Towards the Poincaré Conjecture and the Classification of 3-Manifolds

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The Poincaré Conjecture was posed ninety-nine years ago, and may possibly have been proved in the last few months. This note will be an account of some of the major results over the past hundred years which have paved the way towards a proof, and towards the even more ambitious project of classifying all compact 3-dimensional manifolds. The final paragraph provides a brief description of the latest developments, due to Grigory Perelman. A more serious discussion of Perelman's work will be provided in a subsequent note by Michael Anderson.

Poincaré's Question.

At the very beginning of the twentieth century, Henri Poincaré (1854-1912) made an unwise claim which can be stated in modern language as follows.

> *If a closed 3-dimensional manifold has the homology of the sphere S^3, then it is necessarily homeomorphic to S^3.*

(See Poincaré 1900.[1]) However, within four years he had developed the concept of "fundamental group", and hence the machinery needed to disprove this statement. In Poincaré 1904, he presented a counterexample which can be described as the coset space $\mathrm{SO}(3)/\mathrm{I}_{60}$. Here $\mathrm{SO}(3)$ is the group of rotations of Euclidean 3-space, and I_{60} is the subgroup consisting of those rotations which carry a regular icosahedron or dodecahedron onto itself (the unique simple group of order 60). This manifold has the homology of the 3-sphere, but its fundamental group $\pi_1\Big(\mathrm{SO}(3)/\mathrm{I}_{60}\Big)$ is a perfect group of order 120. He concluded the discussion by asking, again translated into modern language:

> *If a closed 3-dimensional manifold has trivial fundamental group, must it be homeomorphic to the 3-sphere?*

The conjecture that this is indeed the case has come to be known as the *Poincaré Conjecture*. It has turned out to be an extraordinarily difficult question, much harder than the corresponding question in dimension five or more,[2] and is a key stumbling block in the effort to classify 3-dimensional manifolds.

During the next fifty years the field of topology grew from a vague idea to a well developed discipline. However, I will call attention only to a few developments which have played an important role in the classification problem for 3-manifolds. (For further information see: Gordon for a history of 3-manifold theory up to 1960; Hempel for a presentation of the theory up to 1976; Bing for a description of some of the difficulties associated with 3-dimensional topology; James for a general history of topology; Whitehead for homotopy theory; and Devlin for the Poincaré Conjecture as a Millennium Prize Problem.)

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[1] Names in small caps refer to the list of references at the end. Poincaré's terminology may confuse modern readers who use the phrase "simply-connected" to refer to a space with trivial fundamental group. In fact, he used "simply-connected" to mean homeomorphic to the simplest possible model, that is, to the 3-sphere.

[2] Compare Smale 1960, Stallings, Zeeman, and Wallace for dimension five or more, and Freedman for dimension four.

Results based on Piecewise-Linear Methods.

Since the problem of characterizing the 3-sphere seemed so difficult, Max Dehn (1878-1952) tried the simpler problem of characterizing the unknot within S^3.

> **Theorem claimed by Dehn (1910).** *A piecewise-linearly embedded circle $K \subset S^3$ is unknotted if and only if the fundamental group $\pi_1(S^3 \smallsetminus K)$ is free cyclic.*

This is a true statement. However Kneser, 19 years later, pointed out a serious gap in Dehn's proof. The question remained open for nearly fifty years, until the work of Papakyriakopoulos.

One basic step was taken by James Waddel Alexander (1888-1971):

> **Theorem of Alexander (1924).** *A piecewise-linearly embedded 2-sphere in S^3 cuts the 3-sphere into two closed piecewise-linear 3-cells.*

Alexander also showed that a piecewise-linearly embedded torus must bound a solid torus on at least one of its two sides.

