Local connectivity of Julia sets: expository lectures

John Milnor

Abstract

An exposition of results of Yoccoz, Branner, Hubbard and Douady concerning polynomial Julia sets. The contents are as follows:

Introduction

The following notes provide an introduction to work of Branner, Hubbard, Douady, and Yoccoz on the geometry of polynomial Julia sets. They are an expanded version of lectures given in Stony Brook in Spring 1992.

Section 1 describes unpublished work by J.-C. Yoccoz on local connectivity of quadratic Julia sets. (Compare [Hu3].) It presents only the "easy" part of his theory, in the sense that it considers only non-renormalizable polynomials, and makes no effort to describe the more difficult arguments which are needed to deal with local connectivity in parameter space. It is based on second hand sources (Hubbard [Hu1] together with lectures by Branner and Douady), and uses the language of Branner and Hubbard. The presentation is quite different from that of Yoccoz. (Compare Problem 1-e at the end of §1.)

Section 2 describes the analogous arguments used by Branner and Hubbard [BH2] to study higher degree polynomials for which all but one of the critical orbits escape to infinity. In this case, the associated Julia set J is never locally connected. The basic problem is rather to decide when J is totally disconnected. This Branner-Hubbard work came before Yoccoz, and its technical details are not as difficult. However, in these notes their work is presented simply as another application of the same geometric ideas.

Section 3 complements the Yoccoz results by describing a family of examples, due to Douady and Hubbard, showing that an infinitely renormalizable quadratic polynomial may have non-locally-connected Julia set. An Appendix describes needed tools from complex analysis, including the Grötzsch inequality.

We will assume that the reader is familiar with the basic properties of Julia sets and the Mandelbrot set. (For general background, see for example [Be], [Br2], [CG], [D1], [D2], [EL], [L1], or [St]. A brief survey of the Mandelbrot set is given in §3.) In particular, we will make use of *external rays* for a polynomial Julia set $J(f) \subset \mathbb{C}$. (Compare [DH1], [DH2], [D5], [GM], [M2], [M3].)

1. Local Connectivity of Quadratic Julia Sets

This section will prove the following.

Theorem 1.1 (Yoccoz). *If $f_c(z) = z^2 + c$ is a quadratic polynomial such that:*

(1) *the Julia set $J(f_c)$ is connected,*

(2) *both fixed points are repelling, and*

(3) *f_c is not simply renormalizable,*

then $J(f_c)$ is locally connected.

(For the definition of renormalizability, see 1.4, as well as §3.) In terms of the familiar parameter space picture for the family of quadratic maps $f_c(z) = z^2 + c$, condition (1) says that the parameter value c belongs to the Mandelbrot set M, while (2) says that c does not belong to the closure of the central region bounded by the cardioid, and (3) says that c does not belong to any one of the many small copies of M which are scattered densely around the boundary of M. (Compare Figure 1.)

Remarks: The proof gives much more, since it effectively describes the Julia set by a new kind of symbolic dynamics. With a little more work, the Yoccoz method can also deal with the finitely renormalizable case. Since the case of a map with attracting or parabolic fixed point had been understood much earlier [DH2], we see that conditions (2) and (3) can be actually replaced by the following weaker pair of conditions:

(2′) f has no irrationally indifferent periodic points (Cremer or Siegel points), and

(3′) f is not *infinitely* renormalizable.

These modified conditions (1), (2′) and (3′) are all essential. In fact:

Condition (1): For $c \notin M$ the Julia set $J(f_c)$ is a Cantor set, which is certainly not locally connected.

Condition (2′): Sullivan and Douady showed that a polynomial Julia set with a Cremer point is never locally connected. (Compare [Su], [D1],

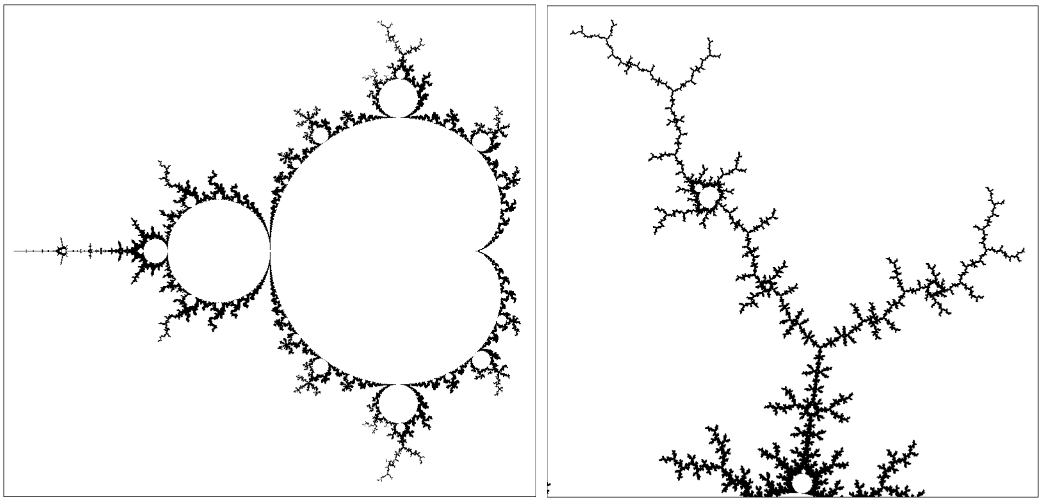


Figure 1. Boundary of the Mandelbrot set M *, and a detail of the one-third limb showing several small copies of* M *. (Compare* §3.)

[M2]. For a more explicit description of non local connectivity, see [Sø1-2].) Similarly, Herman has constructed quadratic polynomials with a Siegel disk having no critical point on its boundary. The corresponding Julia set cannot be locally connected. (See [He,§17.1], [D1,II,5], [D4].)

Condition (3′): In §3, following unpublished work by Douady and Hubbard, we will describe infinitely renormalizable polynomials for which J is not locally connected.

Thus the sharpened version of Theorem 1.1 comes fairly close to deciding exactly which quadratic polynomial Julia sets are locally connected. Yoccoz has also proved a corresponding result in parameter space: For c in the Mandelbrot set M , if the polynomial $f_c(z) = z^2 + c$ is not infinitely renormalizable, then M is locally connected at c . For proofs of this more difficult result, see [Hu3], [K].

The Yoccoz puzzle. The proof of Theorem 1.1 begins as follows. If a connected quadratic Julia set has two repelling fixed points, then one fixed point (to be called β) is the landing point of the zero external ray, and the other (called α) is the landing point of a cycle of q external rays, where $q \geq 2$. (Compare [M2], [Pe1].) Let $0 = c_0 \mapsto c_1 \mapsto \cdots$ be the critical orbit. The *Yoccoz puzzle* of *depth zero* consists of q non-overlapping closed topological disks $P_0(c_0)$, $P_0(c_1)$, … , $P_0(c_{q-1})$, called "*puzzle pieces*", which are obtained by cutting the region $G \leq 1$ along the q external rays landing at α . Here G is the Green's function (or canonical potential function) which vanishes only on the filled Julia set, and satisfies $G\big(f(z)\big) = 2G(z)$; while $P_0(c_i)$ denotes the unique puzzle piece of depth zero which contains

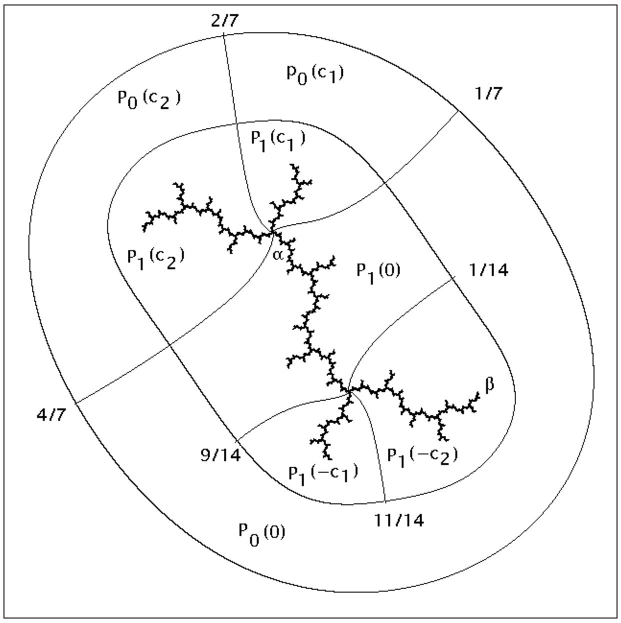


Figure 2. Julia set for $f(z) = z^2 + i$ *, showing the Yoccoz puzzles of depth zero and one. Here* $q = 3$ *. The* $1/7$ *,* $2/7$ *and* $4/7$ *rays land at the fixed point* α *.*

the post-critical point $c_i = f^{\circ i}(0)$.

Inductive Construction: If $P_d^{(1)}, \ldots, P_d^{(m)}$ are the puzzle pieces of depth d , then the connected components of the sets $f^{-1}(P_d^{(i)})$ are the puzzle pieces $P_{d+1}^{(j)}$ of depth $d+1$. As an example, there are always $2q-1$ pieces of depth one. These consist of q pieces $P_1(c_0)$, $P_1(c_1)$, $\ldots$, $P_1(c_{q-1})$ which touch at the fixed point α , together with $q-1$ additional pieces $P_1(-c_1)$, $\ldots$, $P_1(-c_{q-1})$ which touch at the pre-image $-\alpha$.

Since the α-fixed point is used to construct this puzzle, the points in its grand orbit will have to be dealt with by a special argument. (See 1.9.) It will be convenient to introduce the abbreviated notation

$$J_0 \;=\; \bigcup_{n\geq 0} f^{-n}(\alpha)$$

for the set of all points in the Julia set which belong to this excluded grand orbit. For any $z \in J \smallsetminus J_0$, it is easy to see that there is a unique sequence of puzzle pieces

$$P_0(z) \;\supset\; P_1(z) \;\supset\; P_2(z) \;\supset\; \cdots$$

which contain z .

Main Problem. Let $z \in J \smallsetminus J_0$ be any point which belongs to the Julia set $J(f)$ but not to the countable set $\bigcup f^{-n}(\alpha) \subset J(f)$, and let $P_d(z)$ be the unique puzzle piece of depth d which contains this point z . *Does the intersection* $\bigcap_d P_d(z)$ *of these puzzle pieces consist of the single point* z *?*

The associated annuli. Following Branner and Hubbard, consider the puzzle pieces $P_d(z) \supset P_{d+1}(z)$ of two consecutive depths containing some given $z \in J \smallsetminus J_0$. With luck, the smaller puzzle piece $P_{d+1}(z)$ will be contained in the interior of $P_d(z)$. In this case the difference set

$$A_d(z) \;=\; \mathrm{Interior}\big(P_d(z)\big) \smallsetminus P_{d+1}(z)$$

will be an annulus, whose modulus $\mathbf{mod}\, A_d(z)$ is a well defined positive real number. (Compare the Appendix.) However, we must also allow for the possibility that the boundary of $P_d(z)$ intersects P_{d+1} . In that case, we describe $A_d(z)$ as a "*degenerate annulus*", and define $\mathbf{mod}\, A_d(z)$ to be zero.[1] As an example, in Figure 2 the annulus $A_0(-c_1)$ has positive modulus, but the annulus $A_0(0)$ around the critical point is degenerate.

With these definitions we can reformulate the Main Problem as follows:

Modified Main Problem (Branner and Hubbard). *Given a point* $z \in J \smallsetminus J_0$ *, is the sum*

$$\sum_d \mathbf{mod}\, A_d(z)$$

infinite? If so, using the Grötzsch inequality, it is not difficult to prove that the intersection $\bigcap P_d(z)$ must consist of the single point z . (See A.7 in the Appendix.)

The sequence of puzzle pieces

$$P_0(0) \;\supset\; P_1(0) \;\supset\; P_2(0) \;\supset\; \cdots$$

containing the critical point will play a special role. For this sequence to be well defined, we need the following.

Standing Hypothesis 1.2. *We assume that* $0 \notin J_0$ *, so that the orbit of the critical point does not eventually land on the fixed point* α *.*

This is no real restriction. For in the contrary case where $0 \in J_0$ so that the critical orbit does end at α , we are in the post-critically finite case, and local connectivity can be established by more classical methods. (Compare [DH2] or [M2].)

---

[1] It will be convenient to refer to $A_d(z)$ loosely as an "*annulus*" even when it may be degenerate, and hence consist of one or more simply connected regions. In fact it might be clearer to work with the closed ring $\widehat{A}_d(z) = P_d(z) \smallsetminus \mathrm{Interior}\big(P_{d+1}(z)\big)$ which has interior $A_d(z)$, and which has the homotopy type of a circle even in the degenerate case.

We can compare the sequence $\{P_d(z)\}$ for an arbitrary point $z \in J \smallsetminus J_0$ with the sequence $\{P_d(0)\}$ as follows. *Define the* ***semi-critical depth*** $\mathrm{S}(z)$ *to be the largest integer* $d \ge 0$ *for which* $P_d(z) = P_d(0)$, *taking* $\mathrm{S}(z)$ *to be* $+\infty$ *if* $P_d(z) = P_d(0)$ *for all* d , *and setting* $\mathrm{S}(z) = -1$ *if* $P_d(z) \ne P_d(0)$ *for all* d . Thus $\mathrm{S}(z)$ is large whenever z is close to the critical point.

To each point $z_0 \in J \smallsetminus J_0$ we can associate the orbit $z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots$, and the sequence of numbers $\mathrm{S}(z_0)$, $\mathrm{S}(z_1)$, $\mathrm{S}(z_2)$, ... (The sequence associated with the critical orbit $0 = c_0 \mapsto c_1 \mapsto \cdots$ will be of particular interest.) Following Branner and Hubbard, this sequence can be displayed graphically by introducing the associated ***tableau***. (Compare Figures 4, 6, 9.)

Definition. The ***tableau*** associated with an orbit $z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots$ in J_0 is an array with one column associated with each z_i and one row associated with each depth in the Yoccoz puzzle. We will draw a solid vertical line at depth d in the j-th column whenever $d < \mathrm{S}(z_i)$, and a double vertical line when $d = \mathrm{S}(z_i)$. Such single or double vertical line segments correspond to puzzle pieces $P_d(z_i)$ which coincide with the critical puzzle piece $P_d(0)$, and a long chain of such vertical segments indicates an orbit point z_i which is very close to the critical point. Diagonal arrows of length m , pointing north-east, correspond to iterates $f^{\circ m}$, mapping some puzzle piece $P_d(z_i)$ by a branched covering map onto $P_{d-m}(z_{i+m})$.

Given any orbit $z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots$ in J_0 , note that f carries any puzzle piece $P_d(z_i)$ of depth $d > 0$ onto the puzzle piece $P_{d-1}(z_{i+1})$. This map $P_d(z_i) \to P_{d-1}(z_{i+1})$ is either a conformal isomorphism or a two-fold branched covering according as $P_d(z_i)$ does or does not contain the critical point. Now consider the restriction of f to the annulus

$$A_d(z_i) \;=\; \mathrm{Interior}\big(P_d(z_i)\big) \smallsetminus P_{d+1}(z_i) \;.$$

We can distinguish three cases, as follows.

Critical Case. If $d < \mathrm{S}(z_i)$ so that $0 \in P_{d+1}(z_i)$ (indicated by a vertical line segment in the tableau), then $A_d(z_i)$ coincides with $A_d(0)$, and will be called a ***critical annulus***. In the non-degenerate case, note that $A_d(z_i)$ surrounds the critical point 0, and that f maps the annulus $A_d(z_i)$ onto $A_{d-1}(z_{i+1})$ by a 2-fold covering map. It follows easily that

$$\mathbf{mod}\, A_d(z_i) \;=\; \tfrac{1}{2}\,\mathbf{mod}\, A_{d-1}(z_{i+1}) \;.$$

Off-Critical Case. If $d > \mathrm{S}(z_i)$ so that $0 \notin P_d(z_i)$ (indicated by a blank in the tableau), then f maps $A_d(z_i)$ biholomorphically onto $A_{d-1}(z_{i+1})$ and the moduli are equal. In this case, we will say that the annulus $A_d(z_i)$ is ***off-critical***.

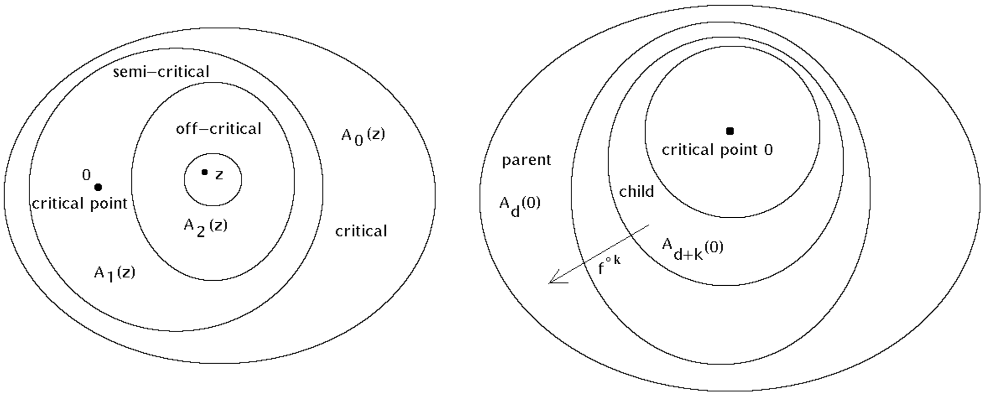


Figure 3. Critical, semi-critical and off-critical annuli (left); and a "child" (right).

Semi-Critical Case. Finally, if $d = \mathrm{S}(z_i)$ so that $0 \in A_d(z_i)$ (corresponding to a double line in the tableau), we will say that $A_d(z_i)$ is *semi-critical*. Here the topology is more complicated, since f maps the annulus $A_d(z_i)$ onto the entire puzzle piece $P_{d-1}(z_{i+1})$. However, it is easy to check that $A_d(z_i)$ is non-degenerate if and only if the annulus $A_{d-1}(z_{i+1})$ is non-degenerate. For an inequality between moduli, see Problem 1-b.

