Fifty Years Ago:
Topology of Manifolds in the 50's and 60's

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The 1950's and 1960's were exciting times to study the topology of manifolds. This lecture will try to describe some of the more interesting developments. The first two sections describe work in dimension 3, and in dimensions $n \geq 5$, while §3 discusses why it is often easier to work in higher dimensions. The last section is a response to questions from the audience.

1 3-Dimensional Manifolds.

A number of mathematicians worked on 3-dimensional manifolds in the 50's. (I was certainly one of them.) But I believe that the most important contribution was made by just one person. Christos Papakyriakopoulos had no regular academic position, and worked very much by himself, concentrating on old and difficult problems. We were both in Princeton during this period, and I saw him fairly often, but had no idea that he was doing such important work. (In fact, I don't really remember talking to him—perhaps we were both too shy.)

Let me try to explain what he accomplished. In 1910, Max Dehn had claimed a proof of the following lemma:

> *If a piecewise linear map from a 2-simplex Δ into a triangulated 3-manifold is one-to-one near $\partial\Delta$, and if the image of the interior is disjoint from the image of the boundary, then there exists a piecewise linear embedding of Δ which agrees with the original map near $\partial\Delta$.*

As an easy corollary, he concluded that if the fundamental group of a knot complement is free cyclic, then there exists a spanning disk, so that the curve must be unknotted. This was a happy state of affairs for twenty years or so until 1929

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when Helmut Kneser studied the proof and discovered that Dehn's argument was seriously incomplete. The lemma remained as an unsolved problem for another thirty years or so, until Papakyriakopoulos, working by himself and using classical methods for finite simplicial complexes and their covering spaces, gave a correct proof. Closely related is his "Loop Theorem":

> *If M^2 is a boundary component of a 3-dimensional manifold-with-boundary W^3, and if the homomorphism $\pi_1(M^2) \to \pi_1(W^3)$ has a non-trivial kernel, then some essential simple closed curve in M^2 bounds an embedded disk in W^3.*

Another fundamental result which Papakyriakopoulos proved at the same time and with similar methods is the "Sphere theorem":

> *Consider a map from the sphere S^2 to an orientable 3–manifold which is essential (i.e., not homotopic to a constant map). Then there exists an essential embedding of S^2 in the manifold.*

Conversely, if there is no such essential embedded sphere (for example if M^3 is the complement of a knot in S^3), then it follows that $\pi_2(M^3) = 0$.

In the 60's, the major progress in 3-manifold theory was again by individuals, working by themselves and using classical piecewise linear methods. Wolfgang Haken pioneered the study of 3-manifolds which contain what we now call *incompressible surfaces*, that is two-sided embedded surfaces of genus ≥ 1 whose fundamental groups map injectively into the fundamental group of the manifold. Whenever such a surface exists, he showed that the manifold could be simplified by cutting along it. This technique was strong enough, for example, to give an effective procedure for deciding whether or not a simple closed curve in S^3 is unknotted. Friedhelm Waldhausen demonstrated the usefulness of these ideas by a number of important applications.

2 Higher Dimensions.

The progress in the study of higher dimensional manifolds was quite different, It involved many people, making use of many different techniques. Some of the necessary tools had been established much earlier, and some were just coming into being:

- **Cohomology theory** had been created by Alexander, Whitney, Čech, and others.
- **Cohomology operations** had been studied by Steenrod, and others.
- **Fiber bundles** were developed by Whitney, Hopf, Steenrod, and others.

- **Characteristic classes** were introduced by Whitney, Stiefel, Pontrjagin, and Chern.
- **Homotopy groups** had been studied but were very poorly understood.
- **Morse theory** had been developed, but its many applications had not been realized.

The first big step in the 50's was by Jean-Pierre Serre, who showed in 1951 that the machinery of spectral sequences has very important applications in homotopy theory. He proved for example that the group $\pi_m(S^n)$ of homotopy classes of maps of S^m to S^n is finite except in the special cases $\pi_n(S^n) \cong \mathbb{Z}$ and $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} +$ (finite). Although this seems to have nothing to do with manifolds, it played a big part in what follows.

