On Lattès Maps

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Dedicated to Bodil Branner.

Abstract. An exposition of the 1918 paper of Lattès, together with its historical antecedents, and its modern formulations and applications.

§**1. The Lattès paper.** In 1918, some months before his death of typhoid fever, Samuel Lattès published a brief paper describing an extremely interesting class of rational maps. Similar examples had been described by Schröder almost fifty years earlier (see §6), but Lattès' name has become firmly attached to these maps, which play a basic role as exceptional examples in the holomorphic dynamics literature.

His starting point was the "*Poincaré function*" $\theta : \mathbb{C} \to \widehat{\mathbb{C}}$ associated with a repelling fixed point $z_0 = f(z_0)$ of a rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. This can be described as the inverse of the Kœnigs linearization around z_0 , extended to a globally defined meromorphic function.[1] Assuming for convenience that $z_0 \neq \infty$, it is characterized by the identity

$$f(\theta(t)) \;=\; \theta(\mu\, t)$$

for all complex numbers t , with $\theta(0) = z_0$, normalized by the condition that $\theta'(0) = 1$. Here $\mu = f'(z_0)$ is the *multiplier* at z_0 , with $|\mu| > 1$. This Poincaré function can be computed explicitly by the formula

$$\theta(t) \;=\; \lim_{n\to\infty} f^{\circ n}\Big(z_0 + t/\mu^n\Big) \;.$$

Its image $\theta(\mathbb{C}) \subset \widehat{\mathbb{C}}$ is equal to the Riemann sphere $\widehat{\mathbb{C}}$ with at most two points removed. In practice, we will always assume that f has degree at least two. The complement $\widehat{\mathbb{C}} \smallsetminus \theta(\mathbb{C})$ is then precisely equal to the *exceptional set* $\mathcal{E}_f$, consisting of all points with finite grand orbit under f .

In general this Poincaré function θ has very complicated behavior. In particular, the Poincaré functions associated with different fixed points or periodic points are usually quite incompatible. However, Lattès pointed out that in special cases θ will be periodic or doubly periodic, and will give rise to a simultaneous linearization for all of the periodic points of f . (For a more precise statement, see the proof of 3.9 below.)

[1] Compare [La], [P], [K]. For general background material, see for example [M3] or [BM].

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We will expand on this idea in the following sections. Section 2 will introduce rational maps which are *finite quotients of affine maps*. (These are more commonly described in the literature as rational maps with *flat orbifold metric*—see §4.) They can be classified into *power maps*, *Chebyshev maps*, and *Lattès maps* according as the Julia set is a circle, a line or circle segment, or the entire Riemann sphere. These maps will be studied in Sections 3 through 5, concentrating on the Lattès case. Section 6 will describe the history of these ideas before Lattès; and §7 will describe some of the developments since his time. Finally, §8 will describe a number of concrete examples.

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§**2. Finite Quotients of Affine Maps.** It will be convenient to make a very mild generalization of the Lattès construction, replacing the linear map $t \mapsto \mu t$ of his construction by an affine map $t \mapsto a t + b$. Let Λ be a discrete additive subgroup of the complex numbers $\mathbb{C}$. In the cases of interest, this subgroup will have rank either one or two, so that the quotient surface $\mathbb{C}/\Lambda$ is either a cylinder $\mathcal{C}$ or a torus $\mathcal{T}$.

Definition 2.1. A rational map f of degree two or more will be called a *finite quotient of an affine map* if there is a flat surface $\mathbb{C}/\Lambda$, an affine map $L(t) = a t + b$ from $\mathbb{C}/\Lambda$ to itself, and a finite-to-one holomorphic map $\Theta : \mathbb{C}/\Lambda \to \widehat{\mathbb{C}} \smallsetminus \mathcal{E}_f$ which satisfies the semiconjugacy relation $f \circ \Theta = \Theta \circ L$. Thus the following diagram must commute:

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{L} & \mathbb{C}/\Lambda \\ \Theta \downarrow & & \Theta \downarrow \\ \widehat{\mathbb{C}} \smallsetminus \mathcal{E}_f & \xrightarrow{f} & \widehat{\mathbb{C}} \smallsetminus \mathcal{E}_f\,. \end{array} \qquad (1)$$

We can also write $f = \Theta \circ L \circ \Theta^{-1}$. It follows for example that any periodic orbit of L must map to a periodic orbit of f, and conversely that every periodic orbit of f outside of the exceptional set $\mathcal{E}_f$ is the image of a periodic orbit of L. (However, the periods are not necessarily the same.) Here the finite-to-one condition is essential. In fact it follows from Poincaré's construction that any rational map of degree at least two can be thought of as an infinite-to-one quotient of an affine map of $\mathbb{C}$.

These finite quotients of affine maps can be classified very roughly into three types, as follows. The set of postcritical points of f plays an important role in all cases. (Compare Lemma 3.4.)

Power maps. These are the simplest examples. By definition, a rational map will be called a *power map* if it is holomorphically conjugate to a map of the form

$$f_a(z) \quad = \quad z^a$$

where a is an integer. Note that f_a, restricted to the punctured plane $\mathbb{C} \smallsetminus \{0\} = \widehat{\mathbb{C}} \smallsetminus \mathcal{E}_{f_a}$, is conjugate to the linear map $t \mapsto at$ on the cylinder $\mathbb{C}/2\pi\mathbb{Z}$. In fact $f_a(e^{it}) = e^{i\,at}$, where the conjugacy $t \mapsto e^{it}$ maps $\mathbb{C}/2\pi\mathbb{Z}$ diffeomorphically onto $\mathbb{C} \smallsetminus \{0\}$. The degree of f_a is equal to $|a|$, the Julia set $J(f_a)$ is equal to the unit circle, and the exceptional set $\mathcal{E}_{f_a} = \{0\,,\,\infty\}$ consists of the two critical points, which are also the two postcritical points.

Chebyshev maps. These are the next simplest examples. A rational map will be called a *Chebyshev map* if it is conjugate to $\pm Ч_n(z)$ where $Ч_n$ is the degree n *Chebyshev polynomial*, defined by the equation[2]

$$Ч_n(u+u^{-1}) \;=\; u^n+u^{-n}\;.$$

For example:

$$Ч_2(z) \;=\; z^2-2\;, \qquad Ч_3(z) \;=\; z^3-3z\;, \quad Ч_4(z) \;=\; z^4-4z^2+2\;, \quad \cdots\;.$$

We will see in §3.8 that power maps and Chebyshev maps are the only finite quotients of affine maps for which the lattice $\Lambda\subset\mathbb{C}$ has rank one.

If we set $u=e^{it}$, then the map $\Theta(t)=u+u^{-1}=2\cos(t)$ is a proper map of degree two from the cylinder $\mathbb{C}/2\pi\mathbb{Z}$ to the plane $\mathbb{C}$, satisfying

$$Ч_n(\Theta(t)) \;=\; \Theta(n\,t) \qquad \text{or equivalently} \qquad Ч_n(2\cos t) \;=\; 2\cos(nt)\;,$$

and also $-Ч_n(\Theta(t)) \;=\; \Theta(n\,t+\pi)$. These identities show that both $Ч_n$ and $-Ч_n$ are finite quotients of affine maps. The Julia set $J(\pm Ч_n)$ is the closed interval $[-2\,,\,2]$, and the exceptional set for $\pm Ч_n$ is the singleton $\{\infty\}$. The postcritical set of $\pm Ч_n$ consists of the three points $\{\pm 2\,,\,\infty\}$. In fact, if $2\cos(t)$ is a finite critical point of $Ч_n$ then by differentiating the equation $Ч_n(2\cos\,t) \;=\; 2\cos(n\,t)$ we see that $\sin(n\,t)\;=\;0$ and hence that $2\cos(n\,t)=\pm 2$.

Note: If n is even, the equation $-Ч_n(z)=Ч_n(kz)/k$ with $k=-1$ shows that $-Ч_n$ is holomorphically conjugate to $Ч_n$. However, for n odd the map $z\mapsto -Ч_n(z)$ has a postcritical orbit $\{\pm 2\}$ of period two, and hence cannot be conjugate to $z\mapsto Ч_n(z)$ which has only postcritical fixed points.

Lattès maps. In the remaining case where the lattice $\Lambda\subset\mathbb{C}$ has rank two so that the quotient $\mathcal{T}=\mathbb{C}/\Lambda$ is a torus, the map $f=\Theta\circ L\circ\Theta^{-1}$ will be called a *Lattès map*. Here L is to be an affine self-map of the torus, and Θ is to be a holomorphic map from $\mathcal{T}$ to the Riemann sphere $\widehat{\mathbb{C}}$. These are the most interesting examples, and exhibit rather varied behavior. Thus we can distinguish between *flexible* Lattès maps which admit smooth deformations, and rigid Lattès maps which do not. (See 5.5 and 5.6, as well as §7 and 8.3.) Another important distinction is between the Lattès maps with three postcritical points, associated with triangle groups acting on the plane, and those with four postcritical points. (See §4.)

For any Lattès map f, since Θ is necessarily onto, there are no exceptional points. Furthermore, since periodic points of L are dense on the torus it follows that periodic points of f are dense on the Riemann sphere. Thus the Julia set $J(f)$ must be the entire sphere.

§3 Cyclic Group Actions on $\mathbb{C}/\Lambda$.

The following result provides a more explicit description of all of the possible Lattès maps, as defined in §2.

Theorem 3.1. *A rational map is Lattès if and only if it is conformally conjugate to a map of the form* $L/G_n\,:\,\mathcal{T}/G_n\;\to\;\mathcal{T}/G_n$ *where:*

- $\mathcal{T}\cong\mathbb{C}/\Lambda$ *is a flat torus,*
- G_n *is the group of* n*-th roots of unity acting on* $\mathcal{T}$ *by rotation around a base point, with* n *equal to either 2, 3, 4, or 6,*

[2] The Russian letter Ч is called "chi", pronounced as in "chicken".

- *$\mathcal{T}/G_n$ is the quotient space provided with its natural structure as a smooth Riemann surface of genus zero,*
- *L is an affine map from $\mathcal{T}$ to itself which commutes with a generator of G_n, and*
- *L/G_n is the induced holomorphic map from the quotient surface to itself.*

Remark 3.2. The map $\mathcal{T} \to \mathcal{T}/G_n \cong \widehat{\mathbb{C}}$ can of course be described in terms of classical elliptic function theory. In the case $n = 2$ we can identify this map with the Weierstrass function $\wp : \mathbb{C}/\Lambda \to \widehat{\mathbb{C}}$ associated with the period lattice Λ. Here the lattice Λ or the torus $\mathcal{T}$ can be completely arbitrary, but in the cases $n \geq 3$ we will see that $\mathcal{T}$ is uniquely determined by n, up to conformal isomorphism. For $n = 3$ we can take the derivative $\wp'$ of the associated Weierstrass function as the semiconjugacy $\wp' : \mathcal{T} \to \widehat{\mathbb{C}}$, while for $n = 6$ we can use either $(\wp')^2$ or $\wp^3$ as semiconjugacy. (For any lattice with G_3-symmetry, these two functions are related by the identity $(\wp')^2 = 4\wp^3 + \text{constant}$. The two alternate forms correspond to the fact that $\mathcal{T}/G_6$ can be identified either with $(\mathcal{T}/G_3)/G_2$ or with $(\mathcal{T}/G_2)/G_3$.) Finally, for $n = 4$ we can use the square $\wp^2$ of the associated Weierstrass function, corresponding to the factorization $\mathcal{T} \to \mathcal{T}/G_2 \to \mathcal{T}/G_4$.

Remark 3.3. This theorem is related to the definition in §2 as follows. Let us use the notation $\Theta^\star : \mathcal{T}^\star = \mathbb{C}/\Lambda^\star \to \widehat{\mathbb{C}} \smallsetminus \mathcal{E}_f$ for the initial semiconjugacy of Definition 2.1, formula (1). The degree of this semiconjugacy $\Theta^\star$ can be arbitrarily large. However, the proof of 3.1 will show that $\Theta^\star$ can be factored in an essentially unique way as a composition $\mathcal{T}^\star \to \mathcal{T} \to \mathcal{T}/G_n \cong \widehat{\mathbb{C}}$ for some torus $\mathcal{T}$, with n equal to 2, 3, 4, or 6.

