Tsujii's Monotonicity Proof for Real Quadratic Maps

A restatement, in the language of quadratic differentials, of Tsujii's proof that the topological entropy of a real quadratic map varies monotonically with its parameter.

Let $\mathcal{Q}(\mathbb{C})$ be the space of meromorphic quadratic differentials on $\mathbb{C}$ which have at most simple poles. I will use the notation: $Q \in \mathcal{Q}(\mathbb{C})$, where

$$Q(z) \;=\; q(z)\,dz^2\;.$$

For any bounded region $U \subset \mathbb{C}$ the norm

$$\|Q\|_U \;=\; \iint_U |q(z)\,dz^2|$$

is defined and finite. If $f : \mathbb{C} \to \mathbb{C}$ is a non-constant polynomial map then the 'push-forward'

$$f_* Q(z) \;=\; \sum_{f(w)=z} \frac{q(w)\,dz^2}{f'(w)^2} \tag{1}$$

is defined and also belongs to $\mathcal{Q}(\mathbb{C})$. (Compare [Gardiner], [Epstein], [McMullen].) Note that

$$\|f_*Q\|_U \;=\; \iint_{z\in U} \left|\sum_{f(w)=z} q(w)\,dw^2\right| \;\le\; \iint_{z\in U} \sum_{f(w)=z} \left|q(w)\,dw^2\right| \;=\; \|Q\|_{f^{-1}(U)}\;.$$

As an example, let f be a non-linear polynomial and let D be a disk large enough so that $f^{-1}(\overline{D}) \subset D$. If A is the annulus $D \smallsetminus f^{-1}(\overline{D})$, then the $f^{-n}(A)$ are disjoint open subsets of D, hence

$$\sum_{n\ge 0} \|f_*^{\circ n} Q\|_A \;\le\; \sum_{n\ge 0} \|Q\|_{f^{-n}(A)} \;\le\; \|Q\|_D \;<\; \infty\;,$$

which implies that

$$\lim_{n\to\infty} \|f_*^{\circ n} Q\|_A \;=\; 0\;. \tag{2}$$

Now consider a quadratic map

$$f(z) \;=\; az^2 + bz + c \tag{3}$$

from $\mathbb{C}$ to itself, with derivative $f'(z) = 2az + b$. The orbit of the critical point $z_0 = -b/2a$ under f will be denoted by $z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots$, and the first derivatives along this critical orbit by $\delta_k = f'(z_k)$. Let

$$Q_k(z) \;=\; \frac{dz^2}{z - z_k}\;.$$

It is easy to see that the push-forward $f_*(Q_k)$ can have poles at most at the critical value z_1 (where the denominator of the expression (1) vanishes), and at the point z_{k+1} (where the numerator of one of the terms in (1) is infinite). The precise formula is as follows.

Lemma 1. *This push-forward is given by*

$$f_*(Q_k) \;=\; \frac{1}{\delta_k}\Big(Q_{k+1} - Q_1\Big) \tag{4}$$

provided that $\delta_k \ne 0$ *; while* $f_*(Q_k) = 0$ *when* $\delta_k = 0$.

Proof. To simplify the computation, let me temporarily assume that f has the special form

$$f(z) \;=\; z^2/2 \,+\, c \;=\; z^2/2 \,+\, z_1\;,$$

so that $f'(z) = z$ and $\delta_k = z_k$. (The general case will then follow easily.)

Choosing $w \in f^{-1}(z)$, we have

$$f_*Q_k(z) \;=\; \Big(\frac{1}{w-z_k} + \frac{1}{-w-z_k}\Big)\frac{dz^2}{f'(w)^2} \;=\; \frac{2z_k}{(w^2-z_k^2)}\frac{dz^2}{w^2} \;=\; \frac{z_k\,dz^2}{2(z-z_{k+1})(z-z_1)}\;, \quad (5)$$

using the fact that $w^2 = 2(z-z_1)$ and that $z_k^2 = 2(z_{k+1}-z_1)$. If $z_k = 0$, then this expression is zero. Otherwise, since

$$\frac{1}{z-z_{k+1}} - \frac{1}{z-z_1} \;=\; \frac{z_{k+1}-z_1}{(z-z_{k+1})(z-z_1)} \;=\; \frac{z_k^2}{2(z-z_{k+1})(z-z_1)}\;,$$

the expression (5) reduces to

$$f_*Q_k \;=\; \frac{1}{z_k}(Q_{k+1}-Q_1) \;=\; \frac{1}{\delta_k}(Q_{k+1}-Q_1)\;,$$

as asserted. The more general case (3) then follows by a straightforward change of variable argument. □