In 1954, René Thom developed *cobordism theory:* If two compact manifolds cobound a smooth, compact manifold of one dimension higher, they are said to be cobordant. One can make an analogous definition for oriented cobordism. Thom showed that cobordism classes form an additive group, which isn't so surprising; but then he proved very sharp results about these groups, using ingenious geometric constructions together with the algebraic techniques which had been pioneered by Serre, Steenrod, and many others.

The n-dimensional oriented cobordism classes form a finitely generated abelian group Ω_n, and the topological product of manifolds gives rise to a bilinear product operation

$$\Omega_m \otimes \Omega_n \to \Omega_{m+n} ,$$

thus making the direct sum Ω_* into a ring. In order to eliminate the problem of torsion, Thom tensored this ring with the rationals, and proved that the result is a polynomial algebra with generators represented by the complex projective spaces of even complex dimension:

$$\Omega_* \otimes \mathbb{Q} \ = \ \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \mathbb{CP}^6, \ldots] .$$

In particular, he showed that $\Omega_n \otimes \mathbb{Q}$ is non-zero only in dimensions divisible by 4.

Thom also provided an effective test to decide when a $4n$-manifold is zero in this group: Take any product of Pontrjagin classes of the correct total dimension, and apply it to the fundamental homology class to obtain an integer, called a *Pontrjagin number*. Thom showed that a manifold is cobordant to zero (i.e., is a boundary) modulo torsion if and only if all of these Pontrjagin numbers are zero.

He also defined the signature of an oriented $4n$-dimensional manifold. I will describe it in terms of homology: Take two homology classes $\alpha, \beta \in H_{2n}(M^{4n}; \mathbb{Z})$. If we think of these intuitively as being represented by $2n$-dimensional submanifolds, then we can make these manifolds intersect transversally in a finite number of points. The signed count of the resulting intersections is an integer $\alpha \cdot \beta$ called the *intersection number*. Now pass to rational coefficients, and pick a basis $\{\alpha_i\}$ for the vector space $H_{2n}(M^{4n}; \mathbb{Q})$ so that the intersection bilinear form is diagonalized: $\alpha_i \cdot \alpha_j = 0$ for $i \neq j$. The sum of the signs of the diagonal elements $\alpha_i \cdot \alpha_i$ is an integer called the ***signature***, denoted by $\sigma = \sigma(M^{4n})$.

Thom proved by a geometric argument that if the manifold is a boundary, then $\sigma = 0$. It followed easily that the signature of any $4n$-manifold can be expressed as a linear combination of Pontrjagin numbers; *but with rational coefficients.* Hirzebruch had conjectured such a formula, and worked out its exact form, which Thom's proof then established.

As an example, in the 8-dimensional case there are two Pontrjagin numbers, $p_2[M^8]$ and ${p_1}^2[M^8]$, and the formula reads:

$$\sigma(M^8) = \frac{7}{45} p_2[M^8] - \frac{1}{45} {p_1}^2[M^8]\,. \qquad (1)$$

(To prove this formula, one need only evaluate both the Pontrjagin numbers and the signature for the two generators, $\mathbb{CP}^4$ and $\mathbb{CP}^2 \times \mathbb{CP}^2$, and then solve the resulting linear equation.) We can also rearrange terms in formula (1) so that p_2 is expressed in terms of p_1 and σ with rational coefficients:

$$p_2[M^8] = \frac{45\,\sigma(M^8) + {p_1}^2[M^8]}{7}\,. \qquad (2)$$

Suppose that we try to apply this last formula, but with the closed manifold M^8 replaced by a manifold-with-boundary, W^8. If the boundary ∂W^8 is a homology 7-sphere, then the signature still makes sense. Furthermore, the first Pontrjagin class, p_1, is well defined as an element of

$$H^4(W^8) \;\cong\; H^4(W^8, \partial W^8)\,;$$

and p_1^2, considered as an element of $H^8(W^8,\, \partial W^8)$, is non-zero, so that the Pontrjagin number ${p_1}^2[W^8]$ can be defined. However $p_2[W^8]$ cannot be defined.