The proof of 3.1 will be based on the following ideas. Let $\theta : \mathbb{C} \to \widehat{\mathbb{C}}$ be a doubly periodic meromorphic function, and let $\Lambda \subset \mathbb{C}$ be its lattice of periods so that $\lambda \in \Lambda$ if and only if $\theta(t + \lambda) = \theta(t)$ for all $t \in \mathbb{C}$. Then the canonical flat metric $|\mathbf{d}t|^2$ on $\mathbb{C}$ pushes forward to a corresponding flat metric on the torus $\mathcal{T} = \mathbb{C}/\Lambda$. If $\ell(t) = at + b$ is an affine map of $\mathbb{C}$ satisfying the identity $f \circ \theta = \theta \circ \ell$, then for $\lambda \in \Lambda$ and $t \in \mathbb{C}$ we have

$$\theta(at + b) \;=\; f(\theta(t)) \;=\; f(\theta(t + \lambda)) \;=\; \theta(a(t + \lambda) + b) \,.$$

It follows that $a\Lambda \subset \Lambda$. Therefore the maps ℓ and θ on $\mathbb{C}$ induce corresponding maps L and Θ on $\mathcal{T}$, so that we have a commutative diagram of holomorphic maps

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{L} & \mathcal{T} \\ \Theta \downarrow & & \Theta \downarrow \\ \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}} \,. \end{array} \tag{2}$$

We will think of $\mathcal{T}$ as a branched covering of the Riemann sphere with projection map Θ. Since L carries a small region of area A to a region of area $|a|^2 A$, it follows that the map L has degree $|a|^2$. Using Diagram (2), we see that the degree $d_f \geq 2$ of the map f must also be equal to $|a|^2$.

One easily derived property is the following. (For a more precise statement, see 4.5.) Let C_f be the set of critical points of f and let $V_f = f(C_f)$ be the set of critical values. Similarly, let $V_\Theta = \Theta(C_\Theta)$ be the set of critical values for the projection map Θ.

Lemma 3.4. *Every Lattès map f is postcritically finite. In fact the postcritical set*

$$P_f \;=\; V_f \cup f(V_f) \cup f^{\circ 2}(V_f) \cup \cdots$$

for f is precisely equal to the finite set V_Θ consisting of all critical values for the projection $\Theta : \mathcal{T} \to \widehat{\mathbb{C}}$.

Proof. Let $d_f(z)$ be the local degree of the map f at a point z . Thus

$$1 \;\le\; d_f(z) \;\le\; d_f \;,$$

where $d_f(z) > 1$ if and only if z is a critical point of f . Given points $\tau_j \in \mathcal{T}$ and $z_j \in \widehat{\mathbb{C}}$ with

$$\begin{array}{ccc} \tau_1 & \overset{L}{\mapsto} & \tau_0 \\ \Theta \big\downarrow & & \Theta \big\downarrow \\ z_1 & \overset{f}{\mapsto} & z_0 \;, \end{array}$$

since L has local degree $d_L(\tau) = 1$ everywhere, it follows that

$$d_\Theta(\tau_0) \;=\; d_\Theta(\tau_1) \cdot d_f(z_1) \;. \tag{3}$$

Since the maps L and Θ are surjective, it follows that z_0 is a critical value of Θ if and and if it is either a critical value of f or has a preimage $z_1 \in f^{-1}(z_0)$ which is a critical value of Θ or both. Thus $V_\Theta \;=\; V_f \cup f(V_\Theta)$, which implies inductively that

$$f^{\circ n}(V_f) \subset V_\Theta \;, \qquad \text{hence} \qquad P_f \;\subset\; V_\Theta \;.$$

On the other hand, if some critical point τ_0 of Θ had image $\Theta(\tau_0)$ outside of the postcritical set P_f , then all of the infinitely many iterated preimages $\cdots \mapsto \tau_2 \mapsto \tau_1 \mapsto \tau_0$ would have the same property. This is impossible, since Θ can have only finitely many critical points. □

We will prove the following preliminary version of 3.1, with notations as in Diagram (2).

Lemma 3.5. *If f is a Lattès map, then there is a finite cyclic group G of rigid rotations of the torus $\mathcal{T}$ about some base point, so that $\Theta(\tau') = \Theta(\tau)$ if and only if $\tau' = g\,\tau$ for some $g \in G$. Thus Θ induces a canonical homeomorphism from the quotient space $\mathcal{T}/G$ onto the Riemann sphere.*

Remark 3.6. Such a quotient $\mathcal{T}/G$ can be given two different structures which are distinct, but closely related. Suppose that a point $\tau_0 \in \mathcal{T}$ is mapped to itself by a subgroup of G , necessarily cyclic, of order $r > 1$. Any τ close to τ_0 can be written as the sum of τ_0 with a small complex number $\tau - \tau_0$. The power $(\tau - \tau_0)^r$ then serves as a local uniformizing parameter for $\mathcal{T}/G$ near τ_0 . In this way, the quotient becomes a smooth Riemann surface. On the other hand, if we want to carry the flat Euclidean structure of $\mathcal{T}$ over to $\mathcal{T}/G$, then the image of τ_0 must be considered as a singular "cone point", as described in the next section. The integer r , equal to the local degree $d_\Theta(\tau_0)$, depends only on the image point $\Theta(\tau_0)$, and is called the *ramification index* of $\Theta(\tau_0)$.

Proof of 3.5. Let U be any simply connected open subset of $\widehat{\mathbb{C}} \smallsetminus P_f \;=\; \widehat{\mathbb{C}} \smallsetminus V_\Theta$. Then the preimage $\Theta^{-1}(U)$ is the union $U_1 \cup \cdots \cup U_n$ of n disjoint open sets, each of which projects diffeomorphically onto U , where $n = d_\Theta$ is equal to the degree of Θ . Let $\Theta_j : U_j \xrightarrow{\cong} U$ be the restriction of Θ to U_j . We will first prove that each composition

$$\Theta_k^{-1} \circ \Theta_j \;:\; U_j \;\to\; U_k \;, \tag{4}$$

is an isometry from U_j onto U_k , using the standard flat metric on the torus.

Since periodic points of f are everywhere dense, we can choose a periodic point $z_0 \in U$.

Now replacing f by some carefully chosen iterate, and replacing L by the corresponding iterate, we may assume without changing Θ that:

- z_0 is actually a fixed point of f, and that
- every point in the finite L-invariant set $\Theta^{-1}(z_0)$ is either fixed by L, or maps directly to a fixed point.

In other words, each point $\tau_j = \Theta_j^{-1}(z_0)$ is either a fixed point of L or maps to a fixed point. For τ close to τ_j, evidently the difference $\tau - \tau_j$ can be identified with a unique complex number close to zero. Setting $L(\tau_j) = \tau_{j'}$, note that the affine map

$$\tau - \tau_j \quad \mapsto \quad L(\tau) - \tau_{j'} \quad \in \quad \mathbb{C}$$

is actually linear, so that $L(\tau) - \tau_{j'} = \mu\,(\tau - \tau_j)$ where $\mu = L'$ is constant. Similarly, for z close to z_0, the difference

$$\kappa_j(z) \;=\; \Theta_j^{-1}(z) - \tau_j$$

is well defined as a complex number. For each index j we will show that the map $z \mapsto \kappa_j(z) \in \mathbb{C}$ is a Kœnigs linearizing map for f in a neighborhood of z_0. That is,

$$\kappa_j(f(z)) \;=\; \mu\,\kappa_j(z)\;, \qquad \text{with} \qquad \kappa_j(z_0) \;=\; 0\;, \tag{5}$$

where the constant $\mu = L'$ is necessarily equal to the multiplier of f at z_0. In fact the identity $\Theta_{j'}^{-1}(f(z)) = L(\Theta_j^{-1}(z))$ holds for all z close to z_0. Subtracting $\tau_{j'}$ we see that

$$\kappa_{j'}(f(z)) \;=\; \mu\,\kappa_j(z)\;. \tag{6}$$

If τ_j is a fixed point so that $j' = j$, then this is the required assertion (5). But $\tau_{j'}$ is always a fixed point, so this proves that $\kappa_{j'}(f(z)) = \mu\,\kappa_{j'}(z)$. Combining this equation with (6), we see that $\kappa_j(z) = \kappa_{j'}(z)$, and it follows that equation (5) holds in all cases.

Since such a Kœnigs linearizing map is unique up to multiplication by a constant, it follows that every $\kappa_i(z)$ must be equal to the product $c_{ij}\,\kappa_j(z)$ for some constant $c_{ij} \neq 0$ and for all z close to z_0. Therefore $\Theta_i^{-1}(z)$ must be equal to $c_{ij}\,\Theta_j^{-1}(z)$ plus a constant for all z close to z_0. Choosing a local lifting of $\Theta_i^{-1} \circ \Theta_j$ to the universal covering space $\widetilde{\mathcal{T}} \cong \mathbb{C}$ and continuing analytically, we obtain an affine map A_{ij} from $\mathbb{C}$ to itself with derivative $A'_{ij} = c_{ij}$, satisfying the identity $\theta = \theta \circ A_{ij}$, where θ is the composition $\widetilde{\mathcal{T}} \to \mathcal{T} \xrightarrow{\Theta} \widehat{\mathbb{C}}$.

We must prove that $|c_{jk}| = 1$, so that this affine transformation is an isometry. Let $\widetilde{G}$ be the group[3] consisting of all affine transformations $\widetilde{g}$ of $\mathbb{C}$ which satisfy the identity $\theta = \theta \circ \widetilde{g}$. The translations $t \mapsto t + \lambda$ with $\lambda \in \Lambda$ constitute a normal subgroup, and the quotient $G = \widetilde{G}/\Lambda$ acts as a finite group of complex affine automorphisms of the torus $\mathcal{T} = \mathbb{C}/\Lambda$. In fact G has exactly n elements, since it contains exactly one transformation g carrying U_1 to any specified U_j. The derivative map $g \mapsto g'$ is an injective homomorphism from G to the multiplicative group $\mathbb{C} \smallsetminus \{0\}$. Hence it must carry G isomorphically onto the unique subgroup of $\mathbb{C} \smallsetminus \{0\}$ of order n, namely the group G_n of n-th roots of unity. Furthermore, a generator of G must have a fixed point in the torus, so G can be considered as a group of rotations about this fixed point. This completes the proof of 3.5. □

[3] This $\widetilde{G}$ is often described as a *crystallographic group* acting on $\mathbb{C}$. That is, it is a discrete group of rigid Euclidean motions of $\mathbb{C}$, with compact quotient $\mathbb{C}/\widetilde{G} \cong \mathcal{T}/G$.

In fact, if we translate coordinates so that some specified fixed point of the G-action is the origin of the torus $\mathcal{T} = \mathbb{C}/\Lambda$, then clearly we can identify G with the group G_n of n-th roots of unity, acting by multiplication on $\mathcal{T}$.

Lemma 3.7. *The order n of such a cyclic group of rotations of the torus with quotient $\mathcal{T}/G_n \cong \widehat{\mathbb{C}}$ is necessarily either 2, 3, 4 , or 6 .*

Proof. Thinking of a rotation through angle α as a real linear map, it has eigenvalues $e^{\pm i\alpha}$ and trace $e^{i\alpha} + e^{-i\alpha} = 2\cos(\alpha)$. On the other hand, if such a rotation carries the lattice Λ into itself, then its trace must be an integer. The function $\alpha \mapsto 2\cos(\alpha)$ is monotone decreasing for $0 < \alpha \le \pi$ and takes only the following integer values:

$$\begin{array}{rcrrrr} r & = & 6 & 4 & 3 & 2 \\ 2\cos(2\pi/r) & = & 1 & 0 & -1 & -2\,. \end{array}$$

This proves 3.7. □

Now to complete the proof of Theorem 3.1, we must find which affine maps $L(\tau) = a\,\tau + b$ give rise to well defined maps of the quotient surface $\mathcal{T}/G_n$. Let ω be a primitive n-th root of unity, so that the rotation $g(t) = \omega t$ generates G_n . Then evidently the points $L(t) = at + b$ and $L(g(t)) = a\,\omega\,t + b$ represent the same element of $\mathcal{T}/G_n$ if and only if

$$a\,\omega\,t + b \;\equiv\; \omega^k(at + b) \mod \Lambda \qquad \text{for some power} \qquad \omega^k\,.$$

If this equation is true for some generic choice of t , then it will be true identically for all t . Now differentiating with respect to t we see that $k = 1$, and substituting $t = 0$ we see that $b \equiv \omega\, b \mod \Lambda$. It follows easily that $g \circ L = L \circ g$. Conversely, whenever g and L commute it follows immediately that L/G_n is well defined. This completes the proof of 3.1. □

The analogous statement for Chebyshev maps and power maps is the following.