As examples, in the schematic picture Figure 3 (left), the annulus $A_0(z)$ is critical, while $A_1(z)$ is semi-critical, and $A_2(z)$ is off-critical. In Figure 2, the annuli $A_0(-c_1)$ and $A_0(-c_2)$ are both semi-critical.

Definition. The critical annulus $A_{d+k}(0)$ in the Yoccoz puzzle will be called a *child* of the critical annulus $A_d(0)$ if and only if $A_{d+k}(0)$ is an unramified two-fold covering of $A_d(0)$ under the map $f^{\circ k}$. (See Figure 3 (right), and Figure 4. Our terminology is based on [Hu1], but with several modifications.)

Note that the modulus of such a child is always exactly half the modulus of the parent. In a tableau, starting with any critical annulus, represented by a single vertical segment, we can locate the parent (necessarily unique if it exists), by drawing a line to the north-east and extending until it hits another vertical segment. Our strategy for solving the Modified Main Problem can now be summarized as follows:

> *Find a critical annulus of positive modulus, and prove that it has so many descendents, children and grandchildren and so on, that the modulus sum is infinite.*

Branner and Hubbard summarize the tableau properties which are needed to carry out this argument in three rules, as follows.

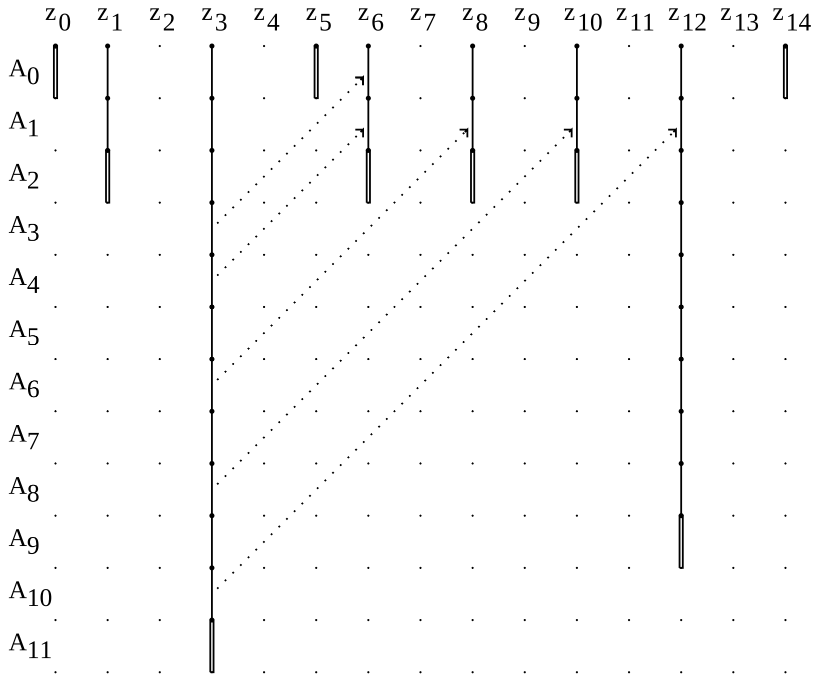


Figure 4. An example: the tableau associated with the orbit of the point $z_0 = 1$ for the map $f(z) = z^2 - 1.6$. Here we see by following the top diagonal arrow that the critical annulus $A_3(z_3) = A_3(0)$ is an unramified two-fold covering of the critical annulus $A_0(z_6) = A_0(0)$. In particular, the critical annulus at depth 3 is a child of the critical annulus at depth 0 . Similarly, the critical annuli at depths 4 , 6 , 8 , 10 are children of the critical annulus of depth 1 .

First tableau rule: *Every column of a tableau is either all critical, or all off-critical, or has exactly one semi-critical depth* $\mathrm{S}(z_i)$ *, and is critical above and off-critical below.*

Now let us compare the tableau of the critical orbit

$$0 = c_0 \mapsto c_1 \mapsto c_2 \mapsto \cdots$$

with the tableau of some given orbit $z_0 \mapsto z_1 \mapsto \cdots$ in J_0 . (The case $z_0 = c_0$ is not excluded.) If the tableau of $\{z_j\}$ is critical or semi-critical at depth d in column m , draw a line "north-east" from this critical or semi-critical annulus, as indicated by the dotted line on the right, and draw a corresponding line north-east from depth d of column zero in the critical tableau.

Second tableau rule: *Everything strictly above the diagonal line on the left must be copied above the diagonal line on the right.*

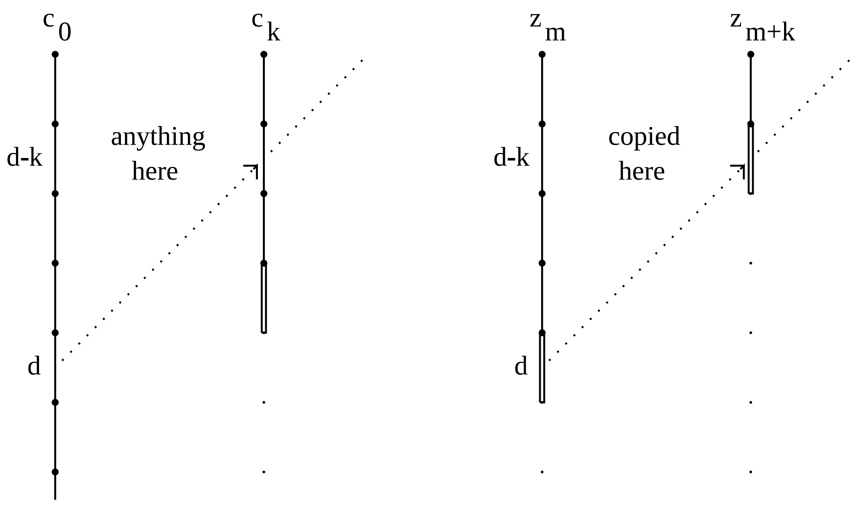


Figure 5. Illustration for the second and third tableau rules, with the critical tableau on the left, and the tableau for $z_0 \mapsto z_1 \mapsto \cdots$ on the right.

The proofs of these two rules are easily supplied. $\square$

Now suppose that the critical annulus of depth d is a child of the critical annulus of depth $d-k$, as indicated in the figure, and suppose that the tableau of $\{z_j\}$ is semi-critical at depth d of column m.

Third tableau rule: *Following the diagonal arrow from this semi-critical annulus of depth d in the tableau of $\{z_j\}$, we must reach a semi-critical annulus at depth $d-k$, as illustrated.*

Proof. According to the hypothesis, $f^{\circ k}$ maps $A_d(0)$ onto $A_{d-k}(0)$, where the point z_m is an element of this annulus $A_d(0)$. Therefore $f^{\circ k}(z_m) = z_{m+k}$ must belong to $A_{d-k}(0)$. $\square$

Definition. We will say that a critical annulus $A_d(0)$ is *excellent* if it contains no post-critical points, or equivalently if there is no semi-critical annulus in the d-th row of the critical tableau.

In Figure 6, note that the critical annuli at depths $1, 3, 4, 6, 7, 8, 9, \ldots$ are excellent. Each one has exactly two children, which are also excellent. On the other hand, $0, 2, 5, 10, 18$ (Fibonacci numbers minus three) are not excellent. Each of these has only one child; however this child is excellent.

Definition: We will say that the critical tableau is *recurrent* if the numbers $\mathrm{S}(c_k)$ with $k>0$ are unbounded, so that there are columns which go arbitrarily far down. It is *periodic* if $\mathrm{S}(c_k)=\infty$ for some $k>0$, so that the

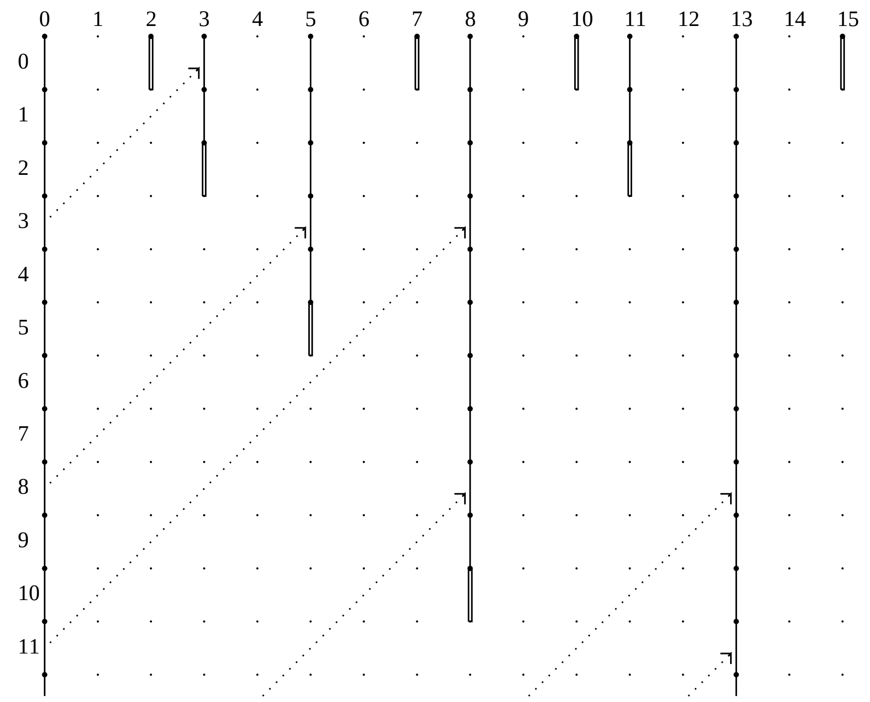


Figure 6. The "Fibonacci" critical tableau. (Compare [BH2].) Here the closest recurrences of the critical orbit come after a Fibonacci number of iterations:

$$1\,,\ 2\,,\ 3\,,\ 5\,,\ 8\,,\ 13\,,\ 21\,,\ 34\,,\ \ 55\,,\ \ldots\ .$$

If u_n is the n-th Fibonacci number, then the semi-critical annulus for column number u_n occurs at depth $u_{n+1}-3$. The diagonal dotted lines illustrate the genealogies

$$\begin{array}{ccccccc} 0 & \leftarrow & 3 & \leftarrow & 8 & \begin{smallmatrix}\leftarrow\\ \leftarrow\end{smallmatrix} & \begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix} \\ & & & \nwarrow & & & \\ & & & & 11 & \begin{smallmatrix}\leftarrow\\ \leftarrow\end{smallmatrix} & \begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\ , \end{array}$$

with arrows from child to parent. Since $A_3(0)$ is the only child of $A_0(0)$, it follows from 1.3 that every descendent of $A_3(0)$ has at least two children.

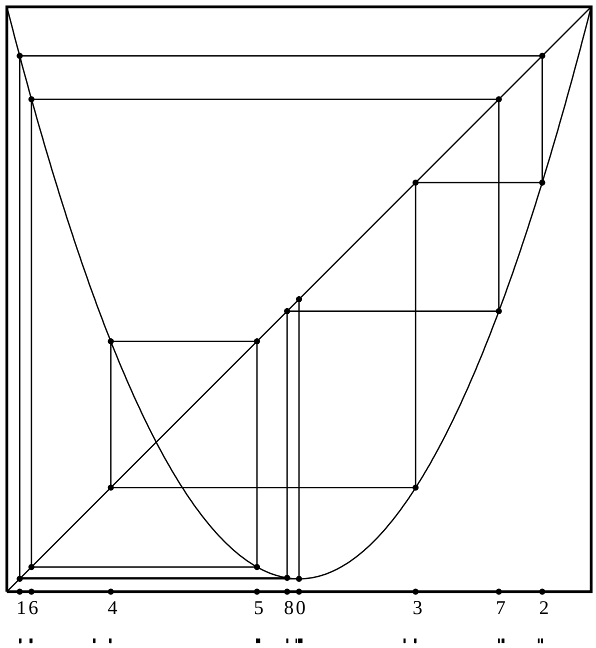


Figure 7. Graph of the unimodal map $x \mapsto x^2 - 1.8705286\cdots$ *which realizes the Fibonacci tableau. (Compare [ML].) The first eight points on the critical orbit are marked. The critical orbit closure is a rather thin Cantor set, which is plotted underneath the graph. (Compare Problem 1-f.)*

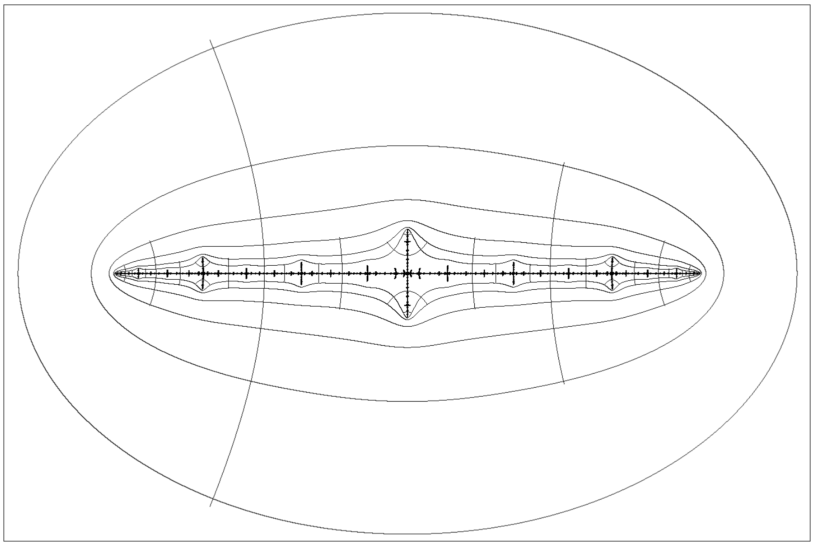


Figure 8. The Yoccoz puzzle at depths zero through five for this quadratic Fibonacci map, drawn to the same scale. Note that the critical pieces are the biggest ones at any specified depth.

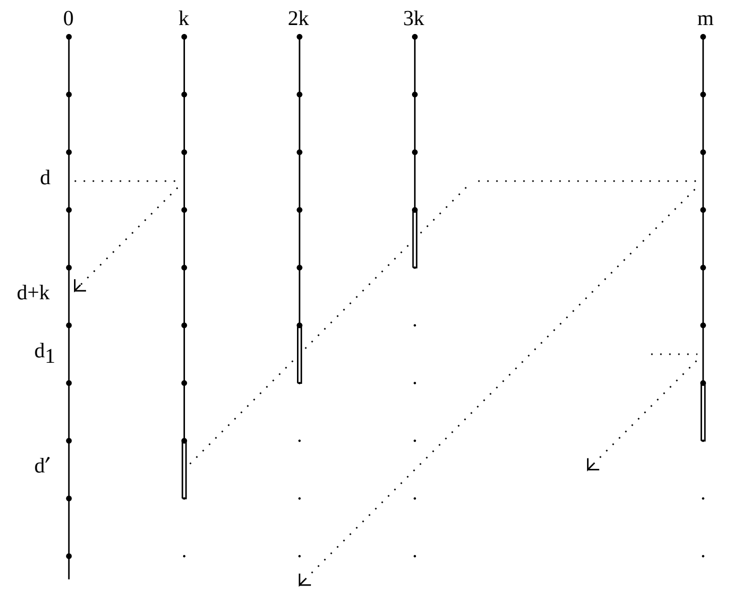


Figure 9. Finding Children. The successive dotted lines illustrate Lemma 1.3 (a), (b) and (d).

corresponding column goes all the way down. In this case, it follows easily that $P_d(c_i) = P_d(c_{i+k})$ for all d and i. (Compare Lemma 1.5 below.)

Lemma 1.3. *If the critical tableau is recurrent but not periodic, then:*

(a) *Every critical annulus has at least one child.*

(b) *Every excellent critical annulus has at least two children.*

(c) *Every child of an excellent parent is excellent.*

(d) *Every only child is excellent.*

Proof of (a). Start at depth d of column zero in the critical tableau, and march to the right until we first meet another critical annulus (= solid vertical line), say at column k. (Figure 9.) Now marching diagonally south-west, we find the first child at depth $d+k$.

Proof of (b). Suppose that the annulus $A_d(0)$ is excellent. Choose k as in (a). By hypothesis, the k-th column cannot be critical all the way down. Hence it must be semi-critical at some depth $d' > d$. Starting at column k and depth d', proceed diagonally right (north-east). By the tableau rules, column $2k$ must be semi-critical at depth $d' - k$. Similarly column $3k$ must be semi-critical at depth $d' - 2k$, and so on, until we again reach depth

d. Furthermore, as we follow this diagonal, we do not meet any other critical or semi-critical annuli. In particular, at the point where we reach depth d, there cannot be a critical or semi-critical annulus. (The hypothesis that d is excellent comes in at this point. Compare Figure 9.) Now let us again march to the right at depth d until we reach a critical annulus, say at column m. Again turning $135°$ and proceeding diagonally south-west, we cannot meet any critical or semi-critical annulus until we are back at column 0. In this way we prove that the critical annulus of depth $m+d$ is also a child of d.

Proof of (c). This is clear, since if $f\big(A_{d+k}(0)\big) = A_d(0)$ where $A_{d+k}(0)$ contains a post-critical point, then $A_d(0)$ does also.

