expansion of ρ is periodic, then McMullen used similar ideas in [Mc9] to show that this Siegel disk is "self-similar" about the critical point 0 (the central point in Figure 5). In fact, his argument can be used to show that the entire Julia set J of f is *asymptotically self-similar* in the following sense: There is a scale factor λ with $|\lambda| > 1$, so that the magnified images $\lambda^n J$ converge to a well defined limit set $\widehat{J} = \lambda\widehat{J} \subset \widehat{\mathbb{C}}$ as $n \to \infty$, using the Hausdorff topology for compact subsets of the Riemann sphere. (In this particular example, $\lambda = 1.8166\cdots$ is real.) The corresponding limit for the boundary of the Siegel disk is a quasicircle contained in $\widehat{J}$, while the corresponding limit for the filled Julia set $K(f)$ (the union of bounded orbits for f) is the entire sphere $\widehat{\mathbb{C}}$.

There has been very significant subsequent work in renormalization, based in part on McMullen's ideas. Compare the discussion in [Me]. Note in particular [L2], which implies that the boundary of the Mandelbrot set is asymptotically self-similar about the Feigenbaum point, and [L3] which proves existence of a full horseshoe structure for the real renormalization operator, and uses it, together with work of Martens and Nowicki, to prove that every real quadratic map outside a set of measure zero has either a periodic attractor or an absolutely continuous asymptotic measure.

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