Lemma 3.8. *If $f = \Theta \circ L \circ \Theta^{-1}$ is a finite quotient of an affine map on a cylinder $\mathcal{C}$, then f is holomorphically conjugate either to a power map $z \mapsto z^a$ or to a Chebyshev map $\pm Ч_d$.*

The proof is completely analogous to the proof of 3.1. In fact any such f is conjugate to a map of the form $L/G_n : \mathcal{C}/G_n \to \mathcal{C}/G_n$, where L is an affine map of the cylinder $\mathcal{C}$ and where n is either one (for the power map case) or two (for the Chebyshev case). Details will be left to the reader. □

The following helps to demonstrate the extremely restricted dynamics associated with finite quotients of affine maps. Presumably nothing like it is true for any other rational map.

Corollary 3.9. *Let $f = \Theta \circ L \circ \Theta^{-1}$ be a finite quotient of an affine map L which has derivative $L' = a$. If $z \in \widehat{\mathbb{C}} \smallsetminus \mathcal{E}_f$ is a periodic point with period $p \ge 1$ and ramification index $r \ge 1$, then the multiplier of $f^{\circ p}$ at z is a number of the form $\mu = (\omega a^p)^r$ where $\omega^n = 1$.*

(The ramification index is described in 3.6 and also in §4.) For example for a periodic orbit of ramification $r = n$ the multiplier is simply a^{pn} . In the case of a generic periodic orbit with $r = 1$, the multiplier has the form ωa^p . In all cases, the absolute value $|\mu|$ is equal to $|a|^{pr}$.

Proof of 3.9. First consider a fixed point $z_0 = f(z_0)$ and let $\Theta(\tau_0) = z_0$. As in 3.6, we can take $\zeta = (\tau - \tau_0)^r$ as local uniformizing parameter near z_0. On the other hand, since $z_0 = f(z_0)$ we have $\tau_0 \sim L(\tau_0)$, or in other words $\tau_0 = \omega\, L(\tau_0)$ for some $\omega \in G_n$. Thus f lifts to the linear map

$$\tau - \tau_0 \;\mapsto\; \omega L(\tau) - \omega L(\tau_0) \;=\; \omega\, a\,(\tau - \tau_0)\;.$$

Therefore, in terms of the local coordinate ζ near z_0, we have the linear map $\zeta \mapsto (\omega\, a)^r \zeta$, with derivative $\mu = (\omega\, a)^r$. Applying the same argument to the p-th iterates of f and L, we get a corresponding identity for a period p orbit. $\square$

§**4. Flat Orbifold Metrics.** We can give another characterization of finite quotients of affine maps as follows.

Definition. By a *flat orbifold metric* on $\widehat{\mathbb{C}} \setminus \mathcal{E}_f$ will be meant a complete metric which is smooth, conformal, and locally isometric to the standard flat metric on $\mathbb{C}$, except at finitely many "cone points", where it has cone angle of the form $2\pi/r$. Here a *cone point* with *cone angle* $0 < \alpha < 2\pi$, is an isolated singular point of the metric which can be visualized by cutting an angle of α out of a sheet of paper and then gluing the two edges together. (A more formal definition will be left to the reader.) In the special case where α is an angle of the form $2\pi/r$, we can identify such a cone with the quotient space $\mathbb{C}/G_r$ where G_r is the group of r-th roots of unity acting by multiplication on the complex numbers, and where the flat metric on $\mathbb{C}$ corresponds to a flat metric on the quotient, except at the cone point.

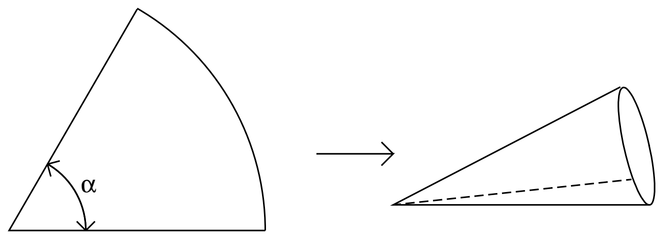

Figure 1. Model for a cone point with cone angle α .

Evidently the canonical flat metric on a torus $\mathcal{T}$ or cylinder $\mathcal{C}$ induces a corresponding flat orbifold metric on the quotient $\mathcal{T}/G_n$ of Theorem 3.1 or the quotient $\mathcal{C}/G_n$ of Lemma 3.8. Thus near any non-cone point we can choose a local coordinate t so that the metric takes the form $|\mathbf{d}t|^2$. I will say that such a metric *linearizes* the map f since, in terms of such preferred local coordinates, f is an affine map with constant derivative.[4] (An equivalent property is that f maps any curve of length δ to a curve of length $k\,\delta$ where $k = |a| > 1$ is constant.) A converse statement is also true:

Theorem 4.1. *A rational map f is a finite quotient of an affine map if and only if it is linearized by some flat orbifold metric, or if and only if there exists*

[4] The more usual terminology for a map f which is linearized by a flat orbifold metric would be that f has "*parabolic*" or "*Euclidean*" *orbifold*. Following Thurston, for any postcritically finite f there is a smallest function $r(z) \ge 1$ on $\widehat{\mathbb{C}} \setminus \mathcal{E}_f$ such that $r(f(z))$ is a multiple of $d_f(z)\, r(z)$ for every z. Furthermore there is an essentially unique complete orbifold metric of constant curvature ≤ 0 on $\widehat{\mathbb{C}} \setminus \mathcal{E}_f$ with $r(z)$ as ramification index function. The curvature is zero if and only if (7) is satisfied. (See [DH] or [M3].)

an integer valued "ramification index" function $r(z)$ *on* $\widehat{\mathbb{C}} \smallsetminus \mathcal{E}_f$ *satisfying the identity*

$$r(f(z)) \;=\; d_f(z)\, r(z) \qquad \textit{for all} \quad z\ , \tag{7}$$

with $r(z) = 1$ *outside of the postcritical set of* f *.*

Proof in the Lattès case. First suppose that f is a finite quotient of an affine map on a torus, conformally conjugate to the quotient map $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$. If τ_0 is a critical point of the projection $\Theta : \mathcal{T} \to \mathcal{T}/G_n \cong \widehat{\mathbb{C}}$ with local degree $d_\Theta(\tau_0) = r$, then the subgroup consisting of elements of G_n which fix τ_0 must be generated by a rotation through $2\pi/r$ about τ_0 . Hence the flat metric on $\mathcal{T}$ pushes forward to a flat metric on $\mathcal{T}/G_n$ with τ_0 corresponding to a cone point $z_0 = \Theta(\tau_0)$ of angle $2\pi/r$. This integer $r = r(z_0) > 1$ is called the ramification index of the cone point. Setting $r(z) = 1$ if z is not a cone point, we see that $r(\theta(\tau))$ can be identified with the local degree $d_\Theta(\tau)$ in all cases. There may be several different points in $\Theta^{-1}(z)$, but Θ must have the same local degree at all of these points, since the angle at a cone point is uniquely defined, or by 3.9. With these notations, the required equation (7) is just a restatement of equation (3) of §3.

Conversely, suppose that (7) is satisfied. *It follows from this equation that* f *is postcritically finite.* In fact we can express the postcritical set P_f as a union $P_1 \cup P_2 \cup \cdots$ of disjoint finite subsets, where

$$P_1 \;=\; f(C_f) \qquad \text{and} \qquad P_{m+1} \;=\; f(P_m) \smallsetminus (P_1 \cup \cdots \cup P_m)\ .$$

Let $|P_m|$ be the number of elements in P_m . Since $f(P_m) \supset P_{m+1}$, the sequence $|C_f| \ge |P_1| \ge |P_2| \ge |P_3| \ge \cdots$ must eventually stabilize. Therefore we can choose an integer m so that P_k maps bijectively onto P_{k+1} for $k \ge m$, and must prove that the number of elements $|P_m| = |P_{m+1}| = \cdots$ is zero. Note that each point of P_{m+1} has d_f distinct preimages, where $d_f \ge 2$ by our standing hypothesis. Thus if $|P_{m+1}| \ne 0$ there would exist some point $z \notin P_m$ with $f(z) \in P_{m+1}$. In fact it would follow that $z \notin C_f \cup P_f$. For if z were in $C_f \cup P_1 \cup \cdots \cup P_{m-1}$ then $f(z)$ would be in some P_k with $k < m+1$, while if z were in P_k with $k > m$ then $f(z)$ would be in P_{k+1} with $k+1 > m+1$, contradicting the hypothesis that $f(z) \in P_{m+1}$ in either case. The existence of a point $z \notin C_f \cup P_f$ with $f(z) \in P_f$ clearly contradicts equation (7).

Let me use the notation M for the Riemann sphere $\widehat{\mathbb{C}}$ together with the "orbifold structure" determined by the function $r : \widehat{\mathbb{C}} \to \{1, 2, 3, \ldots\}$. The *universal covering orbifold* $\widetilde{M}$ can be characterized as a simply connected Riemann surface together with a holomorphic branched covering map $\theta : \widetilde{M} \to M = \widehat{\mathbb{C}}$ such that, for each $\widetilde{z} \in \widetilde{M}$, the local degree $d_\theta(\widetilde{z})$ is equal to the prescribed ramification index $r(\theta(\widetilde{z}))$. Such a universal covering associated with a function $r : M \to \{1, 2, 3, \ldots\}$ exists whenever the number of z with $r(z) > 1$ is finite with at least three elements. (See for example [M3, Lemma E.1].)

For this proof only, it will be convenient to choose some fixed point of f as base point $z_0 \in M$. Using equation (7), there is no obstruction to lifting f to a holomorphic map $\widetilde{f}$ which maps the Riemann surface $\widetilde{M}$ diffeomorphically into itself, with no critical points. Furthermore, we can choose $\widetilde{f}$ to fix some base point $\widetilde{z}_0$ lying over z_0 . The covering manifold $\widetilde{M}$ cannot be a compact surface, necessarily of genus zero, since then $\widetilde{f}$ and hence f would have degree one, contrary to the standing hypothesis that $d_f \ge 2$. Furthermore, since the base point in $\widetilde{M}$ is strictly repelling under $\widetilde{f}$, it follows that $\widetilde{M}$ cannot be a hyperbolic surface. Therefore $\widetilde{M}$ must be conformally isomorphic to the complex numbers $\mathbb{C}$, and $\widetilde{f}$ must correspond to a linear map L from $\mathbb{C}$ to itself. Evidently the standard flat metric on $\mathbb{C} \cong \widetilde{M}$ now gives rise to a flat orbifold metric on $M = \widehat{\mathbb{C}}$ which linearizes the map f.

Finally, suppose that we start with a flat orbifold metric on $\widehat{\mathbb{C}}$ which linearizes the rational map f. The preceding discussion shows that f lifts to a linear map $\widetilde{f}$ on the universal covering orbifold $\widetilde{M}$. Let $\widetilde{G}$ be the group of *deck transformations* of $\widetilde{M}$, that is homeomorphisms $\widetilde{g}$ from $\widetilde{M}$ to itself which cover the identity map of M, so that $\theta \circ \widetilde{g} = \theta$. Then the quotient surface $\widetilde{M}/\widetilde{G}$ can be identified with $M = \widehat{\mathbb{C}}$. If $\Lambda \subset \widetilde{G}$ is the normal subgroup consisting of those deck transformations which are translations of $\widetilde{M} \cong \mathbb{C}$, then the quotient group $G = \widetilde{G}/\Lambda$ is a finite group of rotations with order equal to the least common multiple of the ramification indices. It follows that the quotient $\mathcal{T} = \widetilde{M}/\Lambda$ is a torus, and hence that f is a finite quotient of an affine map of this torus.