Helmut Kneser (1898-1973) carried out a further step which has played a very important role in later developments.[3] He called a closed piecewise-linear 3-manifold *irreducible* if every piecewise-linearly embedded 2-sphere bounds a 3-cell, and *reducible* otherwise. Suppose that we start with such a manifold M^3 which is connected and reducible. Then cutting M^3 along an embedded 2-sphere which does not bound a 3-cell, we obtain a new manifold (not necessarily connected) with two boundary 2-spheres. We can again obtain a closed (possibly disconnected) 3-manifold by adjoining a cone over each of these boundary 2-spheres. Now either each component of the resulting manifold is irreducible, or we can iterate this procedure.

> **Theorem of Kneser (1929).** *This procedure always stops after a finite number of steps, yielding a manifold $\widehat{M}^3$ such that each connected component of $\widehat{M}^3$ is irreducible.*

In fact in the orientable case, if we keep careful track of orientations and the number n of non-separating cuts, then the original connected manifold M^3 can be recovered as the *connected sum* of the components of $\widehat{M}^3$, together with n copies of the "handle" $S^1 \times S^2$. (Compare Seifert 1931, Milnor 1962.)

In 1933, Herbert Seifert (1907-1966) introduced a class of fibrations which play an important role in subsequent developments. For our purposes, a *Seifert fibration* of a 3-manifold can be defined as a circle action which is free except on at most finitely many "short" fibers, as described below. Such an action is specified by a map $(x,t) \mapsto x^t$ from $M^3 \times (\mathbb{R}/\mathbb{Z})$ to M^3 satisfying the usual conditions that $x^0 = x$ and $x^{s+t} = (x^s)^t$. We require that each fiber $x^{\mathbb{R}/\mathbb{Z}}$ should be a circle, and that the action of $\mathbb{R}/\mathbb{Z}$ should be free except on at most finitely many of these fibers. Here is a canonical model for a Seifert fibration in a neighborhood of a short fiber: Let α be a primitive n-th root of unity, and let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk. Form the product $\mathbb{D} \times \mathbb{R}$ and then identify each (z,t) with

[3] Parts of Kneser's paper were based on Dehn's work. In a note added in proof, he pointed out that Dehn's argument was wrong, and hence that parts of his own paper were not proven. However, the result described here was not affected.

$(\alpha z\, ,\, t+1/n)$. The resulting quotient manifold is diffeomorphic to the product $\mathbb{D}\times(\mathbb{R}/\mathbb{Z})$; but the central fiber under the circle action $(z,t)^s=(z\, ,\, t+s)$ is shorter than neighboring fibers, which wrap n times around it, since $(0,t)^{1/n}\equiv(0,t)$.

There were dramatic developments in 3-manifold theory, starting in the late 1950's with a paper by Christos PAPAKYRIAKOPOULOS (1914-1976). He was a quiet person who had worked by himself in Princeton for many years under the sponsorship of Ralph Fox. (I was also working with Fox at the time, but had no idea that Papakyriakopoulos was making progress on such an important project.) His proof of "Dehn's Lemma", which had stood as an unresolved problem since Kneser first pointed out the gap in Dehn's argument, was a tour de force. Here is the statement:

> **Dehn's Lemma (Papakyriakopoulos 1957).** *Consider a piecewise-linear mapping f from a 2-dimensional disk into a 3-manifold, where the image may have many self-intersections in the interior, but is not allowed to have any self-intersections near the boundary. Then there exists a non-singular embedding of the disk which coincides with f throughout some neighborhood of the boundary.*

He proved this by a constructing a tower of covering spaces, first simplifying the singularities of the disk lifted to the universal covering space of a neighborhood, then passing to the universal covering of a neighborhood of the simplified disk, and iterating this construction, obtaining a non-singular disk after finitely many steps. Using similar methods, he proved a result which was later sharpened as follows.

> **Sphere Theorem.** *If the second homotopy group $\pi_2(M^3)$ of an orientable 3-manifold is non-trivial, then there exists a piecewise-linearly embedded 2-sphere which represents a non-trivial element of this group.*

As an immediate corollary, it follows that $\pi_2(S^3\setminus K)=0$ for a completely arbitrary knot $K\subset S^3$. More generally, $\pi_2(M^3)$ is trivial for any orientable 3-manifold which is *irreducible* in the sense of Kneser.