Proof of (d), by contradiction. Consider a child d' which is not excellent, and let $d_1 = n' - k$ be the parent. Then for some $k' \geq k$ the k'-th column has semi-critical annulus at depth d'. (The case $k' = k$ is illustrated in Figure 9.) Following the diagonal up from column k' and depth d', we must meet a semi-critical annulus at depth d_1 by the third tableau rule. (Thus the parent is not excellent.) Now proceed to the right at depth d_1 until we meet a critical annulus, say at column m. Then it follows as above that $d_1 + m$ is a second child; hence d' was not an only child. □

Definition 1.4. (Compare [DH3], [Mc2].) A quadratic polynomial f is *renormalizable* if there exists a closed topological disk Δ_0 around the critical point, and an integer $p \geq 2$ (called the *renormalization period*), with the following four properties:

(1) Δ_0 should be centrally symmetric about the critical point so that f carries Δ_0 onto the image $\Delta_1 = f(\Delta_0)$ by a ramified 2-fold covering map.

(2) f should induce conformal isomorphisms

$$\Delta_1 \xrightarrow{\cong} \Delta_2 \xrightarrow{\cong} \cdots \xrightarrow{\cong} \Delta_p ,$$

where $\Delta_i = f^{\circ i}(\Delta_0)$. In particular, the critical point should be disjoint from $\Delta_1 \cup \cdots \cup \Delta_{p-1}$.

(3) However, the disk Δ_p should contain Δ_0 in its interior.

(4) Finally, the entire orbit of the critical point under the map $f^{\circ p}$ should be contained in the original disk Δ_0.

(Compare the discussion of "polynomial-like mappings" in §2, and the discussion of "tuning" in §3.) If these conditions are satisfied, then the *small filled Julia set* $K_p = K(f^{\circ p}|_{\Delta_0})$ can be defined as the compact connected set consisting of points whose orbits remain in Δ_0 under all iterations of $f^{\circ p}$. This is a proper subset of the filled Julia set $K(f)$ of the original map

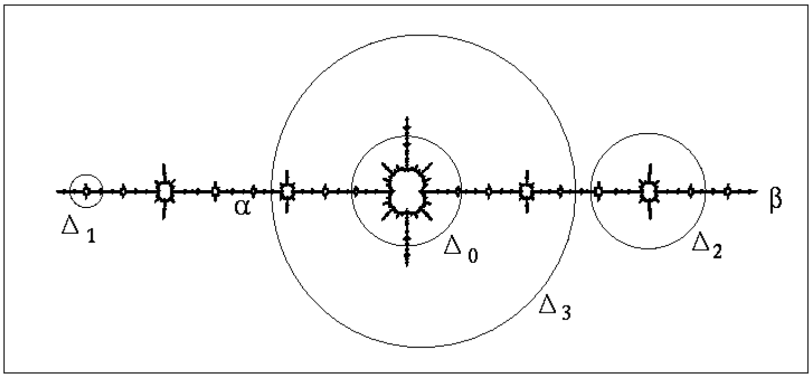


Figure 10. An example of simple renormalization: the Julia set for $f(z) = z^2 - 1.75$, with renormalization period $p = 3$. (The parameter value $c = -1.75$ is the root point of a small copy of the Mandelbrot set.) The filled Julia set $K(f^{\circ 3}|_{\Delta_0})$ is a topological disk bounded by a "cauliflower".

f. Following McMullen [Mc2], the map f is called *simply renormalizable* of period p if also:

(5) Each intersection $f^{\circ i}(K_p) \cap f^{\circ j}(K_p)$ with $0 \le i < j < p$ is either vacuous, or consists of a single point which does not disconnect either of these two sets.

We will be interested only in the simply renormalizable case. See Figure 10 for a simply renormalizable example, and Figure 11 for a map which is renormalizable but not simply renormalizable. (Compare [RS].)

We can now state two basic lemmas.

Lemma 1.5. *If the critical tableau associated with f is periodic, then f is simply renormalizable. More precisely, if $P_d(c_p) = P_d(0)$ for all depths d, with p minimal, and therefore $P_d(c_{i+p}) = P_d(c_i)$ for all i and d by the Second Tableau Rule, then f is simply renormalizable of period p.*

Lemma 1.6. *If the critical orbit lies completely within the union*

$$P_1(c_0) \cup P_1(c_1) \cup \cdots \cup P_1(c_{q-1})$$

of those puzzle pieces of depth one which touch the fixed point α, then f is simply renormalizable of period q. (See Figure 2.)

The following construction will be needed to prove Lemmas 1.5 and 1.6. Recall that each of the puzzle pieces $P_0(c_i)$ of depth zero consists of points in

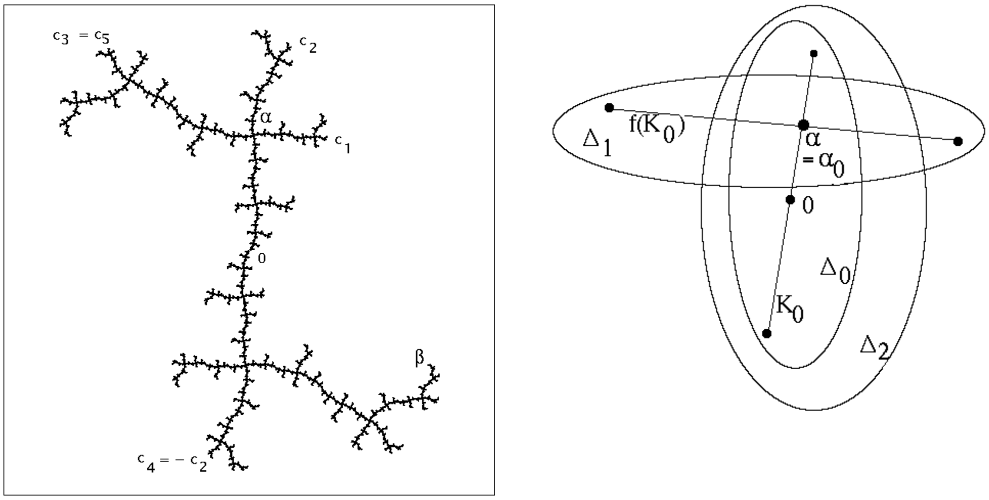


Figure 11. An example of crossed renormalization: The Julia set for $f(z) = z^2 + .41964338 + .60629073\,i$ *is shown on the left, with a schematic diagram of its renormalization on the right. The associated small Julia set* $K(f^{\circ 2}|_{\Delta_0})$ *is a simple arc from* c_2 *to* c_4 *. Our discussion will always exclude such examples.*

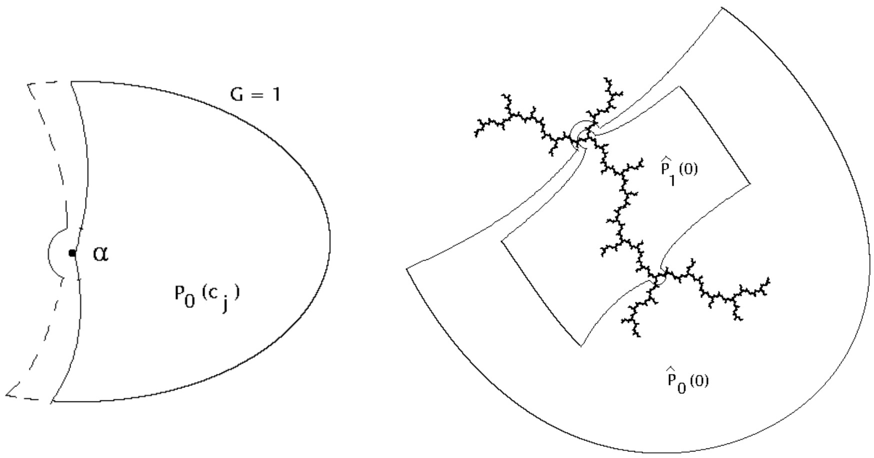


Figure 12 Left: A Puzzle Piece and Thickened Piece of Level 0 *.*
Right: Thickened Pieces of Level 0 *and* 1 *for* $z \mapsto z^2 + i$ *.*

some closed subset of the filled Julia set $K(f)$, together with points outside of $K(f)$ which have potential G and external angle t satisfying inequalities of the form

$$0 < G \le 1 , \qquad t_i \le t \le t'_i .$$

(Here $0 \le i < q$.) We construct a *thickened puzzle piece* $\widehat{P}_0(c_i) \supset P_0(c_i)$ in two steps, as follows. (Figure 12.) First choose a small disk $D_\epsilon(\alpha)$ about the fixed point α. Second, choose $\eta > 0$ so small that every external ray whose angle differs from t_i or t'_i by at most η intersects this disk. Now let $\widehat{P}_0(c_i)$ consist of the disk $D_\epsilon(\alpha)$, together with the region bounded by:

(1) the segment $t_i - \eta \le t \le t'_i + \eta$ of the equipotential $G = 1$,

(2) the external ray segments of angle $t_i - \eta$ and $t'_i + \eta$ which extend from this equipotential $G = 1$ to their first intersection with $D_\epsilon(\alpha)$, and

(3) an arc of the boundary of $D_\epsilon(\alpha)$.

Evidently this thickened puzzle piece $\widehat{P}_0(c_i)$ contains the original $P_0(c_i)$. We now construct thickened puzzle pieces of greater depth by the usual inductive procedure: If $\widehat{P}_d^{(j)}$ is a thickened puzzle piece of depth d, then each component of $f^{-1}(\widehat{P}_d^{(j)})$ is a thickened puzzle piece of depth $d+1$.

The virtue of these thickened pieces is the following statement, which is easily proved by induction: *If a puzzle piece $P_d^{(j)}$ contains $P_{d+1}^{(k)}$, then the corresponding thickened piece $\widehat{P}_d^{(j)}$ contains $\widehat{P}_{d+1}^{(k)}$ in its interior.* In other words, this construction replaces all of our annuli by non-degenerate annuli.

We will only make use of thickened pieces which are small enough to satisfy the following additional restriction: *If $\widehat{P}_d(z)$ contains the critical point, then the interior of $P_d(z)$ must already contain this critical point.* If 1.2 is satisfied, then this requirement is easily satisfied for any bounded values of the depth d, and this will suffice for the applications (Lemmas 1.5 through 1.8). Note however that we cannot expect this condition to be satisfied for unbounded values of d. In fact this would be impossible whenever the fixed point α is an accumulation point for the critical orbit.

Proof of Lemma 1.5. Choose $d \ge p$ so large that the critical annulus $A_d(0)$ is a child of $A_{d-p}(0)$, and let $\Delta_0 = \widehat{P}_d(0)$ be the thickened critical puzzle piece of depth d. Then $\Delta_p = f^{\circ p}(\Delta_0)$ is equal to $\widehat{P}_{d-p}(c_p) = \widehat{P}_{d-p}(0)$, which contains Δ_0 in its interior. The hypothesis guarantees that the successive points $c_p, c_{2p}, c_{3p}, \ldots$ all belong to Δ_0. Since p is minimal, the various puzzle pieces $P_{d-i}(c_i)$ with $0 \le i < p$ cannot overlap, so any intersection point must belong to the boundary of these puzzle pieces, and therefore cannot disconnect the small filled Julia set. Therefore f is simply renormalizable. $\square$

Proof of Lemma 1.6. (suggested by M. Lyubich). Let $\Delta_0 = \widehat{P}_1(0)$. Then it is easy to check that the successive images $\Delta_i = f^{\circ i}(\Delta_0)$ are disjoint from the critical point for $0 < i < q$, and that Δ_q contains Δ_0 in its interior. (Compare Figure 2.) We will prove inductively that $c_{qi} \in P_1(0) \subset \Delta_0$ for every i. It certainly follows from this inductive hypothesis that $c_{qi+1} \in P_0(c_1) \cap K(f) \subset P_1(c_1)$, and similarly that $c_{qi+j} \in P_0(c_j) \cap K(f) \subset P_1(c_j)$ for $0 < j < q$. In particular, c_{qi+q-1} must belong to $P_1(c_{q-1})$, hence $c_{qi+q} \in f\big(P_1(c_{q-1})\big) = P_0(0)$. By the hypothesis of 1.6, the point c_{qi+q} is known to lie in one of the puzzle pieces $P_1(c_h)$ which lie around α. Evidently it can only lie in $P_1(0)$, as required. □

Corollary 1.7. *If f is not simply renormalizable, then there exists a critical annulus of positive modulus.*

Proof. According to 1.6, the critical orbit must visit one of the puzzle pieces $P_1(-c_1)$, $\ldots$, $P_1(-c_{q-1})$ which surround the pre fixed point $-\alpha$. Suppose for example that $c_d \in P_1(-c_i)$. It is easy to check that the corresponding annulus $A_0(c_d) = A_0(-c_i)$ has positive modulus. (Compare Figure 2.) Pulling this annulus back inductively along the critical orbit, it follows that $A_d(0)$ also has positive modulus. □

As another another application of thickened puzzle pieces, we can study orbits which stay away from the critical point.

Lemma 1.8. *Let $z_0 \mapsto z_1 \mapsto \cdots$ be an orbit in $J \setminus J_0$ which is disjoint from some critical puzzle piece $P_N(0)$. Then the intersection $\bigcap P_d(z_0)$ of the puzzle pieces containing z_0 reduces to the single point z_0.*

Proof. Let $U_i = \mathrm{Interior}\big(\widehat{P}^{(i)}_{N-1}\big)$, where the various puzzle pieces of depth $N-1$ have been numbered so that the critical value c_1 belongs to $P^{(0)}_{N-1}$. Each puzzle piece of depth $\geq N$ is contained in some unique $P^{(i)}_{N-1}$, and we give it the Poincaré metric $\mathrm{dist}_i(x,y)$ associated with the corresponding U_i. For $i > 0$, note that there are exactly two branches of f^{-1} on U_i, call them g_1 and g_2, and that each of these maps g_κ on U_i is univalent and carries U_i onto a proper subset of some U_j. Thus each g_κ is Poincaré distance decreasing. Since every puzzle piece of depth N is a compact subset of the associated U_i (we need the thickening at this point), it follows that

$$\mathrm{dist}_j\big(g_\kappa(x)\,,\, g_\kappa(y)\big) \;\leq\; \lambda\,\mathrm{dist}_i(x,y)$$

throughout any such puzzle piece, provided that $i > 0$, where $\lambda < 1$ is a uniform constant. Let δ be the maximum of the Poincaré diameters of the puzzle pieces of depth N. Then for any $h > 0$, since none of the puzzle pieces

$$P_{N+h}(z_0) \;\to\; P_{N+h-1}(z_1) \;\to\; \cdots \;\to\; P_{N+1}(z_{h-1}) \;\to\; P_N(z_h)$$

(except possibly the first) can be contained in the critical value puzzle piece $P^{(0)}_{N-1}$, it follows that the Poincaré diameter of $P_{N+h}(z_0)$ is at most $\lambda^h\delta$, which tends to zero as $h\to\infty$. □

In order to deal with the possibility that $f^{\circ d}(z_0)=\alpha$, so that the puzzle piece $P_d(z_0)$ is not uniquely defined, we will need the following.

Definition. For any point z in the Julia set let $P^*_d(z)$ be the union of the puzzle pieces of depth d containing z . (In most cases, $P^*_d(z)$ is equal to the unique depth d puzzle piece $P_d(z)$ which contains z . However, if $f^{\circ d}(z)=\alpha$ then $P^*_d(z)$ is a union of q distinct puzzle pieces.)

We can now state and prove the principal result of this section.

> **Theorem 1.9.** *Suppose as usual that f is quadratic, with connected Julia set, with both fixed points repelling, and not simply renormalizable. Suppose further that the critical orbit is disjoint from the fixed point α . Then for any $z\in J(f)$ we have $\bigcap_d P^*_d(z)=\{z\}$.*

Proof. Since we assume that f is not simply renormalizable, we know from Corollary 1.7 that there exists some critical annulus $A_m(0)$ which has positive modulus. We will first prove that $\bigcap P_d(0)=\{0\}$, then prove that $\bigcap P_d(z)=\{z\}$ for any $z\in J(f)$ which is not an iterated pre-image of α , and finally prove the corresponding result when $f^{\circ n}(z)=\alpha$.

Critically Recurrent Case. Suppose that the critical orbit is recurrent, so that 1.3 applies. First consider the puzzle pieces $P_d(0)$ around the critical point. If the non-degenerate annulus $A_m(0)$ has at least 2^k descendents in the k-th generation for each k , then each of these contributes exactly $\mathbf{mod}\,A_m(0)/2^k$ to the sum $\sum_d \mathbf{mod}\,A_d(0)$. Hence this sum is infinite, as required. On the other hand, if there are fewer descendents in some generation, then one of them must be an only child, hence excellent by Lemma 1.3d. Using Lemma 1.3b and 1.3c, we again see that $\sum_d \mathbf{mod}\,A_d(0)$ is infinite. Therefore in either case the intersection $\bigcap P_d(0)$ reduces to the single point 0 .

Now consider a point $z_0\neq 0$ of the Julia set. We assume that the orbit $z_0\mapsto z_1\mapsto\cdots$ is disjoint from α , so that the puzzle pieces $P_d(z_0)$ are well defined. If the orbit of z_0 does not accumulate at zero, then we have $\bigcap P_d(z_0)=\{z_0\}$ by Lemma 1.8. Suppose, on the other hand, that the origin is an accumulation point of $\{z_n\}$. In other words, suppose that the tableau of the point z_0 has critical annuli reaching down to all depths. For each depth d , let us start at column zero (corresponding to the point z_0 itself) and advance to the right until we first hit a critical annulus at column n , then proceed diagonally back down until we reach column zero at depth $n+d$.

It follows from this construction that the annulus $A_{n+d}(z_0)$ is conformally isomorphic to $A_d(0)$. Furthermore, distinct values of d must correspond to distinct values of $n+d$. Thus the sum $\sum \mathbf{mod}\, A_d(z_0)$ is also infinite, hence $\bigcap P_d(z_0) = \{z_0\}$, as required.