But suppose this boundary is actually diffeomorphic to S^7. Then we can paste on an 8-ball to get a closed manifold, and compute its Pontrjagin number $p_2[M^8]$ by formula (2). Thus, if the boundary ∂W^8 is a standard 7-sphere then this expression must be an integer, and we can conclude that

$$45\,\sigma + p_1^2 \equiv 0 \pmod 7.$$

If we can find any W^8, bounded by a homology sphere, where this fails, then we've found a homology sphere which cannot be diffeomorphic to the standard sphere.

When I came upon such an example in the mid-50's, I was very puzzled and didn't know what to make of it. At first, I thought I'd found a counterexample to the generalized Poincaré conjecture in dimension seven. But careful study showed that the manifold really was homeomorphic to S^7. Thus, there exists a differentiable structure on S^7 not diffeomorphic to the standard one. (By taking connected sums, it follows easily that there are at least seven distinct differentiable structures on S^7. In fact there are precisely twenty-eight.)

Another important contribution from the 50's is due to Raoul Bott, who exploited Morse theory in a way that no one had thought possible to study homotopy groups of classical groups. Although this seems to have nothing to do with manifold topology, it turned out to be very important for developing the theory. The easiest case to describe is the stable unitary group U, that is, the union of the increasing sequence $\mathrm{U}(1) \subset \mathrm{U}(2) \subset \cdots$. He showed that

$$\pi_n(\mathrm{U}) \;\cong\; \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

This was a fantastic achievement; at this point very few homotopy groups were completely known, and having such a fundamental example with such a simple answer was mind-boggling. One important consequence was the rapid development of topological K-theory.

The study of topology of manifolds came even more alive in the 60's. The first development was Steve Smale's proof of the generalized Poincaré Conjecture in high dimensions. He showed that:

> *If a smooth manifold M^n of dimension $n > 4$ has the homotopy type of S^n, then M^n is homeomorphic to S^n.*

In fact, he actually proved a sharper result (for even dimensions in [1961] and for odd dimensions in [1962]). Smale's argument in dimensions ≥ 6 runs as follows. (Dimension 5 requires a different and very special argument.) Start with a self-indexing Morse function

$$f : M^n \to \mathbb{R}.$$

By careful moves which cancel critical points in pairs, he reduced to the case where f has only two critical points, one of index 0 and one of index n. Now, choose a level set $f^{-1}(r)$ of this Morse function which lies between the two critical points ($0 < r < n$). Looking at the flow lines running to $f^{-1}(r)$ from the index n critical point, we see that the subset $f^{-1}[r, n]$ is diffeomorphic to a disk. Similarly, the set $f^{-1}[0, r]$ is diffeomorphic to a disk. This shows that one can obtain M^n by taking two disks and gluing them together by a diffeomorphism of the boundary spheres.[1] In particular, M^n is homeomorphic to the standard n-sphere. Furthermore, this manifold M^n is determined, up to diffeomorphism, by the diffeomorphism $S^{n-1} \to S^{n-1}$ between boundary spheres. I call such an n-manifold a *twisted sphere*. Thus Smale showed that every smooth manifold of dimension $n > 4$ having the homotopy type of a sphere is in fact, a twisted sphere.

One can form an abelian group Γ_n, equal to the set of twisted n-spheres up to orientation-preserving diffeomorphism, and arrive easily at the exact sequence:

$$\pi_0(\mathrm{Diff}^+(D^n)) \to \pi_0(\mathrm{Diff}^+(S^{n-1})) \to \Gamma_n \to 0\,.$$

[1] Compare Reeb [1952], Milnor [1956].

In 1962, Smale proved the *h-cobordism theorem*. The manifolds M^n and N^n are said to be *h-cobordant* if there is a cobordism W^{n+1} between them such that the inclusion maps

$$i_M : M^n \to W^{n+1} \qquad \text{and} \qquad i_N : N^n \to W^{n+1}$$

are homotopy equivalences. He proved:

> *If W^{n+1} is a compact, smooth h-cobordism between M^n and N^n, and if in addition M^n and N^n are simply connected with $n > 4$, then W is diffeomorphic to $M \times [0,1]$; hence M is diffeomorphic to N.*