Within a few years of Papakyriakopoulos's breakthrough, Wolfgang HAKEN had made substantial progress in understanding quite general 3-manifolds. In 1961, Haken solved the *triviality problem* for knots; that is, he described an effective procedure for deciding whether a piecewise-linearly embedded circle in the 3-sphere is actually knotted. (See SCHUBERT 1961 for further results in this direction, and a clearer exposition.)

Friedhelm WALDHAUSEN made a great deal of further progress based on Haken's ideas. In 1967A, he showed that there is a close relationship between Seifert fiber spaces and manifolds whose fundamental group has non-trivial center. In 1967B he introduced and analyzed the class of *graph manifolds*. By definition, these are manifolds which can be split by disjoint embedded tori into pieces, each of which is a circle bundle over a surface. Two key ideas in the Haken-Waldhausen approach seem rather innocuous, but are actually extremely powerful:

> **Definitions.** For my purposes, a two-sided piecewise-linearly embedded closed surface F in a closed manifold M^3 will be called *incompressible* if the fundamental group $\pi_1(F)$ is infinite, and maps injectively into $\pi_1(M^3)$. The manifold M^3 is *sufficiently large* if it contains an incompressible surface.

As an example of the power of these ideas, Waldhausen showed in 1968 that if two closed orientable 3-manifolds are irreducible and sufficiently large, with the same fundamental group, then they are actually homeomorphic. There is a similar statement for manifolds with boundary. These ideas were further developed in 1979 by JACO AND SHALEN and by JOHANNSON, who emphasized the importance of decomposing a space by incompressible tori.

Another important development during these years was the proof that every topological 3-manifold has an essentially unique piecewise-linear structure (see MOISE), and an essentially unique differentiable structure (see MUNKRES or HIRSCH, together with SMALE 1959). This is very different from the situation in higher dimensions, where it is essential to be clear as to whether one is dealing with differentiable manifolds, piecewise-linear manifolds, or topological manifolds.[4]

Manifolds of Constant Curvature.

The first interesting family of 3-manifolds to be classified were the *flat* Riemannian manifolds—those which are locally isometric to Euclidean space. David Hilbert, in the 18-th of his famous problems, asked whether there were only finitely many discrete groups of rigid motions of Euclidean n-space with compact fundamental domain. Ludwig BIEBERBACH (1886-1982) proved this statement in 1910, and in fact gave a complete classification of such groups. This had an immediate application to flat Riemannian manifolds. Here is a modern version of his result.

> **Theorem (after Bieberbach).** *A compact flat Riemannian manifold M^n is characterized, up to affine diffeomorphism, by its fundamental group. A given group Γ occurs if and only if it is finitely generated, torsionfree, and contains an abelian subgroup of finite index. Any such Γ contains a unique* **maximal** *abelian subgroup of finite index.*

It follows easily that this maximal abelian subgroup N is normal, and that the quotient group $\Phi = \Gamma/N$ acts faithfully on N by conjugation. Furthermore, $N \cong \mathbb{Z}^n$ where n is the dimension. Thus the finite group Φ embeds naturally into the group $\mathrm{GL}(n, \mathbb{Z})$ of automorphisms of N. In particular, it follows that any such manifold M^n can be described as a quotient manifold T^n/Φ, where T^n is a flat torus, where Φ is a finite group of isometries which acts freely on T^n, and where the fundamental group $\pi_1(T^n)$ can be identified with the maximal abelian subgroup $N \subset \pi_1(M^n)$. In the 3-dimensional orientable case, there are just six such manifolds. The group $\Phi \subset \mathrm{SL}(3, \mathbb{Z})$ is either cyclic of order 1, 2, 3, 4, or 6, or is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. (For further information see CHARLAP, as well as ZASSENHAUS, MILNOR 1976A, or THURSTON 1997.)