The proof of 4.1 in the Chebyshev and power map cases is similar and will be omitted. □

Remark 4.2. Note that the construction of the torus $\mathcal{T}$, the group G_n and the affine map L from the rational map f satisfying (7) is completely canonical, except for the choice of lifting for f. For example, when there are four postcritical points, the conformal conjugacy class of the torus $\mathcal{T}$ is completely determined by the set of postcritical points, and in fact by the cross-ratio of these four points.

However, to make an explicit classification we must note the following.

- We want to identify the torus $\mathcal{T}$ with some quotient $\mathbb{C}/\Lambda$. Here, Λ is unique only up to multiplication by a non-zero constant; but we can make an explicit and unique choice by taking Λ to be the lattice $\mathbb{Z} \oplus \gamma\mathbb{Z}$ spanned by 1 and γ, where γ belongs to the Siegel region

$$\begin{aligned} &|\gamma| \geq 1 , \qquad |\Re(\gamma)| \leq 1/2 , \qquad \Im(\gamma) > 0 , \\ &\text{with} \quad \Re(\gamma) \geq 0 \quad \text{whenever} \quad |\gamma| = 1 \quad \text{or} \quad |\Re(\gamma)| = 1/2 . \end{aligned} \tag{8}$$

With these conditions, γ is uniquely determined by the conformal isomorphism class of $\mathcal{T}$. We will describe the corresponding $\Lambda = \mathbb{Z} \oplus \gamma\mathbb{Z}$ as a *normalized lattice*.

- For specified Λ, we still need to make some choice of conformal isomorphism $\upsilon : \mathbb{C}/\Lambda \to \mathcal{T}$. In most cases, υ depends only on a choice of base point $\upsilon(0) \in \mathcal{T}$, up to sign. However, in the special case where $\mathcal{T}$ admits a G_3 (or G_4) action, we can also multiply υ by a cube (or fourth) root of unity. As in §3, it will be convenient to choose one of the fixed points of the G_n action as a base point in $\mathcal{T}$.

- The lifting $L(t) = at + b$ of the map f to the torus is well defined only up to the action of G_n. In particular, we are always free to multiply the coefficients a by an n-th root of unity.

We will deal with all of these ambiguities in §5.

Here is an interesting consequence of 4.1. Let f and g be rational maps.

Corollary 4.3. *Suppose that there is a holomorphic semiconjugacy from f to g, that is, a non-constant rational map h with $h \circ f = g \circ h$. Then f is a finite quotient of an affine map if and only if g is a finite quotient of an affine map.*

Proof. It is not hard to see that $h^{-1}(\mathcal{E}_g)=\mathcal{E}_f$, so that h induces a proper map from $\widehat{\mathbb{C}}\smallsetminus\mathcal{E}_f$ to $\widehat{\mathbb{C}}\smallsetminus\mathcal{E}_g$. Now if f is a finite quotient of an affine map L , say $f=\Theta\circ L\circ\Theta^{-1}$, then it follows immediately that $g=(h\circ\Theta)\circ L\circ(h\circ\Theta)^{-1}$. Conversely, if g is such a finite quotient, then there is a flat orbifold structure on $\widehat{\mathbb{C}}\smallsetminus\mathcal{E}_g$ which linearizes g , and we can lift easily to a flat orbifold structure on $\widehat{\mathbb{C}}\smallsetminus\mathcal{E}_f$ which linearizes f . □

In order to classify all possible flat orbifold structures on the Riemann sphere, we can use a piecewise linear form of the Gauss-Bonnet Theorem. For this lemma only, we allow cone angles which are greater than 2π .

Lemma 4.4. *If a flat metric with finitely many cone points on a compact Riemann surface S has cone angles $\alpha_1,\ldots,\alpha_k$, then*

$$(2\pi-\alpha_1)\;+\;\cdots\;+\;(2\pi-\alpha_k)\;=\;2\pi\,\chi(S)\;, \tag{9}$$

where $\chi(S)$ is the Euler characteristic. In particular, if $\alpha_j=2\pi/r_j$ and if S is the Riemann sphere with $\chi(S)=2$, then it follows that $\sum(1-1/r_j)\;=\;2$.

Proof. Choose a rectilinear triangulation, where the cone points will necessarily be among the vertices. Let V be the number of vertices, E the number of edges, and F the number of faces (i.e., triangles). Then $2E=3F$ since each edge bounds two triangles and each triangle has three edges. Thus

$$\chi(S)\;=\;V-E+F\;=\;V-F/2\;. \tag{10}$$

The sum of the internal angles of all of the triangles is clearly equal to πF . On the other hand, the j-th cone point contributes α_j to the total, while each non-cone vertex contributes 2π . Thus

$$\pi F\;=\;\alpha_1+\cdots+\alpha_k+2\pi(V-k)\;. \tag{11}$$

Multiplying equation (10) by 2π and using (11), we obtain the required equation (9). □

Corollary 4.5. *The collection of ramification indices for a flat orbifold metric on the Riemann sphere must be either $\{2,2,2,2\}$ or $\{3,3,3\}$ or $\{2,4,4\}$ or $\{2,3,6\}$. In particular, the number of cone points must be either four or three.*

Proof. Using the inequality $1/2\le(1-1/r_j)<1$, it is easy to check that the required equation

$$\sum_j(1-1/r_j)\;=\;\chi(\widehat{\mathbb{C}})\;=\;2\;,$$

has only these solutions in integers $r_j>1$. □

Remark 4.6. If $z\in\widehat{\mathbb{C}}$ corresponds to a fixed point for the action of the group G_n on the torus, then the ramification index $r(z)$ is evidently equal to n . For any other point, it is some divisor of n . Thus the order n of the rotation group G_n can be identified with the least common multiple (or the maximum) of the various ramification indices as listed in 4.5.

Remark 4.7. To deal with the case of a map f which has exceptional points, we can assign the ramification index $r(z)=\infty$ to any exceptional point $z\in\mathcal{E}_f$. If we allow such points, then the equation $\sum(1-1/r_j)=2$ has two further solutions, namely: $\{\infty\,,\,\infty\}$ corresponding to the power map case, and $\{2\,,\,2\,,\,\infty\}$ corresponding to the Chebyshev case.

Combining Corollary 4.5 with equation (7), we get an easy characterization of Lattès maps in two of the four cases.

Corollary 4.8. *A rational map with four postcritical points is Lattès if and only if every critical point is simple* (*with local degree two*) *and no critical point is postcritical. Similarly, a rational map with three postcritical points is Lattès of type* $\{3,3,3\}$ *if and only if every critical point has local degree three and none is postcritical.*

The proof is easily supplied. $\square$

We conclude this section with a more precise description of the possible crystallographic groups $\widetilde{G}$ acting on $\mathbb{C}$, and of the corresponding orbifold geometries on $\mathbb{C}/\widetilde{G} \cong \mathcal{T}/G_n$. We first look at the cases $n \geq 3$ where there are exactly three cone points in $\mathcal{T}/G_n$ or equivalently three postcritical points for any associated Lattès map. Thus the collection of ramification indices must be either $\{2,3,6\}$ or $\{2,4,4\}$ or $\{3,3,3\}$. Each of these three possibilities is associated with a rigidly defined flat orbifold geometry which can be described as follows. Join each pair of cone points by a minimal geodesic. Evidently these geodesics cannot cross each other; and no geodesic can pass through a cone point since our cone angles are strictly less than 2π. In this way, we obtain three edges which cut our locally flat manifold into two Euclidean triangles. Since these two triangles have the same edges, they must be precise mirror images of each other. In particular, the two edges which meet at a cone point of angle $2\pi/r_j$ must cut it into two Euclidean angles of π/r_j. Passing to the branched covering space $\mathcal{T}$ or its universal covering $\widetilde{\mathcal{T}}$, we obtain a tiling of the torus or the Euclidean plane[5] by triangles with angles π/r_1, π/r_2 and π/r_3. These tilings are illustrated in Figures 2, 3, 4.

In each case, each pair of adjacent triangles are mirror images of each other, and together form a fundamental domain for the action of the group of Euclidean motions $\widetilde{G}_n$ on the plane, or for the action of G_n on the torus. For each vertex of this diagram, corresponding to a cone point of angle $2\pi/r_j$, there are r_j lines through the vertex, and hence $2r_j$ triangles which meet at the vertex. The subgroup of $\widetilde{G}_n$ (or G_n) which fixes such a point has order r_j and is generated by a rotation through the angle $\alpha_i = 2\pi/r_j$.

The subgroup $\Lambda \subset \widetilde{G}_n$ consists of all translations of the plane which belong to $\widetilde{G}_n$. Recall from 4.4 that the integer n can be described as the maximum of the r_j. The $2n$ triangles which meet at any maximally complicated vertex form a fundamental domain for the action of this subgroup Λ. In the $\{2,3,6\}$ and $\{3,3,3\}$ cases, this fundamental domain is a regular hexagon, while in the $\{2,4,4\}$ case it is a square. In all three cases, the torus $\mathcal{T}$ can be obtained by identifying opposite faces of this fundamental domain under the appropriate translations. Thus when $n \geq 3$ the torus $\mathcal{T}$ is uniquely determined by n, up to conformal diffeomorphism.

In the $\{2,3,6\}$ case, the integers r_j are all distinct, so it is easy to distinguish the three kinds of vertices. However, in the $\{2,4,4\}$ case there are two different kinds of vertices of index 4. In order to distinguish them, one kind has been marked with dots and the other with circles. Similarly in the $\{3,3,3\}$ case, the three kinds of vertices have been marked in

[5] More generally, for any triple of integers $r_j \geq 2$ there is an associated tiling, either of the Euclidean or hyperbolic plane or of the 2-sphere depending on the sign of $1/r_1 + 1/r_2 + 1/r_3 - 1$. See for example [M1].

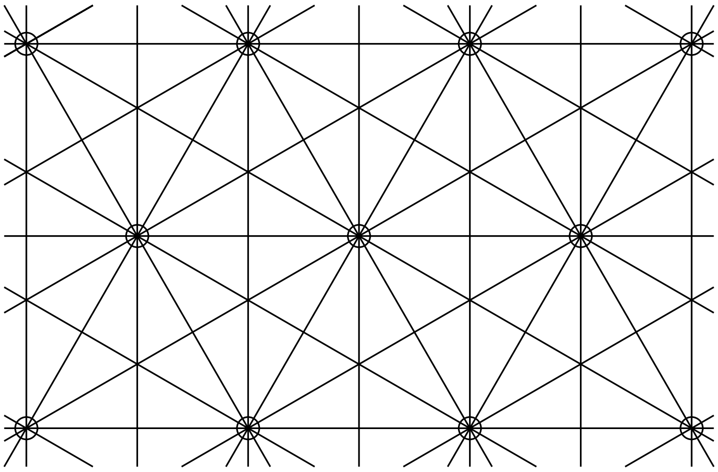

Figure 2. The $\{2,3,6\}$*-tiling of the plane. In each of these diagrams, the points of ramification* n *have been marked, with circles around the lattice points.*