Now suppose that the critical orbit is periodic of period n. Then the quadratic differentials $Q_1\,, Q_2\,, \ldots\,, Q_n$ span an n-dimensional subspace of $\mathcal{Q}(\mathbb{C})$ which is mapped into itself by f_*. In fact f_*Q_k is given by the formula (4) for $1 \le k < n$, while $f_*Q_n = f_*Q_0 = 0$. As an example, for $n = 5$ this transformation can be described by the matrix

$$M_5 \;=\; \begin{pmatrix} -1/\delta_1 & 1/\delta_1 & 0 & 0 & 0 \\ -1/\delta_2 & 0 & 1/\delta_2 & 0 & 0 \\ -1/\delta_3 & 0 & 0 & 1/\delta_3 & 0 \\ -1/\delta_4 & 0 & 0 & 0 & 1/\delta_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\;,$$

with

$$I_5 - M_5 \;=\; \begin{pmatrix} 1+1/\delta_1 & -1/\delta_1 & 0 & 0 & 0 \\ 1/\delta_2 & 1 & -1/\delta_2 & 0 & 0 \\ 1/\delta_3 & 0 & 1 & -1/\delta_3 & 0 \\ 1/\delta_4 & 0 & 0 & 1 & -1/\delta_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. A straightforward induction on n shows that the determinant of the difference $I_n - M_n$ is given by

$$\det(I_n - M_n) \;=\; 1 + \frac{1}{\delta_1} + \frac{1}{\delta_1\delta_2} + \frac{1}{\delta_1\delta_2\delta_3} + \cdots + \frac{1}{\delta_1\delta_2\cdots\delta_{n-1}}\;.$$

On the other hand, if $\lambda_1\,, \ldots\,, \lambda_n$ are the eigenvalues of M_n, then we can write this determinant as the product $\prod_1^n(1-\lambda_j)$. Using (2), since the norm on a finite dimensional vector space is essentially unique, we see that the spectral radius of the matrix M_n is strictly less than one, so that $|\lambda_j| < 1$.

Now suppose that f is a real quadratic map. Then M_n is a real matrix, so the λ_j are either real or complex conjugate in pairs. It follows easily that

$$0 \quad < \quad 1+\frac{1}{\delta_1}+\frac{1}{\delta_1\delta_2}+\frac{1}{\delta_1\delta_2\delta_3}+\cdots+\frac{1}{\delta_1\delta_2\cdots\delta_{n-1}} \quad < \quad 2^n \ . \tag{6}$$

Kneading Theory. By the *kneading sequence* of a real quadratic map f, with critical orbit

$$x_0 \mapsto x_1 \mapsto x_2 \mapsto \cdots \ ,$$

I will mean the sequence of signs

$$\mathbf{k}(f) \ = \ \Big(\mathrm{sgn}(\delta_1)\,,\ \mathrm{sgn}(\delta_1\delta_2)\,,\ \mathrm{sgn}(\delta_1\delta_2\delta_3)\,,\ \cdots\Big)\ ,$$

where $\delta_k = f'(x_k)$. We give such sequences the lexicographical ordering.

Let us specialize to the Douady-Hubbard normal form $f_c(z) = z^2 + c$.

> **Theorem.** *As the parameter c increases from -2 to $1/4$, the corresponding kneading sequence $\mathbf{k}(f_c)$ increases monotonically from the lowest possible sequence of signs $(-,-,-,-,\cdots)$ to the highest possible sequence $(+,+,+,+,\cdots)$.*

Note: Since it is not hard to see that topological entropy of f is a monotone *decreasing* function of the kneading sequence $\mathbf{k}(f)$, it follows that the topological entropy of $f_c(z) = z^2 + c$ is a monotone decreasing function of the parameter c.

The proof, following Tsujii, requires a computation of the sign of the derivative dx_k/dc. (Here we think of $x_k = x_k(c)$ as a function of the parameter c.) Since

$$x_{k+1} \ = \ f_c(x_k) \ = \ x_k^2 + c\ ,$$

with $x_0 = 0$ and $\delta_k = 2x_k$, we have inductively

$$\frac{dx_{k+1}}{dc} \ = \ \frac{dx_k}{dc}\,\delta_k \ + \ 1\ , \qquad \text{with} \qquad \frac{dx_0}{dc} = 0\ .$$

Thus

$$\frac{dx_1}{dc} \ = \ 1\,, \quad \frac{dx_2}{dc} \ = \ \delta_1 + 1\,, \quad \frac{dx_3}{dc} \ = \ \delta_1\delta_2 + \delta_2 + 1\,, \quad \cdots\ ,$$

and in general

$$\frac{dx_{k+1}}{dc} \ = \ \delta_1\delta_2\cdots\delta_k \ + \ \delta_2\delta_3\cdots\delta_k \ + \ \cdots \ + \ \delta_{k-1}\delta_k \ + \ \delta_k \ + \ 1\ .$$

If $\delta_1\,,\ \ldots\,,\ \delta_k$ are all non-zero, then this formula can also be written as

$$\frac{dx_{k+1}}{dc} \ = \ \Big(1+\frac{1}{\delta_1}+\frac{1}{\delta_1\delta_2}+\cdots+\frac{1}{\delta_1\delta_2\cdots\delta_k}\Big)\ \delta_1\delta_2\cdots\delta_k\ .$$

Combining this equation with the inequality (6) we obtain the following.

> **Lemma 2.** *If the critical orbit of $f_c(x) = x^2 + c$ has period exactly n, then the sign of dx_n/dc, evaluated for this value of the parameter c, is equal to the sign of the product $\delta_1\delta_2\cdots\delta_{n-1}$.*

Now suppose that c increases through a very small interval, including a parameter value for which the critical orbit has period exactly n. Then the signs of the orbit points $x_1(c)$, $x_2(c)$, $\cdots$, $x_{n-1}(c)$, or of the corresponding derivatives $\delta_k = 2x_k(c)$, will remain fixed and non-zero. However, the sign of $\delta_n = 2x_n(c)$ will change in such a way that the sign of $\delta_1\delta_2\cdots\delta_{n-1}\delta_n$ increases from -1 to $+1$. The Theorem then follows easily. $\square$

References

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