Compact 3-manifolds of constant *positive* curvature were classified in 1925, by Heinz HOPF (1894-1971). (Compare SEIFERT 1933, MILNOR 1957.) These included, for example, the Poincaré icosahedral manifold which was mentioned earlier. Twenty-five years later, Georges de RHAM (1903-1990) showed that Hopf's classification, up to isometry, actually coincides with the classification up to diffeomorphism.

[4] The statement that a piecewise-linear manifold has an essentially unique differentiable structure remains true in dimensions up to six. (Compare CERF.) However, KIRBY AND SIEBENMANN showed that a topological manifold of dimension four or more may well have several incompatible piecewise-linear structures. The four dimensional case is particularly perilous: Freedman, making use of work of Donaldson, showed that the topological manifold $\mathbb{R}^4$ admits uncountably many inequivalent differentiable or piecewise-linear structures. (See GOMPF.)

The *lens spaces*, with finite cyclic fundamental group, constitute a subfamily of particular interest. These were classified up to piecewise-linear homeomorphism in 1935 by Reidemeister, Franz, and de Rham, using an invariant which they called *torsion*. (See MILNOR 1966 as well as MILNOR AND BURLET 1970 for expositions of these ideas.) The topological invariance of torsion for an arbitrary simplicial complex was proved much later by CHAPMAN. One surprising byproduct of this classification was Horst SCHUBERT'S 1956 classification of knots with "two bridges", that is knots which can be placed in $\mathbb{R}^3$ so that the height function has just two maxima and two minima. He showed that such a knot is uniquely determined by its associated 2-fold branched covering, which is a lens space.

Although 3-manifolds of constant negative curvature actually exist in great variety, few examples were known until Thurston's work in the late 1970's. One interesting example was discovered already in 1912 by H. GIESEKING. Starting with a regular 3-simplex of infinite edge length in hyperbolic 3-space, he identified the faces in pairs to obtain a non-orientable complete hyperbolic manifold of finite volume. SEIFERT AND WEBER described a compact example in 1933: Starting with a regular dodecahedron of carefully chosen size in hyperbolic space, they identified opposite faces by a translation followed by a rotation through 3/10-th of a full turn to obtain a compact orientable hyperbolic manifold. (An analogous construction using 1/10-th of a full turn yields Poincaré's 3-manifold, with the 3-sphere as 120-fold covering space.)

One important property of manifolds of negative curvature was obtained by Alexandre PREISSMANN (1916-1990). (Preissmann, a student of Heinz Hopf, later changed fields and became an expert on the hydrodynamics of river flow.)

> **Theorem of Preissmann (1942).** *If M^n is a closed Riemannian manifold of strictly negative curvature, then any non-trivial abelian subgroup of $\pi_1(M^n)$ is free cyclic.*

The theory received a dramatic impetus in 1975, when Robert RILEY (1935-2000) made a study of representations of a knot group $\pi_1(S^3 \setminus K)$ into $\mathrm{PSL}_2(\mathbb{C})$. Note that $\mathrm{PSL}_2(\mathbb{C})$ can be thought of either as the group of orientation preserving isometries of hyperbolic 3-space, or as the group of conformal automorphisms of its sphere-at-infinity. Using such representations, Riley was able to produce a number of examples of knots whose complement can be given the structure of a complete hyperbolic manifold of finite volume.

Inspired by these examples, Thurston developed a rich theory of hyperbolic manifolds. See the discussion in the following section; together with KAPOVICH 2001 or MILNOR 1982.

The Thurston Geometrization Conjecture.

The definitive conjectural picture of 3-dimensional manifolds was provided by William THURSTON in 1982. It asserts that:

> *The interior of any compact 3-manifold can be split in an essentially unique way by disjoint embedded 2-spheres and tori into pieces which have a geometric structure. Here a "geometric structure" can be defined most easily*[5] *as a complete*

[5] More formally, the canonical model for such a geometric structure is one of the eight possible pairs (X, G) where X is a simply-connected 3-manifold, and G is a transitive group of diffeomorphisms such that G admits a left and right invariant volume form, such that the subgroup fixing any point of X is compact, and such that G is maximal as a group of diffeomorphisms with this last property.