Critically Non-recurrent Case. Now suppose that the critical orbit is not recurrent. Then the critical value $f(0)$ satisfies the hypothesis of Lemma 1.8. Hence $\bigcap P_d\big(f(0)\big) = \{f(0)\}$, and it follows easily that $\bigcap P_d(0) = \{0\}$. Next consider a point $z_0 \neq 0$. Again we may assume that the orbit $\{z_n\}$ accumulates at zero, since otherwise the conclusion would follow from Lemma 1.8. Again 1.7 tells us that there exists one depth m such that $A_m(0)$ has positive modulus. The corresponding depth m for the tableau of z_0 must have infinitely many columns k which are critical. For each of these, let us proceed diagonally back down in the tableau of z_0, ignoring whatever critical or semi-critical annuli we may meet, until we reach column zero at depth $m+k$. Each time we meet a critical or semi-critical annulus, we lose up to half of the modulus. (Problem 1-b below.) However, the hypothesis that the critical orbit is non-recurrent guarantees that such losses will only occur a bounded number of times. Thus $\sum \mathbf{mod}\, A_d(z_0)$ has infinitely many summands which are bounded away from zero. Hence this sum is infinite, and we have proved that $\bigcap P_d(z_0) = \{z_0\}$ in this case also.

Iterated Pre-images of α. If some forward image $z_n = f^{\circ n}(z_0)$ is equal to the fixed point α, then the above arguments do not make sense as stated. In this case, there are q distinct puzzle pieces $P_n^{(i)}$ of depth n which have z_0 as common boundary point. Each of these is contained in a unique sequence of nested puzzle pieces $P_n^{(i)} \supset P_{n+1}^{(i)} \supset \cdots$ which have z_0 as common boundary point. **Assertion:** *For each one of these q nested sequences the intersection $\bigcap_d P_d^{(i)}$ reduces to the single point z_0.* In fact, the proof of Lemma 1.8 applies equally well to this situation. Evidently the statement that $\bigcap_d P_d^*(z_0) = \{z_0\}$ follows immediately.

Thus we have proved Theorem 1.9, $\bigcap_d P_d^*(z) = \{z\}$, in all cases. Theorem 1.1, as stated at the beginning of this section, is a straightforward consequence. (Compare Problem 1-a below.) □

For further developments on the puzzle technique, see for example [F], [L2], [L3], [L4], [Pe3].

Here are some problems for the reader.

Problem 1-a. Local Connectivity. Prove that the intersection of $J(f)$ with each puzzle piece is connected. Conclude that $J(f)$ is locally connected at z whenever $\bigcap P_d^*(z) = \{z\}$.

Problem 1-b. Semi-Critical Annuli. Show that a semi-critical annulus $A_d(z_i)$ of depth $d > 0$ is non-degenerate if and only if $A_{d-1}(z_{i+1})$ is non-degenerate. In the non-degenerate case, show that $A_d(z_i)$ is the union of

(1) a ramified two-fold covering of $A_{d-1}(z_{i+1})$, and

(2) a conformal copy of $P_d(z_{i+1})$.

Splitting $A_{d-1}(z_{i+1})$ by a carefully chosen simple closed curve through the critical value, use the Grötzsch inequality to prove that $\mathbf{mod}\, A_d(z_i) > \frac{1}{2}\mathbf{mod}\, A_{d-1}(z_{i+1})$.

Problem 1-c. Non-degenerate Annuli. Show that an annulus $A_0(z)$ of depth zero is non-degenerate if and only if it is semi-critical, and show that $A_d(z_m)$ is non-degenerate if and only if $A_0(z_{d+m})$ is non-degenerate.

Problem 1-d. Illustrating the Tableau Rules. Suppose that we try to modify the Fibonacci tableau of Figure 6 by changing just one column. For example, suppose that we place the semi-critical annulus for column 5 at depth 3 or 4 or at depth ≥ 6, instead of at depth 5, without changing columns 0 through 4. Show that the tableau rules would then force column 8 to end already at the semi-critical depth 0 or 1 or 2 respectively.

Problem 1-e. The Yoccoz τ-function. For each depth $d > 0$, consider the sequence of puzzle pieces $f^{\circ i}\big(P_d(0)\big) = P_{d-i}(c_i)$ with $0 < i \leq d$. If one of these pieces is equal to the critical puzzle piece $P_{d-i}(0)$, then choose the smallest such i and set $\tau(d) = d - i$. Otherwise, if there is no such i, set $\tau(d) = -1$. Thus $-1 \leq \tau(d) < d$ for all $d > 0$. Show that $\tau(d+1) \leq \tau(d) + 1$ in all cases, and show that the annulus $A_d(0)$ is a child of $A_n(0)$ if and only if $n = \tau(d)$ with $\tau(d+1) = \tau(d) + 1 > 0$.

Problem 1-f. Persistent Recurrence. Show that the critical tableau is recurrent if and only if the function $\tau(d)$ is unbounded, and that it is periodic if and only if $\tau(d{+}1) = \tau(d){+}1$ for large d. By definition, the critical orbit is *persistently recurrent* if the tableau is non-periodic with $\lim_{d\to\infty} \tau(d) = +\infty$. Show that the Fibonacci tableau is persistently recurrent. In the persistently recurrent case, show that the critical orbit is bounded away from the fixed point β. (Otherwise, for any depth d we could choose a post-critical point $c_n \in P_d(\beta)$, then choose the smallest k with $0 \in P_{d+k}(c_{n-k})$, and conclude that $\tau(d{+}k) = 0$.) Show that the critical orbit is bounded away from every iterated pre-image of β, and hence that its closure is nowhere dense in the Julia set. Similarly show that the critical orbit is bounded away from the fixed point α; and use this fact to show that the critical orbit closure is a Cantor set. More generally, show that the critical orbit is bounded away from any periodic or pre-periodic point. (For further information, see [L2].)

Problem 1-g. The Critical Orbit is Generically Dense. It is convenient to say that a property of certain points in a compact set is *generically* true if it is true throughout a countable intersection of dense relatively open subsets. *For a generic point $c \in \partial M$ in the boundary of the Mandelbrot set, show that the critical orbit for the map f_c is everywhere dense in the Julia* *set.* (Proof outline, suggested by McMullen and by J. Kwapisz: Let $M(p,\epsilon)$ be the set of $c \in \partial M$ such that the critical orbit for f_c comes within ϵ of every periodic orbit of period p. Using Montel's Theorem, show that this set is dense for $p \ge 2$. It is clearly open. Therefore a generic c belongs to the intersection of the $M(p,\epsilon)$, which implies that the critical orbit is dense in J.) As one corollary, it follows that a generic f_c is not persistently recurrent, and also there are no excellent annuli in the generic case. If f_c is renormalizable of period k, show that the union of the small Julia sets $f_c^{\circ i}(J_k)$ is a proper subset of the Julia set $J(f_c)$. Conclude that f_c is not renormalizable in the generic case.

2. Polynomials For Which All But One of the Critical Orbits Escape

This section will describe easier applications of ideas from §1. Its object is to describe the Branner-Hubbard theory of polynomials for which only one critical point has bounded orbit. However, we begin with the simpler case where *no* critical point has bounded orbit. The following is well known.

Theorem 2.1. *Let f be a polynomial of degree $n \ge 2$ for which all critical orbits escape to infinity. Then the filled Julia set K (in this case equal to the Julia set $J = \partial K$) is a Cantor set of measure zero*[2]*. Furthermore, the dynamical system $(J\,,\,f|_J)$ is homeomorphic to the one-sided shift on n symbols*[3].

Proof. Let $\omega_1\,,\,\ldots\,,\,\omega_{n-1}$ be the (not necessarily distinct) critical points of f, and let $G:\mathbb{C}\to\mathbb{R}_+$ be the canonical potential function (= Green's function), which satisfies $G\big(f(z)\big) = n\,G(z)$ and vanishes precisely on the filled Julia set K. Thus $G(\omega_i) > 0$ by hypothesis. The critical points of G in $\mathbb{C} \smallsetminus K$ are the points ω_i and also all of their preimages under iterates of f. Hence the critical values of G (other than zero) are the numbers of the form $G(\omega_i)/n^k$ with $k \ge 0$.

The *puzzle* of f is constructed as follows. Choose a number γ, not of the form $G(\omega_i)/n^k$, so that $0 < \gamma < \mathrm{Min}\big\{G\big(f(\omega_i)\big)\big\}$. Then the region $X_0 = G^{-1}(0\,,\,\gamma]$ contains no critical values of f, and is bounded by smooth curves. Similarly, each locus

$$X_d \;=\; f^{-d}(X_0) \;=\; G^{-1}\big[0\,,\,\gamma/n^d\big]$$

[2]Shishikura has shown that the Hausdorff dimension of this Cantor set can be arbitrarily close to two.

[3]Both Przytycki [Pr] and Makienko [Ma] have proved the sharper result that any rational Julia set which is totally disconnected and contains no critical point must be isomorphic to a one-sided shift.

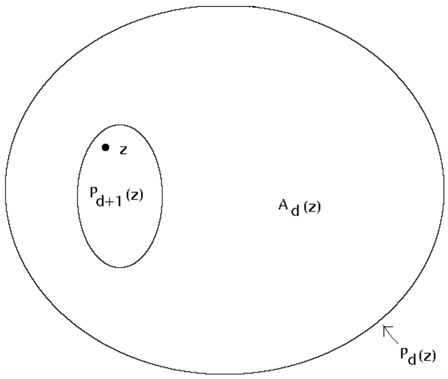


Figure 13. Nested puzzle pieces, and the annulus $A_d(z)$.

is bounded by smooth curves. Note that the complementary region $\mathbb{C} \smallsetminus X_d$ cannot have any bounded component, since the harmonic function G cannot have a local maximum. It follows that the locus X_d is the disjoint union of a finite number of closed topological disks. By definition, each of these closed disks will be called a *puzzle piece* P_d of depth d . *Since these puzzle pieces contain no critical value of f , it follows easily that f maps each P_d of depth $d > 0$ by a conformal isomorphism onto some puzzle piece $f(P_d)$ of depth $d-1$.*

If $P_d \supset P_{d+1}$ are nested puzzle pieces of depths d and $d+1$, then the set

$$A_d \;=\; \mathrm{Interior}(P_d) \smallsetminus P_{d+1}$$

is a well defined annulus of strictly positive modulus. (Figure 13.) We will call such an A_d an *annulus of depth* d in the puzzle. Evidently $f^{\circ d}$ maps each annulus of depth d by a conformal isomorphism onto an annulus of depth zero. *Hence the moduli of all annuli constructed in this way are uniformly bounded away from zero.*

For any point z in the filled Julia set we can form the nested sequence of puzzle pieces $P_0(z) \supset P_1(z) \supset \cdots$, all containing z . Since the moduli of the annuli Interior $P_d(z) \smallsetminus P_{d+1}(z)$ are bounded away from zero, it follows that the intersection $\bigcap P_d(z)$ reduces to the single point z . Since each boundary circle ∂P_d is disjoint from K , this implies that K is totally disconnected.

The proof that J has measure zero will be based on the *McMullen inequality*

$$\mathbf{area}(P_{d+1}) \;\le\; \frac{\mathbf{area}(P_d)}{1 + 4\pi\,\mathbf{mod}\,(A_d)} \;,$$

with $A_d = \mathrm{Interior}(P_d) \smallsetminus P_{d+1}$ as above. (Compare the Appendix.) However, to apply this inequality in a useful way, we will need to construct annuli somewhat differently. Choose a number $\epsilon > 0$ which is small enough so that

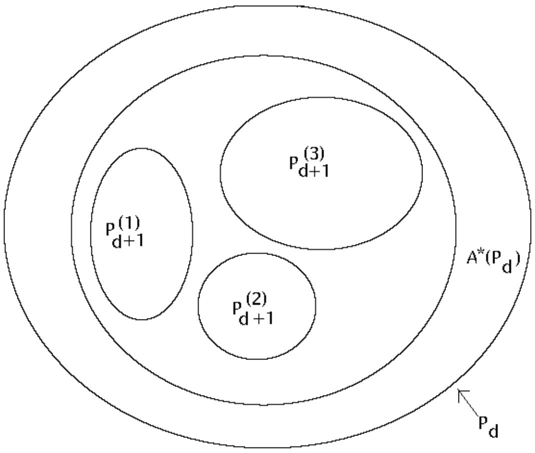


Figure 14. The "thin annulus" $A^*(P_d) \subset A_d \subset P_d$.

there are no critical values of G within the closed interval $[\gamma-\epsilon\,,\,\gamma]$. Then each connected component of $G^{-1}[(\gamma-\epsilon)/n^d\,,\,\gamma/n^d]$ is an annulus, and there is one such "thin annulus"

$$A^*(P_d) \quad = \quad P_d \cap G^{-1}\Big[\frac{\gamma-\epsilon}{n^d}\,,\,\frac{\gamma}{n^d}\Big]$$

within each connected component P_d of $G^{-1}[0\,,\,\gamma/n^d]$. The construction is such that:

(a) these annuli $A^*(P_d)$ have modulus uniformly bounded away from zero, say $\mathbf{mod}\,A^*(P_d) \ge \epsilon > 0$, and

(b) every puzzle piece P_{d+1} which is contained in P_d is actually contained in the smaller disk $P_d \smallsetminus A^*(P_d)$ (the bounded component of $\mathbb{C} \smallsetminus A^*(P_d)$).

Thus the McMullen inequality takes the following sharper form. For each fixed puzzle piece P_d ,

$$\sum_{P_{d+1}\subset P_d} \mathbf{area}(P_{d+1}) \quad \le \quad \frac{\mathbf{area}(P_d)}{1+4\pi\,\mathbf{mod}\,\big(A^*(P_d)\big)} \quad \le \quad \frac{\mathbf{area}(P_d)}{1+4\pi\,\epsilon}\,,$$

to be summed over those puzzle pieces of depth $d+1$ which are contained in the given P_d . It now follows inductively that the sum of the areas of *all* puzzle pieces of depth d satisfies

$$\sum \mathbf{area}(P_d) \;\le\; \sum \mathbf{area}(P_0)/(1+4\pi\,\epsilon)^d\;.$$

Since this tends to zero as $d\to\infty$, and since $J\subset\bigcup P_d$, it follows that J has area zero.

To prove that $(J\,,\,f|_J)$ is isomorphic to the one-sided d-shift, we proceed as follows. We will construct a closed subset $\Gamma\subset\mathbb{C}$, consisting of paths leading out to infinity, such that:

(1) Γ contains all critical values of f,

(2) $f(\Gamma)\subset\Gamma$, and

(3) the complement $U=\mathbb{C}\smallsetminus\Gamma$ is simply-connected, and contains the Julia set.

The construction will be based on the following. By a *loxodrome* in $\mathbb{C}\smallsetminus K$ will be meant a smooth curve which cuts the family of equipotential curves at some constant angle. If we fix this angle, note that there is a unique loxodrome through any point of $\mathbb{C}\smallsetminus K$ which is not a critical point of G. The required Γ will be a locally finite union of loxodromes, all with the same fixed angle, each starting a some postcritical point of f and extending out to infinity. If we choose the angle badly, then such a loxodrome may run into some critical point of G. However, for all but countably many choices of angle this problem will not occur. There remains the possibility that some postcritical point z is itself a precritical point of f, and hence a critical point of G. In this case, there will be finitely many distinct loxodromes of given angle starting at z, and we can simply choose any one of them, making compatible choices along the orbit of z. The required properties (1), (2), (3) are now easily verified.

Since $f(\Gamma)\subset\Gamma$, it follows that $f^{-1}(U)\subset U$. Furthermore, since U is simply connected, and contains no critical values of f, it follows that every one of the d branches of f^{-1} near a point of U extends uniquely to a holomorphic map $g_i:U\to U$. The images

$$g_1(U)\,,\,\ldots\,,\,g_n(U)\ \subset\ U$$

are disjoint open sets covering the Julia set. We will show that the intersections

$$J_i\ =\ J\cap g_i(U)$$

are disjoint compact sets which form the required *Bernoulli partition* $J=J_1\cup\cdots\cup J_n$ of the Julia set. That is:

For each sequence of integers $i_0\,,\,i_1\,,\,\ldots$ between 1 and n there exists one and only one orbit $z_0\mapsto z_1\mapsto z_2\mapsto\cdots$ in the Julia set with $z_k\in J_{i_k}$ for every k.

In fact z_0 can be described as the intersection of the nested sequence of sets $g_{i_0}\circ g_{i_1}\circ\cdots\circ g_{i_k}(J)$. To prove this statement, we use the Poincaré metric on U. Since each g_i restricted to the compact set $J\subset U$ shrinks Poincaré distances by a factor bounded away from one, it follows that each such intersection $\bigcap_k g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_k}(J)$ consists of a single point. This proves 2.1. □

Maps with exactly one bounded critical orbit

This will be an exposition of results due to Branner and Hubbard [BH3]. We now suppose that exactly one of the $n-1$ critical points of f, counted with multiplicity, has bounded orbit, while the orbits of the remaining $n-2$ critical points escape to infinity. (Thus we exclude examples such as $z \mapsto z^3 + i$, for which a double critical point has bounded orbit; however, a double critical point escaping to infinity is fine.) Furthermore, we assume that $n \geq 3$, so that at least one critical orbit does escape. Then, according to Fatou and Julia, the Julia set is disconnected, with uncountably many connected components. Let

$$c_0 \;\mapsto\; c_1 \;\mapsto\; c_2 \;\mapsto\; \cdots$$

be the unique bounded critical orbit.

As in the proof of Theorem 2.1, choose a number $\gamma > 0$ which is not a critical value of G, and so that the region $G^{-1}(0, \gamma]$ contains no critical value of f. Again we define the puzzle pieces of depth d to be the connected components P_d of the locus

$$\bigcup P_d \;=\; G^{-1}\Big[0\,,\,\gamma/n^d\Big]\;.$$

Thus each point $z \in K$ determines a nested sequence $P_0(z) \supset P_1(z) \supset \cdots$, and the central problem is to decide whether or not $\bigcap_d P_d(z) = \{z\}$. Again we look at the intermediate annuli

$$A_d(z) \;=\; \text{Interior}\, P_d(z) \smallsetminus P_{d+1}(z)\;.$$

As in the Yoccoz proof, such an annulus is said to be semi-critical, critical, or off-critical according as the critical point c_0 belongs to the annulus itself, or to the bounded or the unbounded component of its complement. (For this purpose, we ignore the other critical points, whose orbits escape to infinity.) This Branner-Hubbard puzzle is easier to deal with than the Yoccoz puzzle for three reasons:

(a) All of the annuli $A_d(z)$ are non-degenerate, with strictly positive modulus.