Using the group Γ_n of twisted n-spheres, Jim Munkres and Moe Hirsch independently constructed an obstruction theory for the problem of imposing a compatible differentiable structure on a combinatorial manifold[2] of any dimension. The obstructions to the existence of a smooth structure on a combinatorial manifold lie in the groups $H^{k+1}(M^n; \Gamma_k)$; while the obstructions to the uniqueness of such a smooth structures, when it exists, are elements of $H^k(M^n; \Gamma_k)$. The first few coefficient groups are relatively easy to compute: $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$. Jean Cerf proved the much harder result that $\Gamma_4 = 0$ in 1962.[3] Kervaire and I had studied the group Θ_n consisting of homotopy n-spheres up to h-cobordism, showing that these groups are trivial for $n = 4, 5, 6$ and finite in higher dimensions, with $\Theta_7 \cong \mathbb{Z}/28$. Combining all of these results, we see that $\Gamma_n = 0$ for $n < 7$, and that $\Gamma_7 \cong \mathbb{Z}/28$. Furthermore, if we accept Perelman's proof of the Poincaré conjecture, then $\Theta_3 = 0$, and it follows easily that Γ_n maps isomorphically to Θ_n in *all* dimensions.

For dimensions $n \neq 4$, the group $\Gamma_n \cong \Theta_n$ classifies all possible differentiable structures on S^n, up to orientation preserving diffeomorphism. However, in spite of the tantalizing fact that $\Gamma_4 = \Theta_4 = 0$, we don't know anything about possible exotic spheres in dimension 4. That is, we do not know that every 4-dimensional manifold M^4 with the homotopy type of S^4 is actually a differentiable S^4. Furthermore, if an exotic S^4, does exists, we don't know whether the complement of a point is necessarily equal to the standard $\mathbb{R}^4$, or is necessarily an exotic $\mathbb{R}^4$, or whether both cases can occur.

Near the end of the 60's, Kirby and Siebenmann developed an obstruction theory for the much harder problem of passing from topological manifolds to PL-manifolds. For a topological manifold M^n (where the dimension n is assumed to be five or more), there is only one obstruction to existence of a PL-structure, living in $H^4(M^n; \mathbb{Z}/2)$, and only one obstruction to the uniqueness of this structure (when it exists), living in $H^3(M^n; \mathbb{Z}/2)$.

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[2] A *combinatorial* or *PL* n-manifold is a triangulated space which is locally piecewise linearly homeomorphic to Euclidean n-space.

[3] Compare Cerf [1968]. For an alternative proof, see Eliashberg [1992, §2.4].

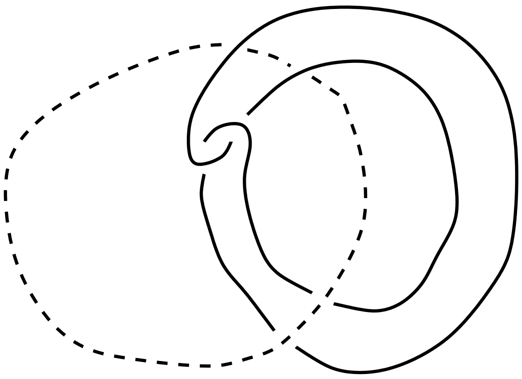

Fig. 1: The black circle does not bound an embedded disk in the complement of the dotted circle.

3 Why are higher dimensions sometimes easier?

Today we know that there are clear differences between low dimensions (< 4), high dimensions (> 4), and dimension 4 which is a jungle! Back in the early 50's, we knew that 1–manifolds were easy to understand, 2–manifolds were fairly easy, 3–manifolds were hard, and we assumed that it would get harder as we went up. So it was a big surprise in the 50's to discover that higher dimensions are often easier than lower, roughly speaking because there is much "more room" to carry out geometric constructions in higher dimensions.

Perhaps a simple example will illustrate the special difficulties which arise in low dimensions. In all dimensions it is important to study embeddings of the circle S^1 into a given manifold M^n. If $n \geq 5$, then using an argument which goes back to Whitney it is not hard to see that such an embedded circle bounds a smooth embedded disk if and only if the inclusion map $S^1 \to M^n$ is homotopic to a constant. But this argument breaks down in dimension 3. As an example, suppose that M^3 is the complement of the dotted circle in Figure 1. Then the black circle can be shrunk to a point in M^3 (if it is allowed to cut across itself); but it does not bound any embedded disk in M^3. One can immerse a disk missing the dotted circle, and one can find an embedded disk that intersects the dotted circle, but one cannot do both.