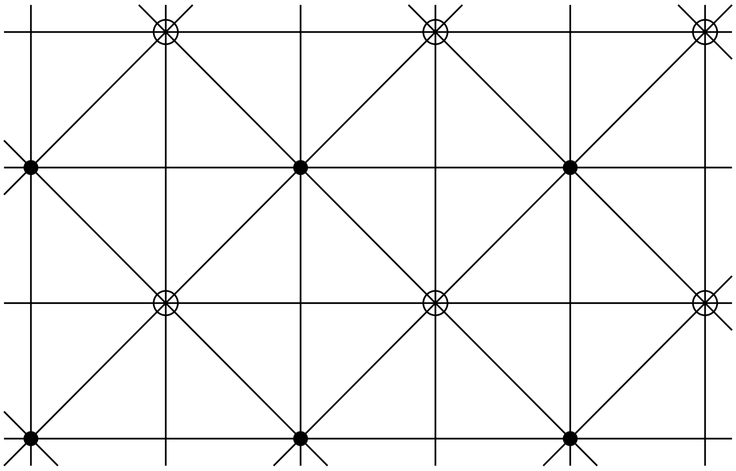

Figure 3. The $\{2,4,4\}$*-tiling.*

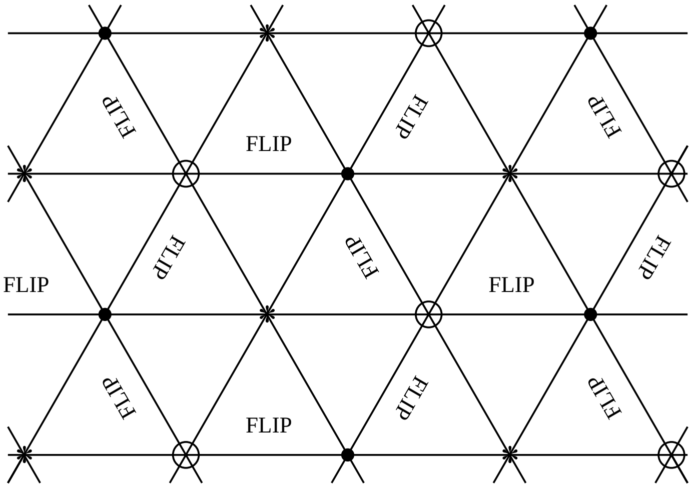

Figure 4. The $\{3,3,3\}$*-tiling, with one tile and its images under* $\widetilde{G}_3$ *labeled.*

three different ways. In this last case, half of the triangles have also been labeled. In all three cases, the points of the lattice Λ , corresponding to the base point in $\mathcal{T}$, have been circled. For all three diagrams, the group $\widetilde{G}_n$ can be descried as the group of all rigid Euclidean motions which carry the marked diagram to itself, and the lattice Λ can be identified with the subgroup consisting of translations which carry this marked diagram to itself.

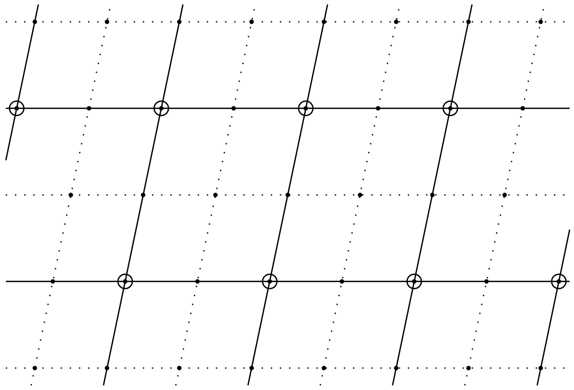

Figure 5. A typical $\{2,2,2,2\}$ *tiling of the plane.*

The analogue of Figures 2, 3, 4 for a typical orbifold of type $\{2,2,2,2\}$ is a tiling of the plane by parallelograms associated with a typical lattice $\Lambda = \mathbb{Z}\oplus\gamma\mathbb{Z}$, as illustrated in Figure 5. All of the vertices in this figure represent critical points for the projection map $\theta : \mathbb{C} \to \widehat{\mathbb{C}}$. Again lattice points have been circled. Any two adjacent small parallelograms form a fundamental region for the action of the group $\widetilde{G}_2$, which consists of $180°$ rotations around the vertices, together with lattice translations. The four small parallelograms adjacent to any vertex forms a fundamental domain under lattice translations.

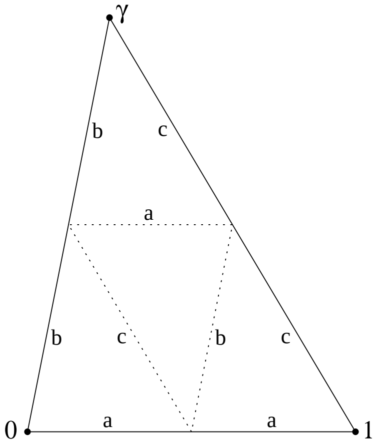

Figure 6. Illustrating the orbifold structure of $\mathcal{T}/G_2$ *.*

In most cases, the corresponding flat orbifold is isometric to some tetrahedron in Euclidean space. (Compare [De].) For example, consider the case where the invariant γ in the Siegel domain (8) satisfies $0 < \Re(\gamma) < 1/2$. Then the triangle with vertices 0 , 1 and γ has all angles acute, and also serves as a fundamental domain for the action of the crystallographic group $\widetilde{G}_2$ on $\mathbb{C}$. Joining the midpoints of the edges of this triangle, as shown in

Figure 6, we can cut this triangle up into four similar triangles. Now fold along the dotted lines and bring the three corner triangles up so that the three vertices $0, 1$ and γ come together. In this way, we obtain a tetrahedron which is isometric to the required flat orbifold $\mathcal{T}/G_2$. The construction when $-1/2 \le \Re(\gamma) < 0$ is the same, except that we use -1 and 0 in place of 0 and 1. The tetrahedrons which can be obtained in this way are characterized by the property that opposite edges have equal length, or by the property that there is a Euclidean motion carrying any vertex to any other vertex. In most cases this Euclidean motion is uniquely determined; but in the special case where we start with an equilateral triangle, with $\gamma = \omega_6$, we obtain a regular tetrahedron which has extra symmetries. !!!!

In the case $\Re(\gamma) = 0$ where γ is pure imaginary, this tetrahedron degenerates and the orbifold can be described rather as the "double" of the rectangle which has vertices

$$0\,,\quad 1/2\,,\quad \gamma/2\,,\quad (1+\gamma)/2\,.$$

Again, in most cases there is a unique orientation preserving isometry carrying any vertex to any other vertex; but in the special case of a square, with $\gamma = i$, there are extra symmetries.

§**5. Classification.** By taking a closer look at the arguments in sections 3 and 4, we can give a precise classification of Lattès Maps. (Compare [DH, §9].) It will be convenient to introduce the notation

$$\omega_n \;=\; \exp(2\pi i/n)\,, \tag{12}$$

for the standard generator of the cyclic group G_n. Thus

$$\omega_2 = -1\,, \qquad \omega_3 = (-1 + i\sqrt{3})/2\,, \qquad \omega_4 = i\,, \qquad \omega_6 = \omega_3 + 1\,.$$

As usual, we consider a Lattès map which is conjugate to $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$, where $\mathcal{T} \cong \mathbb{Z}/\Lambda$ and where $L(t) = at + b$. Here it will be convenient to think of b as a complex number, well defined modulo Λ.

Theorem 5.1. *Such a Lattès map f is uniquely determined up to conformal conjugacy by the following four invariants.*

- *First: the integer n, equal to 2, 3, 4, or 6.*
- *Second: the complex number a^n, with $|a|^2$ equal to the degree of f.*
- *Third: the lattice Λ, which we may take to have the form $\Lambda = \mathbb{Z} \oplus \gamma\mathbb{Z}$ with γ in the Siegel region (8). This lattice must satisfy the conditions that $\omega_n\Lambda = \Lambda$ and $a\Lambda \subset \Lambda$. Let k be the largest integer such that $\omega_k\Lambda = \Lambda$.*
- *Fourth: the product $(1-\omega_n)\,b \in \Lambda$ modulo the sublattice*

$$(1-\omega_n)\Lambda \,+\, (a-1)\Lambda \;\subset\; \Lambda\,,$$

up to multiplication by G_k with k as above. This last invariant is zero if and only if the map f admits a fixed point of maximal ramification index $r = n$, or equivalently a fixed point of multiplier $\mu = a^n$.

For most lattices we have $k = 2$, so that the image of $(1-\omega_n)\,b$ in the quotient group

$$\Lambda\Big/\Big((1-\omega_n)\Lambda \,+\, (a-1)\Lambda\Big) \tag{13}$$

is invariant up to sign. However, in the special case where Λ has G_4 or G_6 symmetry, so that $\gamma = \omega_4$ or $\gamma = \omega_6$, this image is invariant only up to multiplication by ω_4 or ω_6 respectively.

Note that the first invariant n , equal to the greatest common divisor of the ramification indices, can easily be computed by looking at the orbits of the critical points of f , using formula (7) of §4. The invariant a^n can be computed from the multiplier μ at any fixed point, since the equation $\mu = (\omega a)^r$ of Corollary 3.9, with $\omega^n = 1$, implies that $\mu^{n/r} = a^n$. It follows from this equation that $\mu = a^n$ if and only if $r = n$.

The cases with $n \geq 3$ are somewhat easier than the case $n = 2$. In fact a normalized lattice Λ with G_3 or G_6 symmetry is necessarily equal to $\mathbb{Z}[\omega_3] = \mathbb{Z}[\omega_6]$, and the condition $a\Lambda \subset \Lambda$ is satisfied if and only if $a \in \mathbb{Z}[\omega_6]$. Similarly, G_4 symmetry implies that $\Lambda = \mathbb{Z}[i]$, and the possible choices for a are just the elements of $\mathbb{Z}[i]$, always subject to the standing requirement that $|a| > 1$.

Corollary 5.2. *If $n \geq 3$, then the conformal conjugacy class of f is completely determined by the numbers n and a^n where $a \in \mathbb{Z}[\omega_n]$, together with the information as to whether f does or does not have a fixed point of multiplier $\mu = a^n$.*

Evidently there is such a fixed point if and only if $(1-\omega_n)\,b$ is congruent to zero modulo $(1-\omega_n)\Lambda + (a-1)\Lambda$. (When $n = 6$ there is necessarily such a fixed point.)

The proof of 5.1 and 5.2 will be based on the following.

Lemma 5.3. *The additive subgroup of $\mathcal{T} = \mathbb{C}/\Lambda$ consisting of elements fixed by the action of G_n is canonically isomorphic to the quotient group $\Lambda/(1-\omega_n)\Lambda$, of order $|1-\omega_n|^2 = 4\sin^2(\pi/n)$. In fact, the correspondence $t \mapsto (1-\omega_n)\,t$ maps the group of torus elements fixed by G_n isomorphically onto this quotient group.*

Proof. The required equation $\omega_n t \equiv t \mod \Lambda$ is equivalent to $(1-\omega_n)\,t \in \Lambda$, and the conclusion follows easily. □

Note that points of $\mathcal{T}$ fixed by the action of G_n correspond to points in the quotient sphere $\mathcal{T}/G_n$ of maximal ramification index $r = n$. As a check, in the four cases $\{2,2,2,2\}$, $\{3,3,3\}$, $\{2,4,4\}$, and $\{2,3,6\}$, there are clearly 4, 3, 2, and 1 such points respectively. This number is equal to $4\sin^2(\pi/n)$ in each case.

Proof of 5.1. It is clear that the numbers n , γ , a^n , and b completely determine the map $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$. In fact γ determines the torus $\mathcal{T}$, and the power a^n determines a up to multiplication by n-th roots of unity. But we can multiply L and hence a by any n-th root of unity without changing the quotient Ł$/G_n$. Since the numbers n , a^n , and γ are uniquely determined by f (compare the discussion above), we need only study the extent to which b is determined by f .

Recall from Theorem 3.1 that the map $L(t) = at+b$ must commute with multiplication by ω_n . That is

$$L(\omega_n t) \equiv \omega_n L(t) \qquad \mod \Lambda .$$

Taking $t = 0$ it follows that $b \equiv \omega_n b \mod \Lambda$, or in other words $(1-\omega_n)\,b \in \Lambda$, as required.