(b) The various puzzle pieces of depth d are pairwise disjoint.

(c) For each $z \in K$, the intersection $\bigcap_d P_d(z)$ of the puzzle pieces containing z has an immediate topological description: It is equal to the connected component of the filled Julia set K which contains the given point z. For this intersection is clearly connected, and no larger subset can be connected since each boundary $\partial P_d(z)$ is disjoint from K.

The *tableau* of an orbit $z_0 \mapsto z_1 \mapsto \cdots$ is defined just as in §1, and can be described as a record of exactly which of the annuli $A_d(z_i)$ are critical or

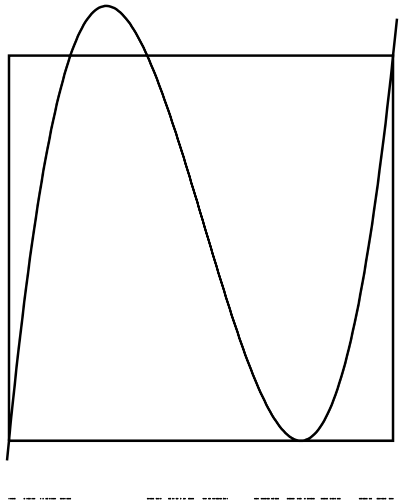


Figure 15. An example: Graph of the function $f(x) = x(x - c_0)^2/(1 - c_0)^2$ on the unit interval $I = [0, 1]$, with critical points $c_0 = .76$ and $c_0/3$. There is just one bounded critical orbit $c_0 \mapsto 0 \mapsto 0 \mapsto \cdots$. Since the iterated pre-images of any point of J are dense in J, and since there are three distinct branches of f^{-1} mapping I into itself, it follows that the Julia set J is completely contained in the real interval I. This Julia set is plotted underneath the graph. Since c_0 and 0 evidently belong to distinct connected components of J, it follows from Theorem 2.2 that J is totally disconnected.

semi-critical or off-critical. First suppose that the tableau of the critical orbit is not periodic. We continue to assume that the degree is $n \geq 3$ and that exactly one of the $n - 1$ critical points has bounded orbit.

Theorem 2.2. *If the critical tableau is not periodic, or equivalently if the post-critical points $c_1, c_2, \ldots$ are all disjoint from the critical component $\bigcap_d P_d(c_0)$, then for every point z_0 of the filled Julia set K the sum $\sum_d \mathbf{mod}\, A_d(z_0)$ is infinite, hence $\bigcap_d P_d(z_0) = \{z_0\}$. It follows that $J = K$ is a totally disconnected set of area zero.*

Thus J is again homeomorphic to a Cantor set. However, in this case $(J, f|_J)$ is not isomorphic to a shift, or even a sub-shift. For there are critical points in J, hence f is not locally one-to-one on J.

The proof of this theorem is quite similar to the proof of the Yoccoz theorem. However there are simplifications, leading to the sharper result which is stated: $\sum \mathbf{mod}\, A_d(z) = \infty$ for *all* $z \in K$. In particular, 1.8 is replaced by the following statement, which is true whether or not the critical tableau is periodic. Consider an orbit $z_0 \mapsto z_1 \mapsto \cdots$.

> **Lemma 2.3.** *Suppose that the points z_1, z_2, ... are all disjoint from some neighborhood $P_N(c_0)$ of the critical point c_0. Then the annuli $A_d(z_0)$ have modulus uniformly bounded away from zero, hence $\sum_d \textbf{mod}\, A_d(z_0) = \infty$.*

Proof. Each annulus $A_d(z_i)$ of depth $d > 0$ has modulus at least half of the modulus of $A_{d-1}(z_{i+1})$. In fact, if $i > 0$ and $d > N$ then these two annuli are conformally isomorphic. Thus

$$\textbf{mod}\, A_d(z_0) \ \geq \textbf{mod}\, A_0(z_d)/2^{N+1} \ ,$$

where the right side is bounded away from zero since there are only finitely many annuli of depth zero. □

Proof of 2.2. If the critical tableau is recurrent, then the proof proceeds exactly as is the Yoccoz argument: Every critical annulus either has at least 2^n descendents in the n-th generation for every n, or else has a descendent with this property. Since all annuli are non-degenerate, it follows that $\sum \textbf{mod}\, A_d(c_0) = \infty$. On the other hand, if the critical tableau is non-recurrent, then it follows from 2.3 that $\sum \textbf{mod}\, A_d(c_0) = \infty$. In particular, it follows that the collection of puzzle pieces $\{P_d(c_0)\}$ forms a fundamental system of neighborhoods of the critical point c_0. The corresponding statements for any iterated pre-image of c_0 follow immediately.

Now consider a point $z_0 \in K$, with orbit $z_0 \mapsto z_1 \mapsto \cdots$ which never meets the critical point c_0. If this orbit does not accumulate at c_0, then the statement that $\sum_d \textbf{mod}\, A_d(z_0) = \infty$ follows from 2.3. On the other hand, if this orbit does accumulate at c_0, then since $\sum \textbf{mod}\, A_d(c_0) = \infty$, an easy tableau argument shows that $\sum_d \textbf{mod}\, A_d(z_0) = \infty$ also. Thus $\bigcap_d P_d(z) = \{z\}$ in all cases.

Since each boundary $\partial P_d(z)$ is disjoint from the filled Julia set K, it follows that K is totally disconnected, and hence that $J = K$. To prove that this set has measure zero, we proceed as in the proof of Theorem 2.1. Choose $\epsilon > 0$ so that the interval $[\gamma - \epsilon, \gamma]$ contains no critical values of the function $G : \mathbb{C} \to \mathbb{R}_+$. Then each puzzle piece $P_d(z)$ contains a unique component $A^*\big(P_d(z)\big)$ of the set $G^{-1}[(\gamma - \epsilon)/n^d, \gamma/n^d]$. These annuli $A^*\big(P_d(z)\big) \subset A_d(z)$ are also non-degenerate, and the proof above shows equally well that $\sum_d \textbf{mod}\, A^*\big(P_d(z)\big) = \infty$. Hence, just as in the proof of 2.1, for each fixed puzzle piece P_d, the total area of the puzzle pieces P_{d+1} of depth $d+1$ which are contained in P_d satisfies

$$\sum_{P_{d+1} \subset P_d} \textbf{area}\, P_{d+1} \ \leq \ \frac{\textbf{area}\, P_d}{1 + 4\pi\, \textbf{mod}\, A^*(P_d)} \ .$$

Define the ratio $\mu(P_d)$ by the formula

$$\textbf{area}\, P_d \;=\; \mu(P_d) \sum_{P_{d+1}\subset P_d} \textbf{area}\, P_{d+1}$$

so that this McMullen inequality takes the form $1+4\pi\,\textbf{mod}\, A^*(P_d) \;\le\; \mu(P_d)$. Then, substituting this formula inductively, we can write

$$\begin{aligned}\sum_{P_0} \textbf{area}\, P_0 \;&=\; \sum_{P_0\supset P_1} \mu(P_0)\, \textbf{area}(P_1) \;=\; \cdots \\ &=\; \sum_{P_0\supset\cdots\supset P_d} \mu(P_0)\,\cdots\,\mu(P_{d-1})\, \textbf{area}(P_d)\ ,\end{aligned}$$

where the left hand expression is to be summed over all puzzle pieces of depth zero, the next over all pairs $P_0 \supset P_1$, and so on. (If $P_0 \supset \cdots \supset P_d$, note that $P_0, \ldots, P_{d-1}$ are uniquely determined by P_d.) Let η_d be the minimum value of the product $\mu(P_0)\cdots\mu(P_{d-1})$ as $P_0, \ldots, P_{d-1}$ varies over all sequences of nested puzzle pieces $P_0 \supset P_1 \supset \cdots \supset P_{d-1}$. Then we see from this last equality that

$$\sum \textbf{area}(P_0) \;\ge\; \eta_d \sum \textbf{area}(P_d)\ ,$$

to be summed over all puzzle pieces of depth zero or d respectively. Thus, if we can prove that $\eta_d \to \infty$ as $d \to \infty$, then it will follow that

$$\textbf{area}(J) \;\le\; \sum \textbf{area}(P_d) \;\le\; \sum \textbf{area}(P_0)/\eta_d \;\to 0\ ,$$

hence $\textbf{area}(J) = 0$ as required.

Clearly $1 < \eta_1 < \eta_2 < \cdots$. If these numbers tended to a finite limit $L < \infty$, then for each d we could find puzzle pieces $P_0(d) \supset P_1(d) \supset \cdots \supset P_{d-1}(d)$ so that $\mu\big(P_0(d)\big)\cdots\mu\big(P_{d-1}(d)\big) \le L$. Hence we could choose a puzzle piece P_0 which occurs infinity often as $P_0(d)$, then choose $P_1 \subset P_0$ which occurs infinitely often as $P_1(d)$, and so on. In this way, we could find a sequence

$$P_0 \;\supset\; P_1 \;\supset\; P_2 \;\supset\; \cdots$$

with $\mu(P_0)\cdots\mu(P_{d-1}) \le L < \infty$ for every d. Since $1+4\pi\,\textbf{mod}\,\big(A^*(P_i)\big) \le \mu(P_i)$, this would imply that

$$1 + 4\pi \sum \textbf{mod}\,\big(A^*(P_i)\big) \;\le\; L < \infty\ ,$$

contradicting our statement that $\sum \textbf{mod}\, A_d(z) = \infty$ for all $z \in K$. This completes the proof of 2.2. □

On the other hand, if the critical tableau is periodic, then we will prove the following.

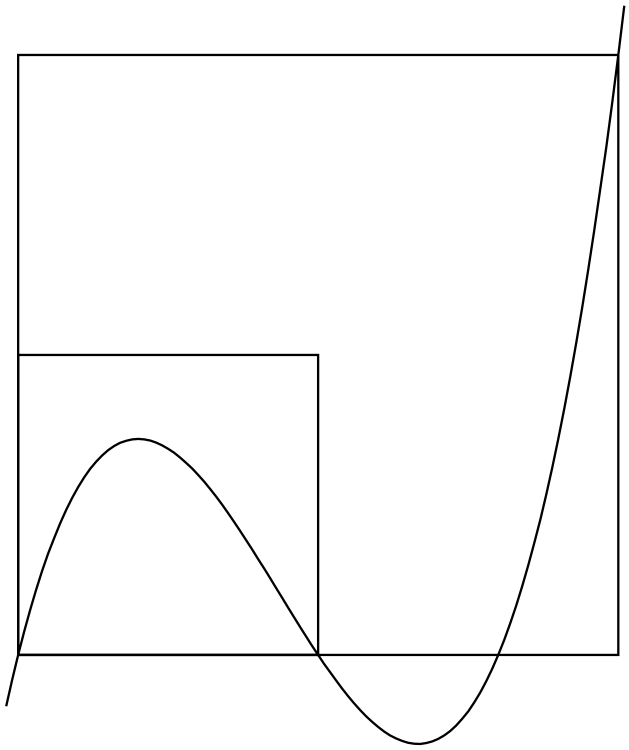


Figure 16. Example for Theorem 2.4: Graph of the map $f(x) = x(2x-1)(5x-4)$ *on the unit interval. In this case the connected interval* $[0, \frac{1}{2}]$ *is contained in the filled Julia set* K *. The orbit of the critical point* $\frac{2}{3}$ *escapes to* $-\infty$ *.*

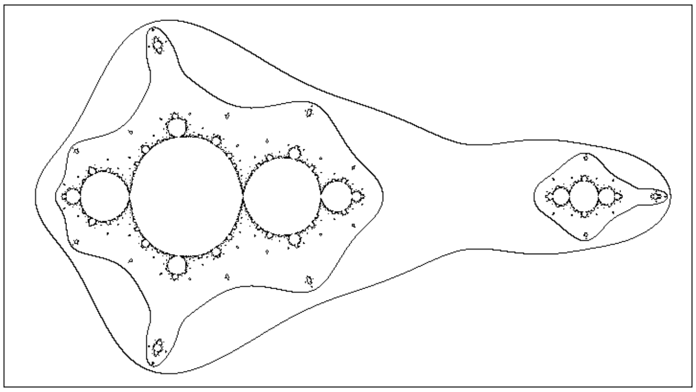


Figure 17. Julia set for this map, drawn to the same scale, showing the puzzle pieces of depth zero and one. Each non-trivial component of J *is homeomorphic to a quadratic Julia set (the "basilica").*

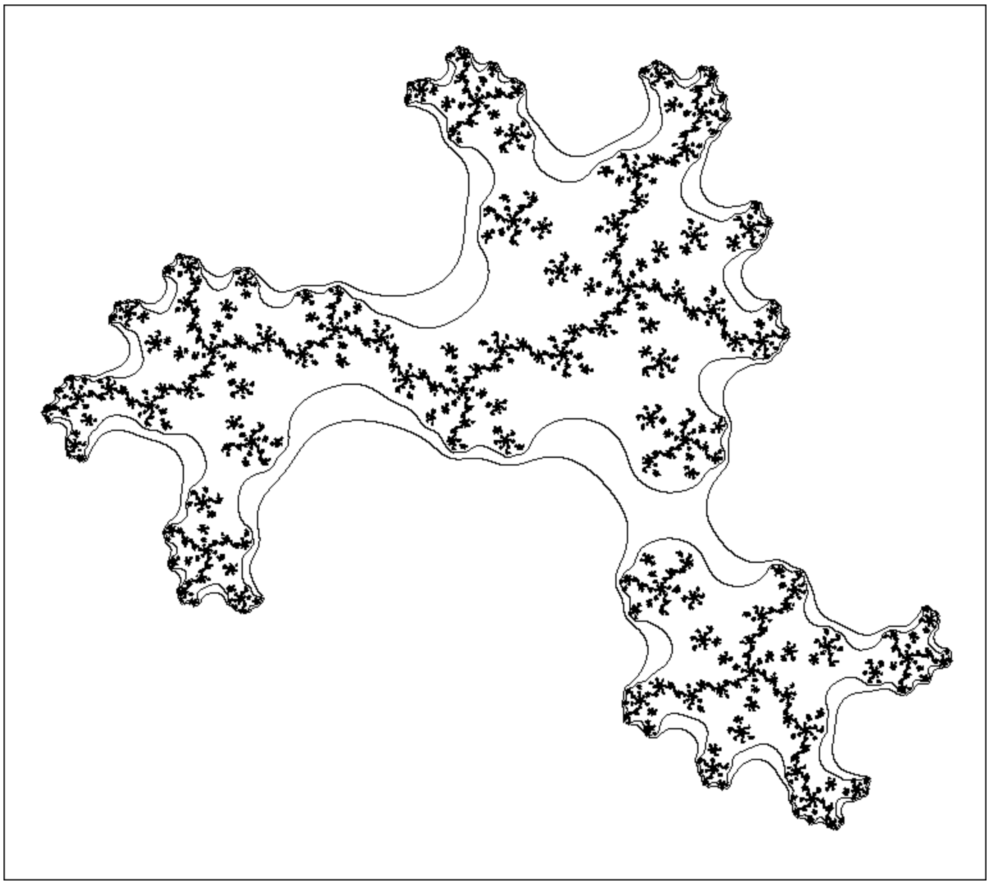


Figure 18. Julia set for $z \mapsto z^3 + a\,z^2 + 1$, with $a = -1.10692 + .63601\,i$, showing the puzzle pieces of depth zero and one. Each non-trivial component of J is homeomorphic to the Julia set for the quadratic map $z \mapsto z^2 + i$.

Theorem 2.4. *Still assuming that just one critical orbit $c_0 \mapsto c_1 \mapsto \cdots$ is bounded, if the associated critical tableau is periodic of period $p \geq 1$, so that $P_d(c_0) = P_d(c_p)$ for all depths d, then the connected component of the filled Julia set $K = K(f)$ which contains c_0 is non-trivial, that is, consists of more than one point. In fact, a component of K is non-trivial if and only if it contains some iterated pre-image of c_0.*

Thus there there are countably many non-trivial components of K. These countably many components are everywhere dense in K, since the iterated pre-images of any point of the Julia set are dense in the Julia set. Since a disconnected Julia set necessarily has uncountably many components, it follows that there are uncountably many single points components. (The Julia set of a rational function may have uncountably many non-trivial components. Compare [Mc1]. However, in the polynomial case no such example is known.)

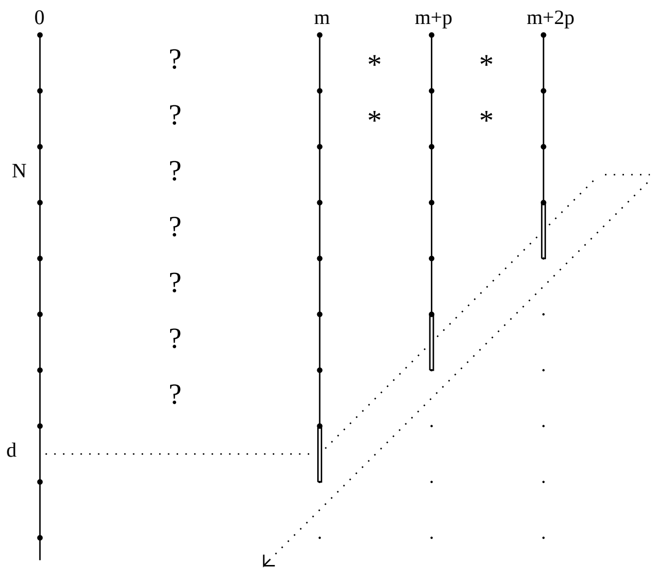


Figure 19. Tableau for an orbit $z_0 \mapsto z_1 \mapsto \cdots$ which intersects every critical puzzle piece. Here the stars stand for entries copied from the periodic critical tableau.