This particular difficulty doesn't seem to arise in dimension 4, since there can be no non-trivial knotted or linked circles in the interior of a 4-manifold. However, circles on the boundary of a 4-manifold again give rise to problems. As an example, a trefoil knot which is embedded in the boundary of the 4-dimensional disk D^4 does not bound any smoothly embedded disk within D^4. (Compare Fox and Milnor [1966].)

4 Questions from the audience.

The five questions which follow have been permuted; and in several cases, the rather brief answers which were given during the lecture have been very substantially augmented.

(1) Are there any themes from the 50's and 60's that have been forgotten and shouldn't have been?

It's hard, I think, to find any branch of mathematics that has been completely neglected.

(2) One thing you're famous for is your list of problems from the 50's. Do you have any problems that you want to see solved now?

The only question in manifold theory which came to mind immediately is the possible existence of exotic differentiable structures on S^4. I have no idea how one can attack that problem. One further important question is the following: *Can every manifold of dimension $n \neq 4$ be triangulated?* (See the discussion on the next page.) John Morgan points out that the *Novikov Conjecture* provides another example of an important unsolved problem in manifold theory. One form of this conjecture states that certain rational cohomology classes

$$f^*(x) \cup L_i(M^n) \ \in \ H^n(M^n; \mathbb{Q})$$

are homotopy type invariants. Here $L_i \in H^{4i}(M^n; \mathbb{Q})$ is the Hirzebruch polynomial which expresses the signature of a $4i$-manifold in terms of Pontrjagin classes, while f is the canonical map from M^n to the classifying space $B\pi_1(M^n)$, and x can be any cohomology class of dimension $n-4i$ in this classifying space. (For an extended discussion, see Ferry et al. [1995].)

(3) Can you give us an update on the list of questions you asked in the 50's ?[4]

Five of the seven questions had been completely answered by the mid 80's. However, the possible existence of a non-triangulable manifold of dimension > 4 remains open; and details for the proof of the 3-dimensional Poincaré Conjecture have not appeared. (Furthermore, the differentiable or PL-version of the 4-dimensional Poincaré Conjecture is untouched.) Here is a more detailed report.

• **Let M^3 be a homology 3-sphere with $\pi_1 \neq 0$. Is the double suspension of M^3 homeomorphic to S^5 ?**

[4]The problem list was probably first circulated during a summer workshop at Seattle in 1963. See Lashof [1965, p. 579].

This was partially proved by Bob Edwards [1975], and in a sharper form by Jim Cannon [1978, 1979]. (Compare Latour [1979].) It provided the first example of a triangulated manifold which is not locally PL-homeomorphic to Euclidean space.

- **Is simple homotopy type a topological invariant?**

Proved independently by T.A. Chapman [1973] and by R.D. Edwards. (Compare Siebenmann [1974], Edwards [1978].)

- **Can rational Pontrjagin classes be defined as topological invariants?**

Proved by Sergei Novikov [1965].

- **(Hauptvermutung.) If two PL-manifolds are homeomorphic, does it follow that they are PL-homeomorphic?**

Answered negatively by the work of Kirby and Siebenmann [1969]. For an explicit counterexample see Siebenmann [1970]. (The Hauptvermutung for 2-dimensional polyhedra had been proved by Papakyriakopoulos [1943], and for 3-manifolds by Moise [1952]. A counterexample for arbitrary simplicial complexes had been given by Milnor [1961]. For the many others who contributed, see Ranicki [1996].)

- **Can topological manifolds be triangulated?**

The same work of Kirby and Siebenmann shows that topological manifolds cannot always be triangulated as PL-manifolds. However, it is possible that every manifold of dimension $n > 4$ does possess some triangulation (which by Kirby-Siebenmann can not always be locally PL-homeomorphic to Euclidean space). This is a fundamental unsolved problem. For further information and partial results, see Matumoto [1978], Galewski and Stern [1980]. (For triangulation of 3-manifolds, see Moise [1952].)