Next recall that we are free to choose any fixed point t_0 for the action of G_n on $\mathcal{T}$ as base point. Changing the base point to t_0 in place of 0 amounts to replacing $L(t)$ by the conjugate map $L(t+t_0) - t_0 = at + b'$ where

$$b' = b + (a-1)\,t_0 , \quad \text{and hence} \quad (1-\omega_n)\,b' = (1-\omega_n)\,b + (a-1)(1-\omega_n)\,t_0 .$$

Since the product $(1-\omega_n)\,t_0$ can be a completely arbitrary element of Λ, this means that we can add a completely arbitrary element of $(a-1)\Lambda$ to the product $(1-\omega_n)\,b$ by a change of base point. Thus the residue class

$$(1-\omega_n)\,b\ \in\ \Lambda\Big/\Big((1-\omega_n)\Lambda\,+\,(a-1)\Lambda\Big)\ ,$$

together with n, γ, and a^n, suffices to determine the conjugacy class of f. However, we have not yet shown that this residue class is an invariant of f, since we must also consider automorphisms of $\mathcal{T}$ which fix the base point. Let ω be any root of unity which satisfies $\omega\Lambda=\Lambda$. Then $L(t)=at+b$ is conjugate to the map $\omega L(t/\omega)=at+\omega b$. In most cases, we can only choose $\omega=\pm 1$. (The fact that we are free to change the sign of b is irrelevant when n is even, but will be important in the case $n=3$.) However, if $\Lambda=\mathbb{Z}[\omega_6]$ then we can choose ω to be any power of ω_6, and if $\Lambda=\mathbb{Z}[i]$ then we can choose ω to be any power of i. Further details of the proof are straightforward. □

Proof of 5.2. In the cases $n\ge 3$, we have noted that Λ is necessarily equal to $\mathbb{Z}[\omega_n]$. Furthermore, for $n=3\,,\,4\,,\,6$, the quotient group $\Lambda/(1-\omega_n)\Lambda$ is cyclic of order $3\,,\,2\,,\,1$ respectively. Thus this group has at most one non-zero element, up to sign. The conclusion follows easily. □

The discussion of Lattès maps of type $\{2,2,2,2\}$ will be divided into two cases according as $a\in\mathbb{Z}$ or $a\notin\mathbb{Z}$. First suppose that $a\notin\mathbb{Z}$.

Definition. A complex number a will be called an *imaginary quadratic integer* if it satisfies an equation $a^2-q\,a+d=0$ with integer coefficients and with $q^2<4d$, so that

$$a\ =\ \Big(q\pm\sqrt{q^2-4d}\Big)/2 \tag{14}$$

is not a real number. Here $|a|^2=d$ is the associated degree. Evidently the imaginary quadratic integers form a discrete subset of the complex plane. In fact for each choice of $|a|^2=d$ there are roughly $4\sqrt{d}$ possible choices for q, and twice that number for a.

> **Lemma 5.4.** *A complex number a can occur as the derivative $a=L'$ associated with an affine torus map if and only if it is either a rational integer $a\in\mathbb{Z}$ or an imaginary quadratic integer. If $a\in\mathbb{Z}$ then any torus can occur, but if $a\notin\mathbb{Z}$ then there are only finitely many possible tori up to conformal diffeomorphism. Furthermore, there is a one-to-one correspondence between conformal diffeomorphism classes of such tori and ideal classes in the ring $\mathbb{Z}[a]$.*

Proof. Let $\mathcal{T}=\mathbb{C}/\Lambda$. The condition that $a\Lambda\subset\Lambda$ means that Λ must be a module over the ring $Z[a]$ generated by a. We first show that a must be an algebraic integer. Without loss of generality, we may assume that $\Lambda=\mathbb{Z}\oplus\gamma\mathbb{Z}$ is a normalized lattice, satisfying the Siegel conditions (8). Thus $1\in\Lambda$ and it follows that all powers of a belong to Λ. If Λ_k is the sublattice spanned by $1\,,\,a\,,\,a^2\,,\,\ldots\,,\,a^k$, then the lattices $\Lambda_1\subset\Lambda_2\subset\cdots\subset\Lambda$ cannot all be distinct. Hence some power a^k must belong to Λ_{k-1}, which proves that a satisfies a monic equation with integer coefficients, and hence is an algebraic integer. On the other hand, a belongs to a quadratic number field since the three numbers $1\,,\,a\,,\,a^2\in\Lambda$ must satisfy a linear relation with integer coefficients. Using the fact that the integer polynomials form a unique factorization domain, it follows that a satisfies a monic degree two polynomial.

Now given $a \notin \mathbb{Z}$ we must ask which normalized lattices Λ are possible. Since $a \in \Lambda$, we can write $a = r + s\gamma$ with $r, s \in \mathbb{Z}$. Changing the sign of a if necessary, we may assume that $s > 0$. Taking real and imaginary parts, it follows that

$$r = \Re(a) - s\,\Re(\gamma) \qquad \text{and} \qquad s = \Im(a)/\Im(\gamma)\,.$$

On the other hand, it follows easily from the Siegel inequalities

$$|\gamma| \ge 1\,, \qquad |\Re(\gamma)| \le 1/2\,, \qquad \Im(\gamma) > 0$$

that $\Im(\gamma) \ge \sqrt{3}/2$. Since a has been specified, this inequality yields an upper bound of $2\,\Im(a)/\sqrt{3}$ for s, and the inequality $|\Re(\gamma)| \le 1/2$ then yields an upper bound for $|r|$. Thus there are only finitely many possibilities for $\gamma = (a-r)/s$.

Next note that the product lattice $\mathcal{I} = s\,\Lambda = s\mathbb{Z} \oplus (a-r)\mathbb{Z}$ is contained in the ring $\mathbb{Z}[a]$, and is an ideal in this ring since $a\mathcal{I} \subset \mathcal{I}$. Clearly the torus $\mathbb{C}/\Lambda$ is isomorphic to $\mathbb{C}/\mathcal{I}$. If $\mathcal{I}'$ is another ideal in $\mathbb{Z}[a]$, note that $\mathbb{C}/\mathcal{I} \cong \mathbb{C}/\mathcal{I}'$ if and only if $\mathcal{I}' = c\mathcal{I}$ for some constant $c \ne 0$. Such a constant must belong to the quotient field $\mathbb{Q}[a]$, so by definition this means that $\mathcal{I}$ and $\mathcal{I}'$ represent the same ideal class. $\square$

For further discussion of maps with imaginary quadratic a see 7.2, 8.1 and 8.2 below. We next discuss the case $a \in \mathbb{Z}$.

Definition: A Lattès map

$$L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n \qquad \text{with} \qquad \mathcal{T} = \mathbb{C}/\Lambda$$

will be called *flexible* if we can vary Λ and L continuously so as to obtain other Lattès maps which are not conformally conjugate to it.

Lemma 5.5. *A Lattès map $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$ is flexible if and only if $n = 2$, and the affine map $L(\tau) = a\tau + b$ has integer derivative, $L' = a \in \mathbb{Z}$.*

Proof. This follows easily from 5.1 and 5.4. $\square$

We can easily classify such maps into a single connected family provided that the degree a^2 is even, and into two connected families when a^2 is odd, as follows. In each case, the coefficients a and b will remain constant but $\mathcal{T}$ will vary through all possible conformal diffeomorphism classes.

- **Maps with postcritical fixed point.** Let $\mathbb{H}$ be the upper half-plane, For each integer $a \ge 2$ there is a connected family of flexible Lattès maps of degree a^2 parametrized by the half-cylinder $\mathbb{H}/\mathbb{Z}$, as follows. Let $\mathcal{T}(\gamma)$ be the torus $\mathbb{C}/(\mathbb{Z} \oplus \gamma\mathbb{Z})$ where γ varies over $\mathbb{H}/\mathbb{Z}$, and let $L : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$ be the map $L(t) = at$. Then

$$L/G_2 : \mathcal{T}(\gamma)/G_2 \to \mathcal{T}(\gamma)/G_2$$

is the required smooth family of maps, with the image of $0 \in \mathcal{T}$ as ramified fixed point. If we restrict γ to the Siegel region (8), then we get a set of representative maps which are unique up to holomorphic conjugacy.

- **Maps without postcritical fixed point.** The construction in this case is identical, except that we take $L(t) = at + 1/2$. If a is even, this construction yields nothing new. In fact, the quotient group (13) of 5.1 is then trivial, and it follows that *every* Lattès map with $L' = a$ must have a postcritical fixed point. However, when a is odd, the period two orbits

$$0 \leftrightarrow 1/2\,, \qquad \gamma/2 \leftrightarrow (\gamma+1)/2$$

in $\mathcal{T}(\gamma)$ map to ramified period two orbits in $\mathcal{T}(\gamma)/G_2$, and there is no postcritical fixed point.

Caution: In this last case, we can no longer realize every conjugacy class of maps by restricting γ to the Siegel region. A larger fundamental domain is needed. For explicitly worked out examples in both cases, see equations (15), (18) and (19) below; and for further discussion see §7.

Here is another characterization.

Lemma 5.6. *A Lattès map is flexible if and only if the multiplier for every periodic orbit is an integer.*

Proof. This follows from Corollary 3.9. If $n > 2$ or if $a \notin \mathbb{Z}$, then we can find infinitely many integers $p > 0$ so that $\omega\, a^p \notin \mathbb{Z}$ for some $\omega \in G_n$. The number of fixed points of the map $\omega\, L^{\circ p}$ on the torus $\mathcal{T}$ grows exponentially with p (the precise number is $|\omega\, a^p - 1|^2$), and each of these maps to a periodic point of the associated Lattès map f. If we exclude the three or four postcritical points, then the derivative of $f^{\circ p}$ at such a point will be $\omega\, a^p$, so that the multiplier of this periodic orbit cannot be an integer. □

It seems very likely that power maps, Chebyshev maps, and flexible Lattès maps are the only rational maps such that the multiplier of every periodic orbit is an integer. (For a related result, see Lemma 7.1 below.)

§**6. Lattès Maps before Lattès.** Although the name of Lattès has become firmly attached to the construction studied in this paper, it actually occurs much earlier in the mathematical literature. Ernst Schröder, in a well known 1871 paper, first described "Chebyshev" type examples using trigonometric functions, and then gave an explicit one-parameter family of "Lattès" type examples as follows. Let $x = \mathrm{sn}(u)$ be the Jacobi sine function with elliptic modulus k, defined by the equation

$$u \;=\; \int_0^x \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}\ .$$

More explicitly, for any parameter $k^2 \neq 0, 1$, let $E_k \subset \mathbb{C}^2$ be the elliptic curve defined by the equation $y^2 = (1-x^2)(1-k^2x^2)$. Then the holomorphic differential dx/y is smooth and non-zero everywhere on E_k (even at the two points at infinity in terms of suitable local coordinates). The integrals $\oint dx/y$ around cycles in E_k generate a lattice $\Lambda \subset \mathbb{C}$, and the integral

$$u(x,y) = \int_{(0,1)}^{(x,y)} dx/y$$

along any path from $(0,1)$ to (x,y) in E_k is well defined modulo this lattice. In fact the resulting coordinate u parametrizes the torus $\mathcal{T} = \mathbb{C}/\Lambda$, and we can set $x = \mathrm{sn}(u)$ and $y = \mathrm{cn}(u)\,\mathrm{dn}(u)$. Here $\mathrm{sn}(u)$ is the Jacobi sine function, and $\mathrm{cn}(u)$ and $dn(u)$ are closely related doubly periodic meromorphic functions which satisfy

$$\mathrm{cn}^2(u) \;=\; 1-\mathrm{sn}^2(u) \qquad \text{and} \qquad \mathrm{dn}^2(u) \;=\; 1-k^2\mathrm{sn}^2(u)\ .$$

Furthermore

$$\mathrm{sn}(2u) \;=\; \frac{2\,\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{1-k^2\mathrm{sn}^4(u)}\ .$$

(Compare [WW, §22.2].) Setting $z = x^2 = \mathrm{sn}^2(u)$ it follows easily that there is a well defined rational function

$$f(z) \;=\; \frac{4z(1-z)(1-k^2z)}{(1-k^2z^2)^2} \tag{15}$$

of degree four which satisfies the semiconjugacy relation

$$\mathrm{sn}^2(2u) \;=\; f\Big(\mathrm{sn}^2(u)\Big) \;.$$

This is Schröder's example (modulo a minor misprint). In the terminology of §5, f is a "flexible Lattès map", described 47 years before Lattès.