Proof of Theorem 2.4. If $P_d(c_0) = P_d(c_p)$ for all d, then it follows from the tableau rules that $P_d(c_i) = P_d(c_{p+i})$ for all i and d. Hence the entire orbit

$$c_0 \mapsto c_p \mapsto c_{2p} \mapsto \cdots$$

of c_0 under $f^{\circ p}$ is contained in the critical component $\bigcap_d P_d(c_0) \subset K(f)$. This intersection certainly has more than one point. For either it contains $c_0 \neq c_p$, or else c_0 is a superattracting point, in which case some entire neighborhood of c_0 belongs to $\bigcap_d P_d(c_0)$.

It follows easily that every pre-critical point in $K(f)$ also belongs to a non-trivial connected component. For if the orbit $z_0 \mapsto z_1 \mapsto \cdots$ intersects the critical component $\bigcap_d P_d(c_0)$, then we can choose the smallest $\ell \geq 0$ for which z_ℓ belongs to this critical component. It follows easily that $f^{\circ \ell}$ maps the component $\bigcap_d P_d(z_0)$ containing z_0 homeomorphically onto this critical component.

Now consider an orbit $z_0 \mapsto z_1 \mapsto \cdots$ in $K(f)$ which is disjoint from this critical component. This means that the tableau of this orbit has no columns which are completely critical. The proof is now divided into two cases:

Case 1. Suppose that there exists a fixed puzzle piece $P_d(c_0)$ which is disjoint from this orbit $\{z_n\}$. Then according to Lemma 2.3 we have $\sum_d \textbf{mod}\, A_d(z_0) = \infty$, hence $\bigcap_d P_d(z_0) = \{z_0\}$.

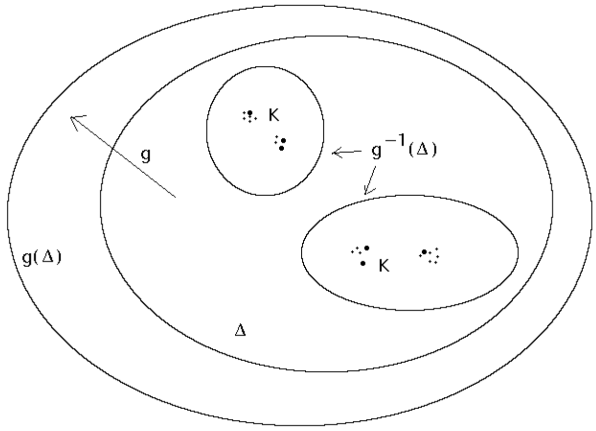


Figure 20. A polynomial-like mapping (g, Δ) *with* $K = K(g, \Delta)$ *totally disconnected.*

Case 2. If the orbit $\{z_n\}$ intersects *every* critical puzzle piece, then we use a tableau argument as follows. Choose a depth N so that the periodic critical tableau has no semi-critical annuli at depths $\geq N$. By hypothesis, there are infinitely many pairs (d, m) , with $d \geq N$, so that the d-th row of the tableau for z_0 is semi-critical in column m and off-critical in earlier columns. (Compare Figure 19.) Using the tableau rules, we can then compute the tableau in column $m+i$ and depth $d-i$ for $0 < i < d-N$. In fact the entries in column $m+jp$ and depth $d-jp$ are semi-critical, and the others are off-critical. It now follows that the annulus $A_{d+m+1}(z_0)$ is conformally isomorphic to an annulus of depth N . Hence its modulus is bounded away from zero. It follows that $\sum_\ell \mathbf{mod}\, A_\ell(z_0) = \infty$, which completes the proof of Theorem 2.4. $\square$

We can understand this proof better by introducing the following concepts, which are due to Douady and Hubbard [DH3].

Definition. By a *polynomial-like map* is meant a pair $(g\,,\,\Delta)$ where $\Delta \subset \mathbb{C}$ is a closed topological disk and g is a continuous mapping, holomorphic on the interior of Δ , which carries Δ onto a closed topological disk $g(\Delta)$ which contains Δ in its interior, such that g maps boundary points of Δ to boundary points of $g(\Delta)$.

The *degree* $\delta \geq 1$ of such a polynomial-like mapping is a well defined topological invariant. Note that almost every point of $g(\Delta)$ has precisely δ pre-images in Δ . The *filled Julia set* $K(g, \Delta)$ is defined to be the compact set consisting of all $z_0 \in \Delta$ such that the entire orbit $z_0 \mapsto z_1 \mapsto \cdots$ of z_0 under g is defined and is contained in Δ .

Lemma 2.5. *Such a polynomial-like map of degree δ has $\delta - 1$ critical points, counted with multiplicity, in the interior of Δ. The filled Julia set $K(g, \Delta)$ is connected if and only if it contains all of these $\delta - 1$ critical points.*

Proof. Consider the nested sequence of compact sets

$$g(\Delta) \supset \Delta \supset g^{-1}(\Delta) \supset g^{-2}(\Delta) \supset \cdots$$

with intersection $K(g, \Delta)$. First suppose that the boundary $\partial\Delta$ contains no post-critical points, that is points $g^{\circ k}(\omega_i)$ with $k > 0$ where ω_i is a critical point of g. Then clearly each $g^{-k}(\Delta)$ is a compact set bounded by one or more closed curves. In fact, each $g^{-k}(\Delta)$ is either a closed topological disk or a finite union of closed topological disks, each of which maps onto the entire disk Δ under $g^{\circ k}$. To see this, note that for each component B of $g^{-k}(\Delta)$ the image $g^{\circ k}(B)$ is compact, and that $g^{\circ k}$ maps the boundary ∂B into $\partial\Delta$ and maps the interior of B onto an open subset of the interior of Δ. Since the interior of Δ is connected, this implies that $g^{\circ k}(B) = \Delta$. If some such component B were not simply-connected, then some component B' of $\mathbb{C} \smallsetminus g^{-k}(\Delta)$ would be bounded. A similar argument would then show that $g^{\circ k}$ must map B' onto the complementary disk $\widehat{\mathbb{C}} \smallsetminus \mathrm{Interior}(\Delta)$, which is impossible.

Applying the Riemann-Hurwitz formula to the ramified covering $\Delta \to g(\Delta)$, we see that the number of critical points of g in the interior of Δ, counted with multiplicity, is equal to $n\,\chi\big(g(\Delta)\big) - \chi(\Delta) = n \cdot 1 - 1$. (Here χ is the Euler characteristic.) Similarly, applying this formula to $g^{-1}(\Delta) \to \Delta$, we see that the number of critical points in $g^{-1}(\Delta)$ is equal to $n\,\chi(g^{-1}(\Delta) - \chi(\Delta)$. Thus all of the $n - 1$ critical points are contained in $g^{-1}(\Delta)$ if and only if $\chi\big(g^{-1}(\Delta)\big) = 1$, so that $g^{-1}(\Delta)$ consists of a single topological disk. Similarly, it follows by induction that all of the critical points are contained in $g^{-k}(\Delta)$ if and only if $g^{-k}(\Delta)$ is a topological disk. If every $g^{-k}(\Delta)$ is a disk, then it follows that the set $K = \bigcap g^{-k}(\Delta)$ is connected. On the other hand, if some $g^{-k}(\Delta)$ consists of two or more disks, then each one of these disks must contain a point of K, which is therefore disconnected.

To complete the proof, we must allow for the possibility that $\partial\Delta$ may contain some post-critical point of g. Clearly there can be at most $d - 1$ post-critical points in the annular region $g(\Delta) \smallsetminus \Delta$. Hence we can choose a disk $\Delta_1 \subset g(\Delta)$ whose boundary avoids these post-critical points. If Δ_1 contains Δ in its interior, and also contains all critical values $g(\omega_i)$ in its interior, then it is easy to check that the pair $\big(g\,,\, g^{-1}(\Delta_1)\big)$ is a polynomial-like map of the same degree, and with the same filled Julia set, but with no post-critical points in the disk boundary. The proof then proceeds as above. $\square$

Remark. Douady and Hubbard prove much sharper statements: If $(g\,,\,\Delta)$ is polynomial-like of degree $\delta \geq 2$, with $K(g\,,\,\Delta)$ connected, then there exists a polynomial map ψ of degree δ so that ψ on some neighborhood of $K(\psi)$ is quasi-conformally conjugate to g on a neighborhood of $K(g\,,\,\Delta)$. Furthermore, this quasi-conformal conjugacy can be chosen so as to satisfy the Cauchy-Riemann equations (in an appropriate sense) on the compact set $K(\psi)$. The polynomial map ψ is then uniquely determined up to affine conjugacy. In the case $\delta = 2$, one has the further statement that ψ depends continuously on $(g\,,\,\Delta)$.

Now let us return to the situation of Theorem 2.4.

Lemma 2.6. *If the critical tableau is periodic of period $p \geq 1$, then for any critical puzzle piece $P_d(c_0)$ with d sufficiently large, the pair $\big(f^{\circ p}\,,\,P_d(c_0)\big)$ is polynomial-like of degree two. Furthermore, the critical orbit*

$$c_0 \;\mapsto\; c_p \;\mapsto\; c_{2p} \;\mapsto\; \cdots$$

under $f^{\circ p}$ is completely contained in $P_d(c_0)$, so that the filled Julia set $K\big(f^{\circ p}\,,\,P_r(c_0)\big)$ is connected. In fact $K\big(f^{\circ p}\,,\,P_d(c_0)\big)$ is equal to the intersection of the critical puzzle pieces $\bigcap_k P_k(c_0)$, and hence is precisely equal to the connected component of $K(f)$ which contains c_0 .

This is proved by a straightforward tableau argument. Details will be left to the reader. □

In this way, Branner and Hubbard show that each non-trivial component of $K(f)$ is homeomorphic to an appropriate quadratic Julia set. As examples, Figures 17 and 18 illustrate the case $p = d = 1$.

3. An Infinitely Renormalizable Non Locally Connected Julia Set

This section describes an unpublished example of Douady and Hubbard. It begins with an outline, with few proofs, of results from Douady and Hubbard [DH1], [DH2], [DH3], [D3]. (See also [M1], [D5], [Sch], [M3].)

Hyperbolic components in the Mandelbrot set

Let $f_c(z) \;=\; z^2 + c$. By definition, the *Mandelbrot set* M is the compact set consisting of all parameter values $c \in \mathbb{C}$ such that

$$\text{the Julia set } J(f_c) \text{ is connected} \qquad \Longleftrightarrow \qquad 0 \text{ has bounded orbit .}$$

However, it is often convenient to identify M with the corresponding set of polynomials f_c. A map $f_c \in M$ is hyperbolic (on its Julia set) if and only if it has a necessarily unique attracting periodic orbit. The hyperbolic maps in M form an open subset of the plane, and each connected component H in this open set is called a ***hyperbolic component*** in M. Let p be the period of the attracting orbit, and let $\lambda_p = \lambda_p(f_c) \in \mathbb{D}$ be its multiplier. The basic facts about hyperbolic components in M are as follows:

> (1) *Any two maps in the same hyperbolic component $H \subset M$ have attracting orbits of the same period p.* We call $p = p_H$ the ***period*** of H.

> (2) *Each hyperbolic component H is conformally isomorphic to the open unit disk $\mathbb{D}$ under the correspondence $f_c \mapsto \lambda_p(f_c)$. In fact this correspondence extends uniquely to a homeomorphism between the closure $\overline{H}$ and the closed unit disk $\overline{\mathbb{D}}$.*

In particular, each H has a unique ***center point*** c_H which maps to $\lambda_p(c_H) = 0$, and each boundary ∂H contains a unique ***root point*** $r_H \in \partial H$ which maps to $\lambda_p(r_H) = 1$. If the map f_c has a superattracting periodic orbit, then evidently c is the center point for one and only one hyperbolic component.

> (3) *Similarly, if f_r has a parabolic periodic orbit, then r is the root point for one and only one hyperbolic component H. If the period of this orbit is p and the multiplier is $\lambda_p = \exp(2\pi i m/p')$ then the period of H is $p\,p'$.*

By definition, the ***principal hyperbolic component*** $H_\heartsuit$ is the set of f_c having an attracting fixed point. If the multiplier is $\lambda_1 \in \mathbb{D}$ then a brief computation shows that $c = \lambda_1(2 - \lambda_1)/4$, hence

$$\lambda_1(c) \;=\; 1 - \sqrt{1 - 4c}\,,$$

taking that branch of the square root which lies in the right half plane. The boundary $\partial H_\heartsuit$ is the ***cardioid***, consisting of all points which have the form $c = e^{i\theta}(2 - e^{i\theta})/4$, so that $\lambda_1(c) = e^{i\theta}$.

Satellites. Given any hyperbolic component H of period $p \ge 1$ and any root of unity $e^{2\pi i m/p'} \ne 1$, it follows from (2) that there is a unique point $r \in \partial H$ with $\lambda_p(r) = e^{2\pi i m/p'}$. According to (3), this r is the root point of a new hyperbolic component H' of period $p\,p'$. We say that H' is a ***satellite***, which is ***attached*** to H at ***internal angle*** m/p'. In the special case where H is the principal hyperbolic component $H_\heartsuit$, we will use the notation $H' = H(m/p')$ for this satellite, and in general we will use the notation $H' = H * H(m/p')$. (Compare the discussion of "tuning" below.)

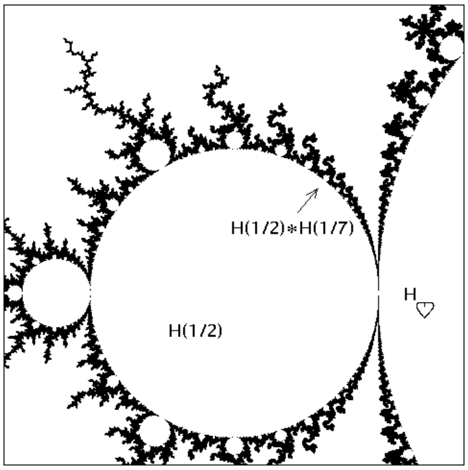


Figure 21. The period two component $H(1/2)$ in the Mandelbrot set, with an arrow pointing to its satellite $H(1/2) * H(1/7)$.

As an example, taking $m/p' = 1/2$, the point $r = -3/4$ is the root point of the period two component $H(1/2)$, consisting of all c with $|c+1| < 1/4$.

External rays. We consider not only external rays for the Julia set in the z-plane (= *dynamic plane*), but also external rays for the Mandelbrot set in the c-plane (= *parameter plane*). Let $H \subset M$ be a hyperbolic component of period $p > 1$. Then exactly two external rays in $\mathbb{C} \smallsetminus M$ land at the root point r_H . Let $0 < a < b < 1$ be their angles. These angles have period exactly p under doubling, and hence can be expressed as fractions of the form $n/(2^p - 1)$. As an example, for the satellite component $H(1/p)$ we have $a = 1/(2^p - 1)$ and $b = 2/(2^p - 1)$.

For any $c \in H \cup \{r_H\}$ the corresponding external rays $R_a(c)$ and $R_b(c)$ in $\mathbb{C} \smallsetminus J(f_c)$ land at a periodic point which can be described as the "root point" of the Fatou component containing c . See Figure 22 for a schematic picture of the way these external rays are arranged in the parameter plane and in the dynamic plane. In particular

$$0 \;<\; a/2 \;<\; b/2 \quad\leq\quad a < b \quad\leq\quad 1 + a/2 \;<\; (1+b)/2 \;<\; 1 \;,$$

where the $a/2$-ray and the $b/2$-ray always land at distinct points: exactly one of these two rays is periodic.

Renormalization and Tuning. Recall that a map $f_c \in M$ is *simply renormalizable* if there is an integer $p \geq 2$ called the renormalization period, and a closed topological disk Δ in the z-plane, centrally symmetric about the origin, so that:

(1) $f^{\circ p-1}$ maps the image $f(\Delta)$ by a conformal isomorphism onto a disk $f^{\circ p}(\Delta)$ which contains Δ in its interior,

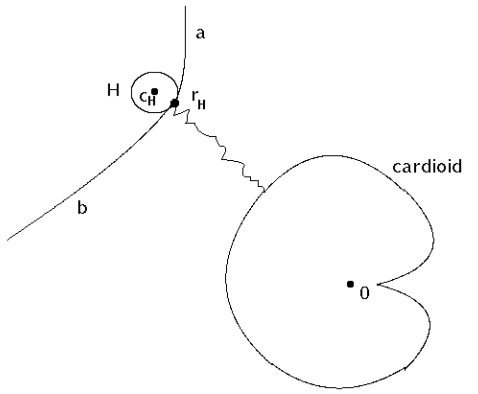
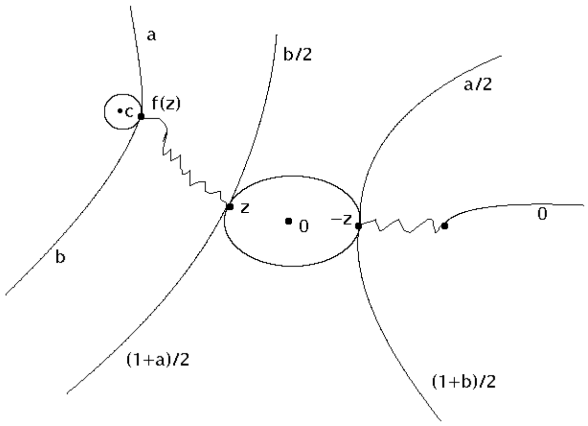


Figure 22. Schematic picture of external rays for a hyperbolic component $H \subset M$ in the c-plane,



and a corresponding picture in the dynamic plane.