In dimension 4, there is a specific counterexample, as follows. V.A. Rokhlin [1952] proved that any closed simply-connected differentiable 4-manifold with Stiefel-Whitney class $w_2 = 0$ (or equivalently with self-intersection form $\alpha \mapsto \alpha \cdot \alpha \in \mathbb{Z}$ which takes only even values) must have signature $\sigma \equiv 0 \pmod{16}$. (Compare Guillou and Marin [1986].) On the other hand, Mike Freedman [1982] constructed a topological 4-manifold with these properties, but with signature $\sigma = 8$. Thus Freedman's manifold is essentially non-differentiable. In fact, it cannot have any PL-structure since there would be no obstruction to smoothing a PL-structure.[5] Andrew Casson, circa 1985, sharpened Rokhlin's theorem by showing that if a homotopy 3-sphere bounds a simply-connected smooth or PL manifold with $w_2 = 0$, then this 4-manifold must have signature $\sigma \equiv 0 \pmod{16}$. (Compare Akbulut and McCarthy [1990].) It follows easily that the Freedman manifold has no triangulation at all. For given a triangulation, a small neighborhood N_v of any vertex v can be described as the cone over

[5] If we assume the 3-dimensional Poincaré Conjecture, then every triangulated 4-manifold is automatically a PL-manifold, so that we can skip Casson's argument, which can be thought of as an early step towards the Poincaré Conjecture.

a homotopy sphere ∂N_v. Now replace each such N_v by a simply connected PL-manifold with $w_2 = 0$ and with the same boundary. According to Casson, each such replacement patch would have signature $\equiv 0 \pmod{16}$. It would then follow easily that the resulting PL-manifold must have signature $\equiv 8 \pmod{16}$, contradicting Rokhlin's theorem.

The Kirby-Siebenmann obstruction class can be defined for a 4-manifold (although it is not the only obstruction to existence of a PL-triangulation); and it is non-zero for the Freedman M^4 discussed above. More generally, Casson showed that any 4-manifold with non-zero Kirby-Siebenmann obstruction can have no triangulation at all. (For other non-triangulation results, see Handel [1978].)

• **The Poincaré hypothesis in dimensions 3, 4.**

Freedman [1982] proved the Poincaré hypothesis for topological 4-manifolds; and Grisha Perelman [2002], [2003a,b] has claimed a proof for 3-manifolds.

In retrospect, it would be clearer to distinguish between three different versions of the Poincaré hypothesis, depending on whether we work in the topological category, the PL category, or the differentiable category. If we accept Perelman's proof, then the topological Poincaré hypothesis is true in *all* dimensions (using Kirby-Siebenmann to reduce to the PL-case in dimensions > 4). The PL-Poincaré hypothesis (that is, the statement that a closed PL-manifold with the homotopy type of S^n is PL-homeomorphic to S^n) is true except possibly for the case $n = 4$. On the other hand, the differentiable Poincaré hypothesis is false in dimensions 7, 8, 9, 10, 11, and in many higher dimensions; but is true in dimension 12. (Compare Kervaire and Milnor.)

• **(The annulus conjecture.) Is the region bounded by two locally flat n-spheres in S^{n+1} necessarily homeomorphic to $S^n \times [0,1]$?**

Proved by Kirby [1969] for dimensions $n+1 \neq 4$, and by Frank Quinn [1982] in the 4-dimensional case.

(4) Is Perelman's work related to the proof that $\Gamma_4 = 0$?

There is no relation. Perelman is concerned with the topology and geometry of 3-manifolds, while the statement that $\Gamma_4 = 0$ says that a very sharply restricted differentiable 4-manifold must be diffeomorphic to the standard 4-sphere.

Recall that Γ_n can be described either as the group of twisted n-spheres up to orientation preserving diffeomorphism, or as the group of orientation preserving diffeomorphisms of the $(n-1)$-sphere, modulo those which extend to diffeomorphisms of the n-disk. Cerf proved that $\Gamma_4 = 0$ back in 1962. (In fact, Cerf proved the stronger result that the group $\mathrm{Diff}^+(S^3)$ is connected. Much later, in 1983, Alan Hatcher proved the much sharper result that $\mathrm{Diff}^+(S^3)$ deformation retracts onto the rotation group SO(4).) Since Kervaire and I had shown that the group Θ_4 of homotopy spheres up to h-cobordism is also trivial, it certainly follows that Γ_4 maps isomorphically onto Θ_4. If we accept Perelman's claimed proof of the classical Poincaré Conjecture, then it follows trivially that $\Theta_3 = 0$, which is all that was missing in order to show that

$$\Gamma_n \xrightarrow{\cong} \Theta_n \tag{3}$$

for all n. However, this uniform statement conceals the fact that we know much less in dimension 4. In all dimensions $n \neq 4$, the statement is much stronger since these mutually isomorphic groups (3) can be identified with the group of *all possible* differentiable structures on S^n, up to orientation preserving diffeomorphism. But in the 4-dimensional case, we don't know whether or not there exist exotic spheres which are not twisted spheres.