It is not hard to see that $\mathrm{sn}(u)$ has critical values ± 1 and $\pm 1/k$, and hence that $\mathrm{sn}^2(u)$ has critical values 1, $1/k^2$, 0, and ∞. On the other hand the map f has three critical values 1, $1/k^2$ and ∞, which all map to the fixed point $0 = f(0)$. Each of these three critical values is the image under f of two distinct simple critical points.

Lucyan Böttcher in 1904 cited the same example (with a different version of the misprint). He was perhaps the first to think of this example from a dynamical viewpoint, and to use the term "chaotic" to describe the behavior of the sequence of iterates of f. In fact he described an orbit $z_0 \mapsto z_1 \mapsto \cdots$ as *chaotic* if for every convergent subsequence $\{z_{n_i}\}$ the differences $n_{i+1} - n_i$ are unbounded. (Note that this includes examples such as irrational rotations which are not chaotic in the modern sense.)

Böttcher actually cited a much earlier paper, written by Charles Babbage in 1815, for a fundamental property of what we now call semiconjugacies. For example, in order to find a periodic point $\psi^n x = x$ of a mapping ψ, Babbage proceeded as follows (see [Ba, p. 412]):

"*Assume as before* $\psi x = \phi^{-1} f \phi\, x$, *then*

$$\psi^2 x = \phi^{-1} f \phi\, \phi^{-1} f \phi\, x = \phi^{-1} f^2 \phi\, x$$
$$\psi^3 x = \phi^{-1} f^2 \phi\, \phi^{-1} f \phi\, x = \phi^{-1} f^3 \phi\, x \;,$$

and generally $\psi^n x = \phi^{-1} f^n \phi\, x$, *hence our equation becomes*

$$\phi^{-1} f^n \phi\, x = x \;. \qquad \cdots \text{''}$$

In modern terminology, we would say that ϕ is a *semiconjugacy* from ψ to f. It follows that any periodic point of ψ maps to a periodic point of f; and furthermore (assuming that ϕ is finite-to-one and onto) any periodic point of f is the image of a periodic point of ψ. Böttcher pointed out that the use of such an intermediary map ϕ to relate the dynamic properties of two maps ψ and f lies at the heart of Schröder's example.

J. F. Ritt carried out many further developments of these ideas in the 1920's. (For further historical information, see [A].)

§**7. More Recent Developments.** This concluding section will outline some of the special properties shared by some or all finite quotients of affine maps.

We first consider the class of *flexible* Lattès maps, as described in 5.5 and 5.6. These are the only known rational maps without attracting cycles which admit a continuous family of deformations preserving the topological conjugacy class. In fact the C^∞ conjugacy class remains almost unchanged as we deform the torus. Differentiability fails only at the postcritical points; and the multipliers of periodic orbits remain unchanged even at these postcritical points.

Closely related is the following:

Fundamental Conjecture. The flexible Lattès maps are the only rational maps which admit an "invariant line field" on their Julia set.

By definition f has an ***invariant line field*** if its Julia set J has positive Lebesgue measure, and if there is a measurable f-invariant field of real one-dimensional subspaces of the tangent bundle of $\widehat{\mathbb{C}}$ restricted to J. The importance of this conjecture is demonstrated by the following. (See [MSS], and compare the discussion in [Mc2] as well as [BM].)

Theorem of Mañé, Sad and Sullivan. *If this Fundamental Conjecture is true, then hyperbolicity is dense among rational maps. That is, every rational map can be approximated by a hyperbolic map.*

To see that every flexible Lattès map has such an invariant line field, note that any torus $\mathbb{C}/(\mathbb{Z}\oplus\gamma\mathbb{Z})$ is foliated by a family of circles $\Im(t) = \text{constant}$ which is invariant under the affine map L. If f is the associated Lattès map L/G_2, then this circle foliation maps to an f-invariant foliation of $J(f) = \widehat{\mathbb{C}}$ which is not only measurable but actually smooth, except for singularities at the four postcritical points.

Let us define the ***multiplier spectrum*** of a degree d rational map f to be the function which assigns to each $p \ge 1$ the unordered list of multipliers at the $d^p + 1$ (not necessarily distinct) fixed points of the iterate $f^{\circ p}$. Call two maps ***isospectral*** if they have the same multiplier spectrum.

Theorem of McMullen. *The flexible Lattès maps are the only rational maps which admit non-trivial isospectral deformations. The conjugacy class of any rational map which is not flexible Lattès is determined, up to finitely many choices, by its multiplier spectrum.*

This is proved in [Mc1, §2]. McMullen points out that the Lattès maps L/G_2 associated with imaginary quadratic number fields provide a rich source of isospectral examples which are not flexible. First note the following.

Lemma 7.1. *Two Lattès maps $L/G_2 : \mathcal{T}/G_2 \to \mathcal{T}/G_2$ are isospectral if and only if they have the same derivative $L' = a$, up to sign, and the same numbers of periodic orbits of various periods within the postcritical set P_f.*

Proof Outline. Let $\omega = \pm 1$. The number of fixed points of the map $\omega L^{\circ p}$ on the torus can be computed as $|\omega a^p - 1|^2$. These fixed points occur in pairs $\{\pm t\}$, and each such pair corresponds to a single fixed point of the corresponding iterate $f^{\circ p}$, where $f \cong L/G_2$. Whenever $t \ne -t$ on the torus, the multiplier of $f^{\circ p}$ at this fixed point is also equal to ωa^p. For the exceptional points $t = -t$ which are fixed under the action of G_2 and correspond to postcritical points of f, the multiplier of $f^{\circ p}$ is equal to a^{2p}. Thus, to determine the multiplier spectrum completely, we need only know how many points of various periods there are in the postcritical set. □

Example 7.2. Let $\xi = i\sqrt{k}$ where $k > 0$ is a square-free integer, and let $a = m\xi + n$. Then for each divisor d of m the lattice $\mathbb{Z}[\xi d] \subset \mathbb{C}$ is a $\mathbb{Z}[a]$-module. Hence the linear map $L(\tau) = a\tau$ acts on the associated torus $\mathcal{T}/\mathbb{Z}[\xi d]$. If m is highly divisible, then there are many possible choices for d. Suppose, to simplify the discussion, that mk and n are both even, so that a^2 is divisible by two in $\mathbb{Z}[a]$. Then multiplication by a^2 acts as the zero map on the group consisting of elements of order two in $\mathcal{T}$. Thus $0 = L(0)$ is the only periodic point under this action, hence the image of 0 in $\mathcal{T}/G_2$ is the only postcritical periodic point of L/G_2. It then follows from 7.1 that these examples are all isospectral.

Berteloot and Loeb [BL1] have characterized Lattès maps in terms of the dynamics and geometry of the associated homogeneous polynomial map of $\mathbb{C}^2$. Every rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree two or more lifts to a homogeneous polynomial map $F : \mathbb{C}^2 \to \mathbb{C}^2$ of the same degree with the origin as an attracting fixed point. They show that f is Lattès if and only if the boundary of the basin of the origin under F is smooth and strictly pseudoconvex within some open set. In fact, this boundary must be spherical in suitable local holomorphic coordinates, except over the postcritical set of f.

Anna Zdunik [Z] has characterized Lattès maps using only measure theoretic properties. It is not hard to see that the standard probability measure on the flat torus pushes forward under Θ to an ergodic f-invariant probability measure on the Riemann sphere. This measure is smooth and in fact real analytic, except for mild singularities at the postcritical points. Furthermore, it is *balanced*, in the sense that the preimage $f^{-1}(U)$ of any simply connected subset of $\widehat{\mathbb{C}} \smallsetminus P_f$ is a union of n disjoint sets of equal measure. Hence it coincides with the unique measure of maximal entropy, as constructed by Lyubich [Ly], or by Freire, Lopez and Mãñé [FLM], [Ma]. The converse theorem is much more difficult:

Theorem of Zdunik. *The Lattès maps are the only rational maps for which the measure of maximal entropy is absolutely continuous with respect to Lebesgue measure.*

We can think of the maximal entropy measure $\mu_{\max}$ as describing the asymptotic distribution of random backward orbits. That is, if we start with any non-exceptional initial point z_0, and then use a fair d-sided coin or die to iteratively choose a backward orbit

$$\cdots \mapsto z_{-2} \mapsto z_{-1} \mapsto z_0 \;,$$

then $\{z_n\}$ will be equidistributed with respect to $\mu_{\max}$. This measure $\mu_{\max}$ always exists. An absolutely continuous invariant measure is much harder to find, and an invariant measure which is ergodic and belongs to the same measure class as Lebesgue measure is even rarer. However Lattès maps are not the only examples: Mary Rees [Re] has shown that for every degree $d \geq 2$ the moduli space of degree d rational maps has a subset of positive measure consisting of maps f which have an ergodic invariant measure μ in the same measure class as Lebesgue measure. Such a measure is clearly unique, since Lebesgue almost every forward orbit $z_0 \mapsto z_1 \mapsto z_2 \cdots$ must be equidistributed with respect to μ.

Using these ideas, an easy consequence of Zdunik's Theorem is the following.

Corollary. *A Lebesgue randomly chosen forward orbit for a Lattès map has the same asymptotic distribution as a randomly chosen backward orbit.*

I don't know whether Lattès maps are the only ones with this property.

In general, different rational maps have different invariant measures, except that every invariant measure for f is also an invariant measure for its iterates $f^{\circ p}$. However, every Lattès map L/G_n shares its measure $\mu_{\max}$ with a rich collection of Lattès maps $\widehat{L}/G_n$

where $\widehat{L}$ ranges over all affine maps of the torus which commute with the action of G_n. This collection forms a semigroup which is not finitely generated. (If we consider only the linear torus maps $\widehat{L}(\tau) = a\tau$, then we obtain a commutative semigroup.) I don't know any other examples, outside of the Chebyshev and power maps, of a semigroup of rational maps which is not finitely generated, and which shares a common non-atomic invariant measure. (See [LP] for related results.)

Closely related is the study of commuting rational maps. Following a terminology introduced much later by Veselov [V], let us call a rational map f *integrable* if it commutes with another rational map, $f \circ g = g \circ f$, where both f and g have degree at least two, and where no iterate of f is equal to an iterate of g.

> **Theorem of Ritt and Eremenko.** *A rational map f of degree $d_f \geq 2$ is integrable if and only if it is a finite quotient of an affine map; that is if and only if it is either a Lattès, Chebyshev, or power map. Furthermore, the commuting map g must have the same Julia set, the same flat orbifold metric, the same measure of maximal entropy, and the same set of preperiodic points as f.*

This is a modern formulation of a statement which was proved by Ritt [R2] in 1923, and by Eremenko [E] using a quite different method in 1989. For higher dimensional analogues, see [DS]. In fact there has been a great deal of interest in higher dimensional analogues of Lattès maps in recent years. Compare [BL2], [Di], [Du], [V].

§**8. Examples.** This concluding section will provide explicit formulas for some particular Lattès maps.

8.1. Degree Two Lattès Maps. Recall from Lemma 5.4 and equation (14) that the derivative $L' = a$ for a torus map of degree d must either be a (rational) integer, so that $d = a^2$, or must be an imaginary quadratic integer of the form $a = \big(q \pm \sqrt{q^2 - 4d}\big)/2$ with $q^2 < 4d$, satisfying $a^2 - q\,a + d = 0$ and $|a|^2 = d$. Furthermore, replacing a by $-a$ if necessary, we may assume that $q \geq 0$. Thus, in the degree two case, the only distinct possibilities are $q = 0, 1, 2$, with

$$a \;=\; i\sqrt{2}\,, \qquad \text{or} \qquad a \;=\; (1 \pm i\sqrt{7})/2\,, \qquad \text{or} \qquad a \;=\; 1 \pm i\,.$$

In each of these cases, the associated torus $\mathcal{T} = \mathbb{C}/\Lambda$ is necessarily conformally isomorphic to the quotient $\mathbb{C}/\mathbb{Z}[a]$. In fact we can assume that $\Lambda = \mathbb{Z} \oplus \gamma\mathbb{Z}$ with γ in the Siegel region (8), and set $a = r + s\gamma$ with $r, s \in \mathbb{Z}$. Let us assume, to fix our ideas, that $\Im(a) > 0$. Then, arguing as in the proof of 5.4, we have

$$0 \;<\; s \;\leq\; \frac{2\,\Im(a)}{\sqrt{3}} \;\leq\; \frac{2|a|}{\sqrt{3}} \;=\; \frac{2\sqrt{2}}{\sqrt{3}} \;\approx\; 1.63\,.$$

Therefore $s = 1$, hence $a \equiv \gamma \pmod{\mathbb{Z}}$; so the lattice Λ must be equal to $\mathbb{Z}[a]$.