(2) the entire orbit of 0 under $f^{\circ p}$ is contained in Δ , and

(3) the images $f^{\circ i}(K_p)$ and $f^{\circ j}(K_p)$ do not disconnect each other for $0 \le i < j < p$. Here K_p denotes the small filled Julia set consisting of points of Δ whose orbits under $f^{\circ p}$ remain in Δ forever.

The following facts are surely true (compare [M1], [Mc2], [D5], [M3]), although I am not aware of any published proof.

The set of all $f_c \in M$ which are simply renormalizable of period p consists of a finite number of small copies of M (sometimes with the root point deleted). Each of these small copies contains a unique hyperbolic component of period p . Conversely, each

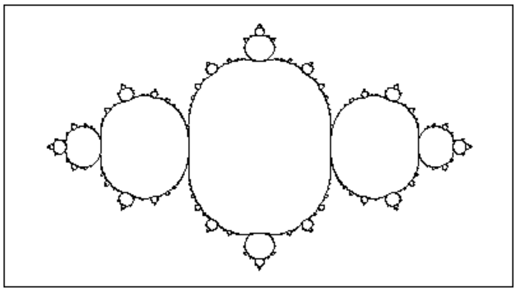
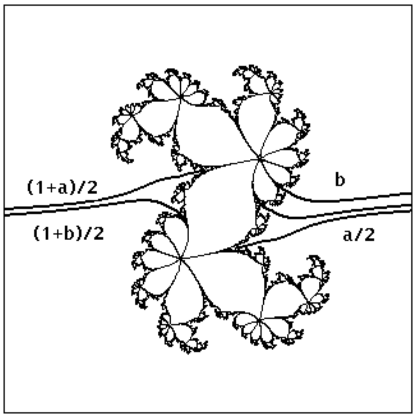
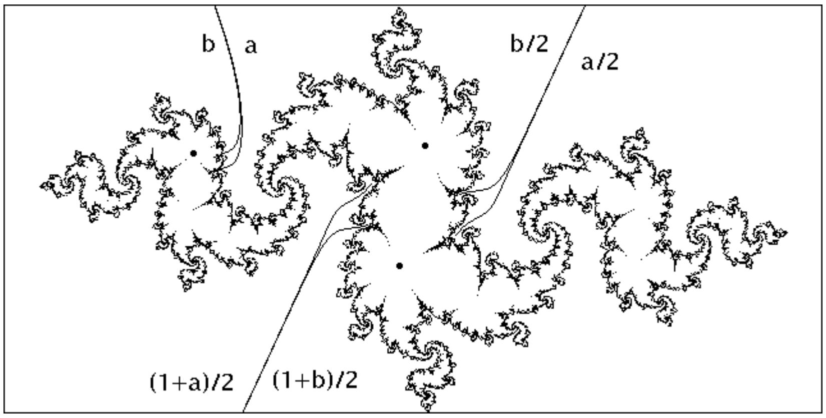


Figure 23. Julia set for the root point of $H(1/2)$,



for the root point of $H(1/7)$ *(here the middle ray* $a = b/2$ *has not been labeled),*



and for the root point of $H(1/2) * H(1/7)$.

hyperbolic component H of period $p \geq 2$ determines a small copy of M which can be described as the image of an associated mapping $c \mapsto H * c$, which embeds M homeomorphically onto a proper subset $H * M \subset M$. The elements of this form (with the possible exception of the root point r_H), are precisely the elements of M which are simply renormalizable of period p.

This embedding $c \mapsto H{*}c$ maps each hyperbolic component H' of period $p' \geq 1$ conformally onto a hyperbolic component $H{*}H'$ of period pp' in such a way that the multiplier $\lambda_{pp'}(H{*}c)$ is equal to $\lambda_{p'}(c)$. In particular, this embedding maps center points to center points and root points to root points. It carries the principal hyperbolic component $H_\heartsuit$ onto H itself, and maps the satellite $H(m/p')$ of $H_\heartsuit$ onto the satellite $H * H(m/p')$ of H. This $*$-product operation between hyperbolic components is associative, and has the principal hyperbolic component $H_\heartsuit$ as two-sided identity element. Thus it makes the collection of hyperbolic components into a monoid (=associative semigroup with identity) which operates as a semigroup of embeddings of M into itself. This monoid is free non-commutative.

Intuitively, the Julia set for $H * c'$ is obtained from the Julia set of a map f_c belonging to the hyperbolic component H by replacing each bounded component of $\mathbb{C} \setminus J(f_c)$ by a copy of $J(f_{c'})$ and defining the dynamics appropriately so that only the copy which is pasted in place of the critical component is mapped non-homeomorphically. The result is described as f_c *tuned by* $f_{c'}$.

In terms of external angles, let $a < b$ be the two external angles for the root point of H. These angles have periodic binary expansions of the form $.a_1 \cdots a_p\, a_1 \cdots a_p \cdots$ and $.b_1 \cdots b_p\, b_1 \cdots b_p \cdots$, both with period exactly equal to p. **Assertion:** If the point $c' \in \partial M$ is the landing point of a ray with angle $t \in \mathbb{R}/\mathbb{Z}$, then the image $H * c'$ is the landing point of a ray whose binary expansion can be obtained by inserting the p-tuple $a_1 \cdots a_p$ in place of each zero in the binary expansion of t and the p-tuple $b_1 \cdots b_p$ in place of each one. (See [D3].)

The example

Consider a sequence of integers $1 < p_1 < p_2 < \cdots$.

Theorem 3.1. *If this sequence diverges to infinity sufficiently rapidly, then the sequence of subsets $H(1/p_1) * H(1/p_2) * \cdots * H(1/p_k) {*} M$ intersects in a single point $\omega \in M$ with the property that the Julia set $J(f_\omega)$ is not locally connected.*

The proof begins as follows. It will be convenient to use the abbreviation $r(k)$ for the root point $r_{H(1/p_k)}$ on the cardioid. Start with any $p_1 > 1$ and consider the embedding $c \mapsto H(1/p_1) * c$ from M into itself, which carries the period 1 root point $r_\heartsuit = 1/4$ to the root point $r(1) = r_{H(1/p_1)}$. Since this embedding is continuous, we can place $H(1/p_1) * c$ as close as we like to $r(1)$ by choosing c close to $1/4$. In particular, if p_2 is sufficiently large, then the entire $1/p_2$-limb will be close to $1/4$ by the Yoccoz inequality, hence every point $c \in H(1/p_1) * H(1/p_2) * M$ will certainly be close to $r(1)$. Under such a small perturbation, note that the parabolic fixed point of $f_{r(1)}$ splits up into a repelling fixed point for f_c, together with a nearby period p_1 orbit. Thus, by choosing p_2 large, we can place this entire period p_1 orbit into an ϵ-neighborhood of the parabolic fixed point of $f_{r(1)}$.

Similarly, choosing p_3 even larger, for any $c \in H(p_1) * H(p_2) * H(p_3) * M$ we can guarantee that the parabolic period p_1 orbit for $H(1/p_1) * r(2)$ is replaced by a nearby period $p_1 p_2$ orbit. In particular, we can guarantee that this new orbit lies within the $\epsilon + \epsilon/2$ neighborhood of the original parabolic fixed point. Continuing inductively, we can guarantee that all of the new orbits which are constructed lie within the 2ϵ neighborhood of the original fixed point. Furthermore, we can easily guarantee that these successive copies of M have diameter shrinking to zero.

Thus, if ω is the unique point in the intersection, we know that the Julia set $J(f_\omega)$ contains orbits of period $1, p_1, p_1 p_2, \ldots$, all lying within the 2ϵ neighborhood of the original parabolic fixed point. These various periodic points can be described as the landing points of external rays of angles $a_k < b_k$ where $0 < a_1 < a_2 < \cdots < b_2 < b_1$. Furthermore, it is easy to check that the difference $b_k - a_k$ tends rapidly to zero, so that $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k$. On the other hand, as noted in the preceding section, the external rays of angle $a_k/2$ and $b_k/2$ land at different points, which are negatives of each other. Each such landing point is either close to the original parabolic fixed point, or close to its negative. Thus we have a sequence of angles $a_k/2$ and $b_k/2$ tending to a common limit, yet the landing points of the corresponding external rays for $J(f_\omega)$ do not tend to a common limit. According to Caratheodory, this implies that $J(f_\omega)$ is not locally connected. □

More generally, if $H_1, H_2, \ldots$ is any sequence of hyperbolic components which are not centered along the real axis, then it is again true that $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k$. Let $\omega \in M$ be any element of the intersection $\bigcap_k H_1 * H_2 * \cdots * H_k * M$. Whenever the landing points in $J(f_\omega)$ of the rays of angle $a_k/2$ fail to converge to the critical point, it follows that $J(f_\omega)$ is not locally connected. In particular, this is true whenever the sequence of components H_k converges sufficiently rapidly to the root point $r_\heartsuit = 1/4$.

This seems to be the only known obstruction to local connectivity in the infinitely renormalizable case. Thus whenever the landing points of the $a_k/2$ rays do converge to the critical point, we can ask whether $J(f_\omega)$ is in fact locally connected.

For further papers related to local connectivity of Julia sets, see for example [Ki], [Pe2], [PM], [Sø1], [Sø2], [Z].

Appendix: Length-Area-Modulus Inequalities

The most basic length-area inequality is the following. Let $I^2 \subset \mathbb{C}$ be the open unit square consisting of all $z = x + iy$ with $0 < x < 1$ and $0 < y < 1$. By a *conformal metric* on I^2 we mean a metric of the form

$$ds \;=\; \rho(z)|dz|$$

where $z \mapsto \rho(z) > 0$ is any strictly positive continuous real valued function on the open square. In terms of such a metric, the *length* of a smooth curve $\gamma : (a, b) \to I^2$ is defined to be the integral

$$L_\rho(\gamma) \;=\; \int_a^b \rho\big(\gamma(t)\big)|d\gamma(t)| \;,$$

and the *area* of a region $U \subset I^2$ is defined to be

$$\mathbf{area}_\rho(U) \;=\; \int\!\!\int_U \rho(x+iy)^2 dx\, dy \;.$$

In the special case of the Euclidean metric $ds = |dz|$, with $\rho(z)$ identically equal to 1, the subscript ρ will be omitted.

Theorem A.1. *If* $\mathbf{area}_\rho(I^2)$ *(the integral over the entire square) is finite, then for Lebesgue almost every* $y \in (0,1)$ *the length* $L_\rho(\gamma_y)$ *of the horizontal line* $\gamma_y : t \mapsto (t, y)$ *at height* y *is finite. Furthermore, there exists* y *so that*

$$L_\rho(\gamma_y)^2 \;\le\; \mathbf{area}_\rho(I^2) \;. \tag{1}$$

In fact, the set consisting of all $y \in (0,1)$ *for which this inequality is satisfied has positive Lebesgue measure.*

Remark 1. Evidently this inequality is best posible. For in the case of the Euclidean metric $ds = |dz|$ we have

$$L(\gamma_y)^2 \;=\; \mathbf{area}(I^2) \;=\; 1 \;.$$

Remark 2. It is essential here that we use a square, rather than a rectangle. If we consider instead a rectangle R with base Δx and height Δy, then the corresponding inequality would be

$$L_\rho(\gamma_y)^2 \;\le\; \frac{\Delta x}{\Delta y}\mathbf{area}_\rho(R) \tag{2}$$

for a set of y with positive measure.

Proof of A.1. We use the Schwarz inequality

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \;\le\; \left(\int_a^b f(x)^2\,dx\right)\cdot\left(\int_a^b g(x)^2\,dx\right) ,$$

which says (after taking a square root) that the inner product of any two vectors in the Euclidean vector space of square integrable real functions on an interval is less than or equal to the product of their norms. We may as well consider the more general case of a rectangle $R = (0,\,\Delta x)\times(0,\,\Delta y)$. Taking $f(x)\equiv 1$ and $g(x) = \rho(x,y)$ for some fixed y, we obtain

$$\left(\int_0^{\Delta x} \rho(x,y)\,dx\right)^2 \;\le\; \Delta x\int_0^{\Delta x}\rho(x,y)^2\,dx ,$$

or in other words

$$L_\rho(\gamma_y)^2 \;\le\; \Delta x\int_0^{\Delta x}\rho(x,y)^2dx ,$$

for each constant height y. Integrating this inequality over the interval $0<y<\Delta y$ and then dividing by Δy, we get

$$\frac{1}{\Delta y}\int_0^{\Delta y} L_\rho(\gamma_y)^2dy \;\le\; \frac{\Delta x}{\Delta y}\,\mathbf{area}_\rho(A) . \tag{3}$$

In other words, the *average* over all y in the interval $(0,\,\Delta y)$ of $L_\rho(\gamma_y)^2$ is less than or equal to $\frac{\Delta x}{\Delta y}$ $\mathbf{area}_\rho(A)$. Further details of the proof are straightforward. □

Now let us form a cylinder $\mathcal{C}$ of circumference Δx and height Δy by gluing the left and right edges of our rectangle together. More precisely, let $\mathcal{C}$ by the quotient space which is obtained from the infinitely wide strip $0<y<\Delta y$ in the z-plane by identifying each point $z = x+iy$ with its translate $z+\Delta x$. Define the *modulus* $\mathbf{mod}(\mathcal{C})$ of such a cylinder to be the ratio $\Delta y/\Delta x$ of height to circumference. By the *winding number* of a closed curve γ in $\mathcal{C}$ we mean the integer

$$w \;=\; \frac{1}{\Delta x}\oint_\gamma dx .$$

Theorem A.2 (Length-Area Inequality for Cylinders). *For any conformal metric $\rho(z)|dz|$ on the cylinder $\mathcal{C}$ there exists some simple closed curve γ with winding number $+1$ whose* *length* $L_\rho(\gamma) = \oint_\gamma \rho(z)|dz|$ *satisfies the inequality*

$$L_\rho(\gamma)^2 \;\le\; \mathbf{area}_\rho(A)/\mathbf{mod}(A)\;. \tag{4}$$

Furthermore, this result is best possible: If we use the Euclidean metric $|dz|$ *then*

$$L(\gamma)^2 \;\ge\; \mathbf{area}(A)/\mathbf{mod}(A) \tag{5}$$

for ***every*** *such curve* γ *.*

Proof. Just as in the proof of A.1, we find a horizontal curve γ_y with

$$L_\rho(\gamma_y)^2 \;\le\; \frac{\Delta x}{\Delta y}\mathbf{area}_\rho(\mathcal{C}) \;=\; \frac{\mathbf{area}_\rho(\mathcal{C})}{\mathbf{mod}(\mathcal{C})}\;.$$

On the other hand in the Euclidean case, for any closed curve γ of winding number one we have

$$L(\gamma) \;=\; \oint_\gamma |dz| \;\ge\; \oint_\gamma dx \;=\; \Delta x\;,$$

hence $L(\gamma)^2 \;\ge\; (\Delta x)^2 \;=\; \mathbf{area}(\mathcal{C})/\mathbf{mod}(\mathcal{C})$. □

Definitions. A Riemann surface A is said to be an ***annulus*** if it is conformally isomorphic to some cylinder. An embedded annulus $A \subset \mathcal{C}$ is said to be ***essentially embedded*** if it contains a curve which has winding number one around $\mathcal{C}$.

Here is an important consequence of Theorem A.2.

Corollary A.3 (An Area-Modulus Inequality). *Let* $A \subset \mathcal{C}$ *be an essentially embedded annulus in the cylinder* $\mathcal{C}$ *, and suppose that* A *is conformally isomorphic to a cylinder* $\mathcal{C}'$ *. Then*

$$\mathbf{mod}(\mathcal{C}') \;\le\; \frac{\mathbf{area}(A)}{\mathbf{area}(\mathcal{C})}\mathbf{mod}(\mathcal{C})\;. \tag{6}$$

In particular:

$$\mathbf{mod}(\mathcal{C}') \;\le\; \mathbf{mod}(\mathcal{C})\;. \tag{7}$$

Proof. Let $\zeta \mapsto z$ be the embedding of $\mathcal{C}'$ onto $A \subset \mathcal{C}$. The Euclidean metric $|dz|$ on $\mathcal{C}$, restricted to A , pulls back to some conformal metric $\rho(\zeta)|d\zeta|$ on $\mathcal{C}'$, where $\rho(\zeta) = |dz/d\zeta|$. According to A.2, there exists a curve γ' with winding number 1 about $\mathcal{C}'$ whose length satisfies

$$L_\rho(\gamma')^2 \le \mathbf{area}_\rho(\mathcal{C}')/\mathbf{mod}(\mathcal{C}')\;.$$

This length coincides with the Euclidean length $L(\gamma)$ of the corresponding curve γ in $A \subset \mathcal{C}$, and $\mathbf{area}_\rho(\mathcal{C}')$ is equal to the Euclidean area $\mathbf{area}(A)$, so we can write this inequality as

$$L(\gamma)^2 \le \mathbf{area}(A)/\mathbf{mod}(\mathcal{C}')\;.$$

But according to (5) we have

$$\mathbf{area}(\mathcal{C})/\mathbf{mod}(\mathcal{C}) \;\le\; L(\gamma)^2 \;.$$

Combining these two inequalities, we obtain

$$\mathbf{area}(\mathcal{C})/\mathbf{mod}(\mathcal{C}) \;\le\; \mathbf{area}(A)/\mathbf{mod}(\mathcal{C}') \;,$$

which is equivalent to the required inequality (6). □

Corollary A.4. *The modulus of a cylinder is a well defined conformal invariant.*

Proof. If $\mathcal{C}'$ is conformally isomorphic to $\mathcal{C}$ then (7) asserts that $\mathbf{mod}(\mathcal{C}') \le \mathbf{mod}(\mathcal{C})$, and similarly $\mathbf{mod}(\mathcal{C}) \le \mathbf{mod}(\mathcal{C}')$. □

It follows that the *modulus* of an annulus A can be defined as the modulus of any conformally isomorphic cylinder. Furthermore, if A is essentially embedded in some other annulus A' , then $\mathbf{mod}(A) \le \mathbf{mod}(A')$.