Riemannian metric which is locally isometric to one of the eight model structures listed below.

For simplicity, I will deal only with closed 3-manifolds. Then we can first express the manifold as a connected sum of manifolds which are *prime* (that is, not further decomposable as non-trivial connected sums). It is claimed that each prime manifold either can be given such a geometric structure, or else can be separated by incompressible tori into open pieces, each of which can be given such a structure. The eight allowed geometric structures are represented by the following examples:

- the sphere S^3 , with constant curvature $+1$,
- the Euclidean space $\mathbb{R}^3$, with constant curvature 0 ,
- the hyperbolic space H^3 , with constant curvature -1 ,
- the product $S^2 \times S^1$,
- the product $H^2 \times S^1$ of hyperbolic plane and circle,
- a left invariant[6] Riemannian metric on the special linear group $\mathrm{SL}(2, \mathbb{R})$,
- a left invariant Riemannian metric on the solvable *Poincaré-Lorentz group* $E(1, 1)$, which consists of rigid motions of a 1+1 dimensional spacetime provided with the flat metric $dt^2 - dx^2$,
- a left invariant metric on the nilpotent *Heisenberg group*, consisting of 3×3 matrices of the form
$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} .$$

In each case, the universal covering of the indicated manifold provides a canonical model for the corresponding geometry. Examples of the first three geometries, were discussed in the section on constant curavture. A closed orientable manifold locally isometric to $S^2 \times S^1$ is necessarily diffeomorphic (but not necessarily isometric) to the manifold $S^2 \times S^1$ itself; but any product of a surface of genus two or more with a circle represents the $H^2 \times S^1$ geometry. The unit tangent bundle of a surface of genus two or more represents the $\mathrm{SL}(2, \mathbb{R})$ geometry. A torus bundle over the circle represents the Poincaré-Lorentz solvegeometry provided that its monodromy is represented by a transformation of the torus with a matrix such as $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ which has an eigenvalue greater than one. Finally, any nontrivial circle bundle over a torus represents the nilgeometry. Six of these eight geometries, all but the hyperbolic and solvegeometry cases, correspond to manifolds with a Seifert fiber space structure.

Two special cases are of particular interest. The conjecture would imply that:

A closed 3-manifold has finite fundamental group if and only if it has a metric of constant positive curvature. In particular, any M^3 with trivial fundamental group must be homeomorphic to S^3 .

[6] See Milnor 1976b §4 for the list of left invariant metrics in dimension 3.

This is a very sharp version of the Poincaré Conjecture. Another consequence would be the following:

> *A closed manifold M^3 is hyperbolic if and only if it is prime, with an infinite fundamental group which contains no $\mathbb{Z}\oplus\mathbb{Z}$.*

In the special case of a manifold which is sufficiently large, Thurston himself proved this statement, and in fact proved the full geometrization conjecture. (See MORGAN, THURSTON 1986, and MCMULLEN 1992.) Another important result by Thurston is that a surface bundle over the circle is hyperbolic if and only if (1) its monodromy is pseudo-Anosov up to isotopy, and (2) its fiber has negative Euler characteristic. See SULLIVAN, MCMULLEN 1996, or OTAL.

The spherical and hyperbolic cases of the Thurston Geometrization Conjecture are extremely difficult. However, the remaining six geometries are well understood. Many authors have contributed to this understanding (see for example GORDON AND HEIL, AUSLANDER AND JOHNSON, SCOTT, TUKIA, GABAI, and CASSON AND JUNGREIS). See THURSTON 1997 and SCOTT 1983B for excellent expositions.

The Ricci Flow.