(5) I've heard you were once challenged to write a limerick involving Papakyriakopoulos.

I've heard many variations over the years, but I believe that my original version went as follows:

The perfidious lemma of Dehn
drove many a good man insane
but Christos Pap-
akyriakop-
oulos proved it without any pain.

References

Akbulut, S. and McCarthy, J.: [1990], *Casson's invariant for oriented homology 3-spheres. An exposition,* Mathematical Notes **36**, Princeton University Press.

Bott, R.: [1959], *The stable homotopy of the classical groups,* Annals Math. **70**, 313–337.

Cannon, J. W.: [1978], *The recognition problem: what is a topological manifold?* Bull. Amer. Math. Soc. **84**, 832–866.

———— : [1979], *Shrinking cell-like decompositions of manifolds. Codimension three,* Annals Math. **110**, 83–112.

Cerf, J.: [1968], "Sur les difféomorphismes de la sphère de dimension trois ($\Gamma_4 = 0$)," Lecture Notes in Math. **53**, Springer-Verlag, Berlin-New York.

Chapman, T.A.: [1973], *Compact Hilbert cube manifolds and the invariance of Whitehead torsion,* Bull. Amer. Math. Soc., **79**, 52–56.

Edwards, R.D.: [1975], *The double suspension of a certain homology 3-sphere is S^5,* Notices Amer. Math. Soc. **22**, A-334.

———— : [1978], *On the topological invariance of simple homotopy type for polyhedra,* Amer. J. Math. **100**, 667–683.

Eliashberg, Y: [1992], *Contact 3-manifolds 20 years since Martinet,* Ann. Inst. Fourier **42**, 165–192.

Ferry, S., Ranicki, A, and Rosenberg, J.: [1995], "A history and survey of the Novikov conjecture, Novikov conjectures, index theorems and rigidity," volumes 1, 2 (Oberwolfach, 1993), London Math. Soc. Lecture Note Ser., 226, 227 Cambridge Univ. Press, Cambridge, 1995.

Fox, R. H. and Milnor, J.: [1966], *Singularities of 2-spheres in 4-space and cobordism of knots,* Osaka J. Math. **3**, 257–267. (Reprinted in Milnor, "Collected Papers 2", Publish or Perish 1995.)

Freedman, M.H.: [1982], *The topology of four-dimensional manifolds,* J. Differential Geom. **17**, no. 3, 357–453.

Galewski, D. and Stern, R.: [1980], *Classification of simplicial triangulations of topological manifolds,* Annals Math. **111**, 1–34.

Guillou, L. and Marin, A. (editors): [1986], "À la recherche de la topologie perdue I. Du côté de chez Rohlin, II. Le côté de Casson", Progress in Mathematics **62**, Birkhäuser, Boston.

Haken, W.: [1962], *Über das Homöomorphieproblem der 3-Mannigfaltigkeiten I*, Math. Z. **80**, 89–120.

Handel, M.: [1978], *A resolution of stratification conjectures concerning CS sets.* Topology **17**, 167–175.

Hatcher, A.: [1983], *A proof of the Smale conjecture,* Annals Math. **117**, 553–607.

Hirsch, M.: [1963], *Obstruction theories for smoothing manifolds and mappings,* Bull. Amer. Math. Soc. **69**, 352–356.

Hirzebruch, F.: [1956], "Neue topologische Methoden in der algebraischen Geometrie," Springer Verlag. (English translation, 3rd edition, 1966.)

Kervaire, M. and Milnor, J.: [1963], *Groups of homotopy spheres: I,* Annals Math. **77**, 504–537.