Corollary A.5 (Grötzsch Inequality). *Suppose that $A' \subset A$ and $A'' \subset A$ are two disjoint annuli, each essentially embedded in A . Then*

$$\mathbf{mod}(A') + \mathbf{mod}(A'') \;\le\; \mathbf{mod}(A) \;.$$

Proof. We may assume that A is a cylinder $\mathcal{C}$. According to (6) we have

$$\mathbf{mod}(A') \;\le\; \frac{\mathbf{area}(A')}{\mathbf{area}(\mathcal{C})}\mathbf{mod}(\mathcal{C}) \;, \qquad \mathbf{mod}(A'') \;\le\; \frac{\mathbf{area}(A'')}{\mathbf{area}(\mathcal{C})}\mathbf{mod}(\mathcal{C}) \;.$$

where all areas are Euclidean. Using the inequality

$$\mathbf{area}(A') + \mathbf{area}(A'') \le \mathbf{area}(\mathcal{C}) \;,$$

the conclusion follows. □

Up to this point, we have only considered cylinders or annuli of finite modulus. If we take an arbitrary Riemann surface with free cyclic fundamental group, then it is always conformally isomorphic to some cylinder, providing that we allow also the possibility of a one-sided infinite or two-sided infinite cylinder (that is, either the upper half-plane or the full complex plane modulo the identification $z \equiv z + 1$). By definition, a Riemann surface conformally isomorphic to such an infinite cylinder will be called an *annulus of infinite modulus*.

Now consider the following situation. Let $U \subset \mathbb{C}$ be a bounded simply connected open set, and let $K \subset U$ be a compact subset such that the difference $A = U \smallsetminus K$ is an annulus (which may have finite or infinite modulus).

Corollary A.6. *Suppose that $K \subset U$ as described above. Then K reduces to a single point if and only if the annulus $A = U \smallsetminus K$ has infinite modulus. Furthermore, the diameter of K is bounded by the inequality*

$$4\,\mathbf{diam}(K)^2 \;\le\; \frac{\mathbf{area}(A)}{\mathbf{mod}(A)} \;\le\; \frac{\mathbf{area}(U)}{\mathbf{mod}(A)}\,. \tag{8}$$

Proof. According to A.2, there exists a curve with winding number one about A whose length satisfies $L^2 \le \mathbf{area}(A)/\mathbf{mod}(A)$. Since K is enclosed within this curve, it follows easily that $\mathbf{diam}(K) \le L/2$, and the inequality (8) follows. Conversely, if K is a single point then using (7) we see easily that $\mathbf{mod}(A) = \infty$. □

Corollary A.7. (Branner-Hubbard). *Let $K_1 \supset K_2 \supset K_3 \supset \cdots$ be compact subsets of $\mathbb{C}$ with each K_{n+1} contained in the interior of K_n. Suppose further that each interior K_n^o is simply connected, and that each difference $A_n = K_n^o \smallsetminus K_{n+1}$ is an annulus. If $\sum_1^\infty \mathbf{mod}(A_n)$ is infinite, then the intersection $\bigcap K_n$ reduces to a single point.*

Proof. It follows inductively from the Grötzsch inequality that the modulus $\mathbf{mod}(K_1^o \smallsetminus K_n)$ tends to infinity as $n \to \infty$. Hence by (7) and A.7 the intersection of the K_n is a point. □

Corollary A.8 (McMullen Inequality). *Again suppose that $K \subset U \subset \mathbb{C}$, and that $A = U \smallsetminus K$ is an annulus. Then*

$$\mathbf{area}(K) \;\le\; \mathbf{area}(U)/e^{4\pi\,\mathbf{mod}(A)}\,. \tag{9}$$

Proof. (Compare [BH, II].) We will first prove the weaker inequality

$$\mathbf{area}(K) \;\le\; \frac{\mathbf{area}(U)}{1 + 4\pi\,\mathbf{mod}(A)}\,. \tag{10}$$

The *isoperimetric inequality* asserts that the area enclosed by a plane curve of length L is at most $L^2/(4\pi)$, with equality if and only if the curve is a round circle. (See for example [CR].) Combining this with the proof of A.6, we see that

$$\mathbf{area}(K) \;\le\; \frac{L^2}{4\pi} \;\le\; \frac{\mathbf{area}(A)}{4\pi\,\mathbf{mod}(A)}\,.$$

Writing this inequality as $4\pi\,\mathbf{mod}(A) \le \mathbf{area}(A)/\mathbf{area}(K)$ and adding $+1$ to both sides we obtain the completely equivalent inequality $1+4\pi\,\mathbf{mod}(A) \le \mathbf{area}(U)/\mathbf{area}(K)$. This in turn is equivalent to (10).

To sharpen this inequality, we proceed as follows. Cut the annulus A up into n concentric annuli A_i, each of modulus equal to $\mathbf{mod}(A)/n$. Let K_i be the bounded component of the complement of A_i, and assume that these annuli are nested so that $A_i \cup K_i = K^o_{i+1}$ with $K_1 = K$. Let $K^o_{n+1} = A \cup K = U$. Then

$$\mathbf{area}(K_{i+1})/\mathbf{area}(K_i) \;\geq\; 1 + 4\pi\,\mathbf{mod}(A)/n$$

by (10), hence

$$\mathbf{area}(U)/\mathbf{area}(K) \;\geq\; (\,1\,+\,4\pi\,\mathbf{mod}(A)/n\,)^n\,,$$

where the right hand side converges to $e^{4\pi\,\mathbf{mod}(A)}$ as $n \to \infty$. □

As a final illustration of modulus-area inequalities, consider a *flat torus* $\mathbf{T} = \mathbb{C}/\Lambda$. Here $\Lambda \subset \mathbb{C}$ is to be a 2-dimensional *lattice*, that is an additive subgroup of the complex numbers, spanned by two elements λ_1 and λ_2 where $\lambda_1/\lambda_2 \notin \mathbb{R}$. Let $A \subset \mathbf{T}$ be an embedded annulus.

By the "*winding number*" of A in $\mathbf{T}$ we will mean the lattice element $w \in \Lambda$ which is constructed as follows. Under the universal covering map $\mathbb{C} \to \mathbf{T}$, the central curve of A lifts to a curve segment which joins some point $z_0 \in \mathbb{C}$ to a translate $z_0 + w$ by the required lattice element. We say that $A \subset \mathbf{T}$ is an *essentially embedded annulus* if $w \neq 0$.

Corollary A.9 (Bers Inequality). *If the annulus A is embedded in the flat torus $\mathbf{T} = \mathbb{C}/\Lambda$ with winding number $w \in \Lambda$, then*

$$\mathbf{mod}(A) \;\leq\; \frac{\mathbf{area}(T)}{|w|^2}\,. \tag{11}$$

Roughly speaking, if A winds many times around the torus, so that $|w|$ is large, then A must be very skinny. A slightly sharper version of this inequality is given in Problem A-3 below.

Proof. Choose a cylinder $\mathcal{C}'$ which is conformally isomorphic to A. The Euclidean metric $|dz|$ on $A \subset \mathbf{T}$ corresponds to some metric $\rho(\zeta)|d\zeta|$ on $\mathcal{C}'$, with

$$\mathbf{area}_\rho(\mathcal{C}') \;=\; \mathbf{area}(A)\,.$$

By A.2 we can choose a curve γ' of winding number one on $\mathcal{C}'$, or a corresponding curve γ on $A \subset \mathbf{T}$, with

$$L(\gamma)^2 \;=\; L_\rho(\gamma')^2 \;\leq\; \frac{\mathbf{area}_\rho(\mathcal{C}')}{\mathbf{mod}(\mathcal{C}')} \;=\; \frac{\mathbf{area}(A)}{\mathbf{mod}(A)} \;\leq\; \frac{\mathbf{area}(T)}{\mathbf{mod}(A)}\,.$$

Now if we lift γ to the universal covering space $\mathbb{C}$ then it will join some point z_0 to $z_0 + w$. Hence its Euclidean length $L(\gamma)$ must satisfy $L(\gamma) \geq |w|$. Thus

$$|w|^2 \;\leq\; \frac{\mathbf{area}(T)}{\mathbf{mod}(A)}\,,$$

which is equivalent to the required inequality (11). $\square$

Concluding Problems:

Problem A-1. In the situation of Theorem A.1, show that more than half of the horizontal curves γ_y have length $L_\rho(\gamma_y) \le \sqrt{2\,\mathbf{area}_\rho(I^2)}$. (Here "more than half" is to be interpreted in the sense of Lebesgue measure.)

Problem A-2. Show that the converse to A.7 is false: A nest $K_1 \supset K_2 \supset K_3 \supset \cdots$ of compact subsets of $\mathbb{C}$ may intersect in a point even if $\sum_1^\infty \mathbf{mod}(K_n^o \smallsetminus K_{n+1})$ is finite. (For example, do this by showing that a closed disk $\overline{\mathbb{D}}'$ of radius $1/2$ can be embedded in the open unit disk $\mathbb{D}$ so that the complementary annulus $A = \mathbb{D} \smallsetminus \overline{\mathbb{D}}'$ has modulus arbitrarily close to zero.)

Problem A-3 (Sharper Bers Inequality). If the flat torus $\mathbf{T} = \mathbb{C}/\Lambda$ contains several disjoint annuli A_i , all with the same "winding number" $w \in \Lambda$, show that

$$\sum \mathbf{mod}(A_i) \;\le\; \mathbf{area}(\mathbf{T})/|w|^2\ .$$

If two essentially embedded annuli are disjoint, show that they necessarily have the same winding number.

References

[A] L. Ahlfors, "Conformal Invariants", McGraw-Hill 1973.

[Be] A. Beardon, "Iteration of Rational Functions", Grad. Texts Math. **132**, Springer 1991.

[Bl] P. Blanchard, Disconnected Julia sets, pp. 181-201 of "Chaotic Dynamics and Fractals", ed. Barnsley and Demko, Academic Press 1986.

[BDK] P. Blanchard, R. Devaney and L. Keen, The dynamics of complex polynomials and automorphisms of the shift, Inv. Math **104** (1991) 545-580.

[Br1] B. Branner, The parameter space for complex cubic polynomials, pp. 169-179 of "Chaotic Dynamics and Fractals", ed. Barnsley and Demko, Academic Press 1986.

[Br2] B. Branner, The Mandelbrot set, pp. 75-105 of "Chaos and Fractals", edit. Devaney and Keen, Proc. Symp. Applied Math. **39**, Amer. Math. Soc. 1989.

[BH1] B. Branner and J. H. Hubbard, The iteration of cubic polynomials, Part I: the global topology of parameter space, Acta Math. **160** (1988), 143-206;

[BH2] B. Branner and J. H. Hubbard, The iteration of cubic polynomials, Part II: patterns and parapatterns, Acta Math. **169** (1992), 229-325.

[CG] L. Carleson and T. Gamelin, "Complex Dynamics", Springer 1993.

[CR] R. Courant and H. Robbins, "What is Mathematics?", Oxford U. Press 1941.

[D1] A. Douady, Systèmes dynamiques holomorphes, Séminar Bourbaki, 35^e année 1982-83, n^o 599; Astérisque **105-106** (1983) 39-63.

[D2] A. Douady, Julia sets and the Mandelbrot set, pp. 161-173 of "The Beauty of Fractals", edit. Peitgen and Richter, Springer 1986.

[D3] A. Douady, Algorithms for computing angles in the Mandelbrot set, pp. 155-168 of "Chaotic Dynamics and Fractals", ed. Barnsley & Demko, Acad. Press 1986.

[D4] A. Douady, Disques de Siegel et anneaux de Hermann, Séminar Bourbaki, 39^e année 1986-87, n^o 677; Astérisque **152-153** (1987-88) 151-172.

[D5] A. Douady, Chirurgie sur les applications holomorphes, Proc. Int. Cong. Math. Berkeley, A.M.S. (1987) 724-738.

[DH1] A. Douady and J. H. Hubbard, Itération des polynômes quadratiques complexes, C. R. Acad. Sci. Paris **294** (1982) 123-126.

[DH2] A. Douady and J. H. Hubbard, Étude dynamique des polynômes complexes I & II, Publ. Math. Orsay (1984-85).

[DH3] A. Douady and J. H. Hubbard, On the dynamics of polynomial-like mappings, Ann. Sci. Ec. Norm. Sup. (Paris) **18** (1985), 287-343.

[EL] A. Eremenko and M. Lyubich, The dynamics of analytic transformations, Leningr. Math. J. **1** (1990) 563-634.

[F] D. Faught, Local connectivity in a family of cubic polynomials, thesis, Cornell 1992.

[GM] L. Goldberg and J. Milnor, Fixed point portraits of polynomial maps II, Ann. Sci. Éc. Norm. Sup. **26** (1993), 51–98.

[He] M. Herman, Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of C^n near a fixed point, pp. 138-198 of Proc 8^{th} Int. Cong. Math. Phys., World Sci. 1986.

[Hu1] J. H. Hubbard, Puzzles and quadratic tableaux (according to Yoccoz), preprint 1990.

[Hu2] J. H. Hubbard, Parapuzzles and local connectivity in the parameter plane (according to Yoccoz), preprint 1990.

[Hu3] J. H. Hubbard, Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz, pp. 467-511 of "Topological Methods in Modern Mathematics", ed. Goldberg and Phillips, Publish or Perish 1993.

[Ka] J. Kahn, Holomorphic Removability of Julia Sets, Stony Brook I.M.S. Preprint 1998#11.

[Ki] J. Kiwi, Non-accessible Critical Points of Cremer Polynomials, Stony

Brook I.M.S. Preprint 1995#2.

[L1] M. Lyubich, The dynamics of rational transforms: the topological picture, Russian Math. Surveys **41:4** (1986), 43-117.

[L2] M. Lyubich, Milnor's attractors, persistent recurrence and renormalization, pp. 513-541 of "Topological Methods in Modern Mathematics", ed. Goldberg and Phillips, Publish or Perish 1993.

[L3] M. Lyubich, Renormalization ideas in conformal dynamics. Current developments in mathematics, 1995 (Cambridge, MA), 155–190, Internat. Press, Cambridge, MA, 1994.

[L4] M. Lyubich, Dynamics of quadratic polynomials. I, II. Acta Math. **178** (1997) 185–247, 247–297.

[LM] M. Lyubich and J. Milnor, The Fibonacci unimodal map, J. Amer. Math. Soc. **6** (1993), 425–457.

[Ma], P. Makienko, Totally disconnected Julia sets, MSRI preprint 042-95.

[Mc1] C. McMullen, Automorphisms of rational maps, pp. 31-60 of "Holomorphic Functions and Moduli", edit. Drasin et al., Springer 1988.

[Mc2] C. McMullen, "Complex Dynamics and Renormalization," Ann. Math. Studies **135**, Princeton U. Press, 1994

[Mc3] C. McMullen, "Renormalization and 3-manifolds which fiber over the circle", Ann. Math. Studies **142**, Princeton U. Press, 1996.

[M1] J. Milnor, Self-similarity and hairiness in the Mandelbrot set, pp. 211-257 of "Computers in Geometry and Topology", edit. Tangora, Lect. Notes Pure Appl. Math. **114**, Dekker 1989

[M2] J. Milnor, "Dynamics in One Complex Variable, Introductory Lectures", Vieweg 1999, to appear.

[M3] J. Milnor, Periodic orbits, external rays and the Mandelbrot set: an expository account, Stony Brook I.M.S. Preprint 1999#3; Astérisque to appear.

[Pe1] C. L. Petersen, On the Pommerenke-Levin-Yoccoz inequality, Ergodic Theory Dynamical Systems **13** (1993) 785–806.

[Pe2] C. L. Petersen, Local connectivity of some Julia sets containing a circle with an irrational rotation. Acta Math. 177 (1996), 163–224.

[Pe3] C. L. Petersen, Puzzles and Siegel disks, "Progress in Holomorphic Dynamics", pp. 50–85, Pitman Res. Notes Math. Ser., 387, Longman, Harlow, 1998.

[PM] R. Perez-Marco, Fixed points and circle maps, Acta Math. **179** (1997), 243-294.

[Pr] F. Przytycki, Iterations of rational functions: which hyperbolic components contain polynomials? Fund. Math. **149** (1996), 95-118.

[RS] J. Riedl and D. Schleicher, On Crossed Renormalization of Quadratic Polynomials, "Problems on Complex Dynamical Systems" (Proceedings of conference on holomorphic dynamics), Kyoto 1997.

[Sch] D. Schleicher, Rational parameter rays of the Mandelbrot set, Stony Brook I.M.S. Preprint 1997#13.

[Sh] M. Shishikura, The Hausdorff Dimension of the Boundary of the Mandelbrot Set and Julia Sets, Ann. of Math. **147** (1998), 225-267.

[Sø1] D. E. K. Sørensen, Accumulation theorems for quadratic polynomials, Ergodic Theory Dynam. Systems **16** (1996), 555–590.

[Sø2] D. E. K. Sørensen, Describing quadratic Cremer point polynomials by parabolic perturbations. Ergodic Theory Dynam. Systems **18** (1998), 739–758.

[St] N. Steinmetz, "Rational Iteration: Complex Analytic Dynamical Systems", de Gruyter 1993.

[Su] D. Sullivan, Conformal dynamical systems, pp. 725-752 of "Geometric Dynamics", edit. Palis, Lecture Notes Math. **1007**, Springer 1983.

[Z] S. Zakeri, Biaccessibility in Quadratic Julia Sets II: The Siegel and Cremer Cases, Stony Brook I.M.S. Preprint 1998#1.

Institute for Mathematical Sciences,
State University of New York at
Stony Brook NY 11794-3660, U.S.A.
e-mail: jack@math.sunysb.edu