A quite different method was introduced by Richard HAMILTON 1982. Consider a Riemannian manifold with local coordinates $u^1, \ldots, u^n$, and with metric $ds^2=\sum g_{ij}du^i\,du^j$. The associated *Ricci flow* is a one parameter family of Riemannian metrics $g_{ij}=g_{ij}(t)$ satisfying the differential equation

$$\partial g_{ij}/\partial t \;=\; -2\,R_{ij}\;,$$

where $R_{ij}=R_{ij}(\{g_{hk}\})$ is the associated Ricci tensor. This particular differential equation was chosen by Hamilton for much the same reason that Einstein introduced the Ricci tensor into his theory of gravitation[7] —he needed a symmetric 2-index tensor which arises naturally from the metric tensor and its derivatives $\partial g_{ij}/\partial u^h$ and $\partial^2 g_{ij}/\partial u^h\partial u^k$. The Ricci tensor is essentially the only possibility. The factor of 2 in this equation is more or less arbitrary, but the negative sign is essential. The reason for this is that the Ricci flow equation is a kind of non-linear generalization of the heat equation

$$\partial\phi/\partial t \;=\; \Delta\phi$$

of mathematical physics. For example, as g_{ij} varies under the Ricci flow, the associated *scalar curvature* $R=\sum g^{ij}R_{ij}$ varies according to a non-linear version

$$\partial R/\partial t \;=\; \Delta R\,+\,2\sum R^{ij}R_{ij}$$

of the heat equation. Like the heat equation, the Ricci flow equation is well behaved in forward time and acts as a kind of smoothing operator, but is usually impossible to solve in backward time. If some parts of a solid object are hot and others are cold then, under the heat equation, heat will flow from hot to cold so that the object gradually attains a uniform temperature. To some extent the Ricci flow behaves similarly, so that the curvature "tries" to become more uniform; but there are many complications which have no easy resolution.

To give a very simple example of the Ricci flow, consider a round sphere of radius r in Euclidean $(n+1)$-space. Then the metric tensor takes the form

$$g_{ij} \;=\; r^2\widehat{g}_{ij}$$

[7] For relations between the geometrization problem and general relativity, see ANDERSON.

where $\widehat{g}_{ij}$ is the metric for a unit sphere, while the Ricci tensor

$$R_{ij} \;=\; (n-1)\widehat{g}_{ij}$$

is independent of r. The Ricci flow equation reduces to

$$dr^2/dt \;=\; -2(n-1) \qquad \text{with solution} \qquad r^2(t) \;=\; r^2(0) \,-\, 2(n-1)\,t\;.$$

Thus the sphere collapses to a point in finite time. More generally, Hamilton was able to prove the following.

> **Theorem of Hamilton.** *Suppose that we start with a compact 3-dimensional manifold whose Ricci tensor is everywhere positive definite. Then, as the manifold shrinks to a point under the Ricci flow, it becomes rounder and rounder. If we rescale the metric so that the volume remains constant, then it converges towards a manifold of constant positive curvature.*

Hamilton tried to apply this technique to more general 3-manifolds, analyzing the singularities which may arise, but was able to prove the geometrization conjecture only under very strong supplementary hypotheses. (For a survey of such results, see Cao and Chow.)

In a remarkable pair of preprints, Grigory Perelman has announced a resolution of these difficulties, and promised a proof of the full Thurston conjecture based on Hamilton's ideas; with further details to be provided in a third preprint. One way in which singularities may arise during the Ricci flow is that a 2-sphere in M^3 may collapse to a point in finite time. Perelman shows that such collapses can be eliminated by performing a kind of "surgery" on the manifold, analogous to Kneser's construction as described earlier. After a finite number of such surgeries, he asserts that each component either:

(1) converges towards a manifold of constant positive curvature which shrinks to a point in finite time, or possibly

(2) converges towards an $S^2 \times S^1$ which shrinks to a circle in finite time, or

(3) admits a Thurston "thick-thin" decomposition, where the thick parts correspond to hyperbolic manifolds and the thin parts correspond to the other Thurston geometries.

I will not attempt to comment on the details of Perelman's arguments, which are ingenious and highly technical. However, it is clear that he has introduced new methods which are both powerful and beautiful, and made a substantial contribution to our understanding.

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