Kirby, R.C.: [1969], *Stable Homeomorphisms and the Annulus Conjecture*, Annals Math. **89**, 575–582.

Kirby, R.C. and Siebenmann, L.C.: [1969], *On the triangulation of manifolds and the Hauptvermutung,* Bull. Amer. Math. Soc. **75**, 742–749.

Lashof, R.: [1965], *Problems in Differential and Algebraic Topology. Seattle Conference, 1963,* Annals Math., 2nd. Ser., **81**, No. 3., 565–591.

Latour, F.: [1979], *Double suspension d'une sphère d'homologie [d'après R. Edwards],* Séminaire Bourbaki, 1977/78, No. 515, 169–186; Lecture Notes in Math., **710**, Springer, Berlin.

Matumoto, T.: [1978], *Triangulation of manifolds*, Algebraic and Geometric Topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, 3–6, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I.

Milnor, J.: [1956], *On manifolds homeomorphic to the* 7*-sphere.* Annals Math. **64**, 399–405.

———— : [1961], *Two complexes which are homeomorphic but combinatorially distinct,* Annals Math. **74**, 575–590.

Moise, E.E.: [1952], *Affine structures in* 3*-manifolds. V. The triangulation theorem and Hauptvermutung,* Annals Math. (2) **56**, 96–114.

Munkres, J.: [1960], *Obstructions to the smoothing of piecewise-differentiable homeomorphisms,* Annals Math. **72**, 521–554.

———— : [1964], *Obstructions to imposing differentiable structures*, Illinois J. Math. **8**, 361–376.

Novikov, S.P.: [1965], *Topological invariance of rational Pontrjagin classes,* Dokl. Akad. Nauk. SSSR **163**, 298–300. (Soviet Math. Dokl. **6**, 921–923).

Papakyriakopoulos, C. D.: [1943], *A new proof for the invariance of the homology groups of a complex* (Greek), Bull. Soc. Math. Grèce **22**, 1–154.

———— : [1957], *On Dehn's lemma and the asphericity of knots,* Proc. Nat. Acad. Sci. U.S.A. **43**, 169–172.

Perelman, G.: [2002], *The entropy formula for the Ricci flow and its geometric applications,* arXiv: math.DG/0211159v1, November.

———— : [2003a], *Ricci flow with surgery on three-manifolds,* arXiv: math.DG/0303109, March.

———— : [2003b], *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv: math.DG/0307245, July.

Quinn, F.: [1982], *Ends of maps. III. Dimensions* 4 *and* 5*,* J. Differential Geom. **17**, 503–521.

Ranicki, A. (editor): [1996], "The Hauptvermutung Book", K-Monogr. Math. **1**, Kluwer Acad. Publ., Dordrecht.

Reeb, G.: [1952], *Sur certaines propriétés topologiques des variétés feuilletées,* Publ. Inst. Math. Univ. Strasbourg **11**, 5–89, 155–156. Actualités Sci. Ind., no. **1183**.

Rokhlin (=Rohlin), V. A.: [1952], *New results in the theory of four-dimensional manifolds* (Russian) Doklady Akad. Nauk SSSR **84**, 221–224.

Serre, J.-P.: [1951], *Homologie singulière des espaces fibrés. III. Applications homotopiques.* C. R. Acad. Sci. Paris **232**, 142–144.

Siebenmann, L.C.: [1970], *Topological manifolds,* Actes du Congrès International des Mathématiciens, Tome **2**, 133–163. Gauthier-Villars, Paris, 1971.

———— : [1974], *L'invariance topologique du type simple d'homotopie* [*d'aprés T. Chapman et R. D. Edwards*]. Séminaire Bourbaki 1972/1973, No. 428, pp. 186–209; Lecture Notes in Math. **383**, Springer, Berlin, 1974.

Smale, S.: [1961], *Generalized Poincaré's conjecture in dimensions greater than four*, Annals Math. **74**, 391–406.

———— : [1962], *On the structure of manifolds,* Amer. J. Math. **84**, 387–399.

Thom, R.: [1954], *Quelques propriétés globales des variétés différentiables,* Comment. Math. Helvet., **28**, 17–86.

Waldhausen, F.: [1968], *On irreducible 3-manifolds which are sufficiently large*, Annals Math. **87**, 56–88.