First suppose that $f \cong L/G_2$ is a Lattès map of type $\{2, 2, 2, 2\}$, with $L(t) = at + b$. The four points of the form $\lambda/2$ in $\mathcal{T}$ map to the four postcritical points of f. Hence the action of the Lattès map f on its postcritical set is mimicked by the action of L on this group of elements of the form $\lambda/2$ in $\mathcal{T}$, or equivalently by the action of $t \mapsto at + 2b$ on the four element group $\mathbb{Z}[a]/2\mathbb{Z}[a]$. A brief computation shows that the quotient group $\mathbb{Z}[a]/\big(2\mathbb{Z}[a] + (a-1)\mathbb{Z}[a]\big)$ of Theorem 5.1 is trivial when q is even but has two elements when q is odd. Thus, in the two cases $a = i\sqrt{2}$ and $a = 1+i$ where q is even, we may assume that $L(t) = at$. In these cases, the equation $a^2 - qa + 2 = 0$ implies that $a^2 \equiv 0 \mod 2\mathbb{Z}[a]$, and hence that

$$1 \mapsto a \mapsto 0 \qquad \text{and} \qquad 1 + a \mapsto a \mapsto 0$$

in this quotient group. Thus in these two cases there is a unique postcritical fixed point, which must have mutiplier a^2 by 3.9. In fact, the diagram of critical and postcritical points for the Lattès map f necessarily has the following form.

$$\begin{array}{ccccccccc} & & & & \odot & & & & \\ & & & & \uparrow & & & & \\ * & \mapsto & \bullet & \mapsto & \bullet & \mapsfrom & \bullet & \mapsfrom & * \end{array}$$

Here each star stands for a simple critical point, each heavy dot stands for a ramified (or postcritical) point, and the heavy dot with a circle around it stands for a postcritical fixed point. If we put the two critical points at ± 1 and put the postcritical fixed point at infinity, then f will have the form

$$f(z) \;=\; (z + z^{-1})/a^2 \,+\, c$$

for some constant c . In fact we easily derive the forms

$$\begin{array}{lclcl} f(z) & = & -(z+z^{-1})/2 + \sqrt{2} & \quad\text{when}\quad & a = i\sqrt{2}\ , \quad \text{and} \\ f(z) & = & \pm(z+z^{-1})/2i & \quad\text{when}\quad & a = 1 \pm i\ . \end{array}$$

On the other hand, for $a = (1 \pm i\sqrt{7})/2$, a similar argument shows that there are two possible critical orbit diagrams, as follows. Either:

$$* \mapsto \bullet \mapsto \odot \qquad * \mapsto \bullet \mapsto \odot$$

with two postcritical fixed points, or

$$* \mapsto \bullet \mapsto \bullet \longleftrightarrow \bullet \mapsfrom \bullet \mapsfrom *$$

with no postcritical fixed points. In the first case, if we put the postcritical fixed points at zero and infinity, and another fixed point at $+1$, then the map takes the form

$$f(z) \;=\; z\frac{z + a^2}{a^2 z + 1}\ .$$

This commutes with the involution $z \mapsto 1/z$, and we can take the composition

$$z \;\mapsto f(1/z) \;=\; 1/f(z) \;=\; \frac{az^2 + 1}{z(z + a^2)}$$

as the other Lattès map with the same value of a , but with $\{0, \infty\}$ as postcritical period two orbit. (See [M5, §B.3] for further information on these maps.[6])

We can also ask for Lattès maps of degree two of the form L/G_n with $n > 2$. However, only the type $\{2, 4, 4\}$ with n=4 can occur, since, of the lattices $\mathbb{Z}[a]$ described above, only $Z[1+i] = \mathbb{Z}[i]$ admits a rotation of order greater than 2 . In fact the rotation $t \mapsto it$ of the torus $\mathbb{C}/\mathbb{Z}[i]$ corresponds to an involution $z \mapsto -z$ which commutes with the associated

[6] Caution: In both [M3, 2000] and [M5], the term "Lattès map" was used with a more restricted meaning, allowing only maps of type $\{2, 2, 2, 2\}$ with $n = 2$.

Lattès map $f(z)=(z+z^{-1})/2i$. To identify z with $-z$, we can introduce the new variable $w=z^2$ and set

$$g(w) \;=\; g(z^2) \;=\; f(z)^2 \;=\; -(z^2+2+z^{-2})/4 \;=\; -(w+2+w^{-1})/4\ .$$

Up to holomorphic conjugacy, this is the unique degree two Lattès map of the form L/G_4 . Its critical points ± 1 have orbit $1 \mapsto -1 \mapsto 0 \mapsto \infty \circlearrowleft$, so that -1 is both a critical point and a critical value, yielding the following schematic diagram.

$$\begin{array}{ccccccc} * & \mapsto & * & \mapsto & \bullet & \mapsto & \odot \\ & & 2 & & 4 & & 4 \end{array}$$

Here the ramification index is indicated underneath each ramified point. Thus the map has type $\{2,4,4\}$, as expected. (Alternatively, the map $z\mapsto 1-2/z^2$, with critical points zero and infinity and with critical orbit $0 \mapsto \infty \mapsto 1 \mapsto -1 \circlearrowleft$, could also be used as a normal form for this same conjugacy class.)

8.2. Degree Three. If the torus map $L(t)=at+b$ has degree $|a|^2=3$, then according to equation (14) we can write $a=\big(q\pm\sqrt{q^2-12}\big)/2$ for some integer q with $q^2<12$ or in other words $|q|\le 3$. I will try to analyze only a single case, choosing $q=0$ with $a=i\sqrt{3}$ so that $a^2=-3$.

For this choice $a=i\sqrt{3}$, setting $a=r+s\gamma$ as in the proof of 5.4, we find that s can be either one or two, and it follows easily that there are exactly two essentially distinct tori which admit an affine map L with derivative $L'=a$. We can choose either the hexagonally symmetric torus $\mathcal{T}=\mathbb{C}/\mathbb{Z}[\omega_6]=\mathbb{C}/\mathbb{Z}[(a+1)/2]$, or its 2-fold covering torus $\mathcal{T}'=\mathbb{C}/\mathbb{Z}[a]$.

For the torus $\mathbb{C}/\mathbb{Z}[a]$, since there is no G_3 or G_4 symmetry, we are necessarily in the case $n=2$. A brief computation shows that the quotient group $\mathbb{Z}[a]/(2\mathbb{Z}[a]+(a-1)\mathbb{Z}[a])$ of Theorem 5.1 has two elements, so there are two possible Lattès maps, corresponding to the two affine maps $L(t)=at$ and $L(t)=at+1/2$. The corresponding critical orbit diagrams have the form

$$* \mapsto \odot \qquad * \mapsto \odot \qquad * \mapsto \bullet \leftrightarrow \bullet \mapsfrom *$$

with two postcritical fixed points, and

$$\begin{array}{ccccccc} * & \mapsto & \bullet & \mapsto & \bullet & \mapsfrom & * \\ & & \uparrow & & \downarrow & & \\ * & \mapsto & \bullet & \mapsfrom & \bullet & \mapsfrom & * \end{array}$$

with no postcritical fixed points. In the first case, if we place the postcritical fixed points at zero and infinity, and place a fixed point with multiplier $+a$ at $+1$, then the map takes the form

$$f(z) \;=\; \frac{z(z-a)^2}{(az-1)^2}\ .$$

The remaining fixed point then lies at -1 and has multiplier $-a$. The two remaining critical points $\pm 2i-a$ map to the period two postcritical orbit $-2i+a \longleftrightarrow 2i+a$.

We can construct the other Lattès maps with the same a and the same lattice $\mathbb{Z}[a]$ by composing f with the Möbius involution g which satisfies

$$g\ :\ 0 \leftrightarrow 2i+a\ , \qquad g\ :\ \infty \leftrightarrow -2i+a\ .$$

The critical orbit diagram for this composition permutes the four postcritical points cyclically, as required.

A beautifully symmetric example. Now consider the torus $\mathcal{T} = \mathbb{C}/\mathbb{Z}[\omega_6]$. As noted at the end of §4, the quotient $\mathcal{T}/G_2$, with its flat orbifold metric, is isometric to a regular tetrahedron with the four cone points as vertices. Again there are two distinct Lattès maps with invariant $a^2 = -3$. The map $L(t) = at$ induces a highly symmetric piecewise linear map L/G_2 of this tetrahedron. (Compare [DMc] for a discussion of symmetric rational maps.) The four vertices are postcritical fixed points of this map, and the midpoints of the four faces are the critical points, each mapping to the opposite vertex. Thus the critical orbit diagram has the following form.

$$* \mapsto \odot \qquad * \mapsto \odot \qquad * \mapsto \odot \qquad * \mapsto \odot$$

Similarly, the midpoint of each edge maps to the midpoint of the opposite edge, forming a period two orbit.

If we place these critical points on the Riemann sphere at the cube roots of -1 and at infinity, then this map takes the form

$$f(z) \;=\; \frac{6\,z}{z^3-2}\,, \tag{16}$$

with a critical orbit $\omega \;\mapsto\; -2\omega \circlearrowleft$ whenever $\omega^3 = -1$, and also $\infty \;\mapsto\; 0 \circlearrowleft$.

The affine map $L(t) = at + 1/2$ yields a Lattès map L/G_2 with the same critical and postcritical points, but with the following critical orbit diagram.

$$* \mapsto \bullet \leftrightarrow \bullet \leftarrow\!\shortmid * \qquad * \mapsto \bullet \leftrightarrow \bullet \leftarrow\!\shortmid *$$

Such a map can be constructed by composing the map f of (16) with the Möbius involution

$$g(z) = (2-z)/(1+z)$$

which satisfies $-1 \;\leftrightarrow\; \infty$ and $\omega_6 \;\leftrightarrow\; \overline{\omega}_6$. This corresponds to a 180° rotation of the tetrahedron about an axis joining the midpoints of two opposite faces.

Now consider Lattès maps L/G_n with $n \ge 3$ and with $a = i\sqrt{3}$. Evidently the lattice must be $\mathbb{Z}[\omega_6]$, and n must be either 3 or 6, so the type must be either $\{3,3,3\}$ or $\{2,3,6\}$. Using 5.2, we can easily check that there is just one possible map in each case, corresponding to the linear map $L(t) = at$. Since both G_2 and G_3 are subgroups of G_6 , this torus map $L(t) = at$ gives rise to maps of type $\{2,2,2,2\}$ and $\{3,3,3\}$ and $\{2,3,6\}$ which are related by the commutative diagram

$$\begin{array}{ccc} L & \mapsto & L/G_2 \\ \downarrow & & \downarrow \\ L/G_3 & \mapsto & L/G_6\,. \end{array}$$

Here L/G_2 is the "beautifully symmetric example" of equation (16). The corresponding Lattès map L/G_6 of type $\{2,3,6\}$ can be constructed from (16) by identifying each z with $\omega\,z$ for $\omega \in G_3$. If we introduce the new variable $\zeta = z^3$, then the corresponding map $L/G_6 = f/G_3$ is given by mapping $\zeta = z^3$ to $g(\zeta) = f(z)^3$, so that

$$g(\zeta) \;=\; \left(\frac{6\,z}{z^3-2}\right)^3 \;=\; \frac{6^3\,\zeta}{(\zeta-2)^3}\,. \tag{17}$$

The three critical points at the cube roots of -1 now coalesce into a single critical point at

-1 , with $g(-1)=g(8)=8$. There is still a critical point at infinity with $g(\infty)=g(0)=0$. But now infinity is also a critical value. In fact there if a double critical point at $\zeta=2$, with $g(2)=\infty$. The corresponding diagram for the critical and postcritical points $2\mapsto\infty\mapsto 0$ and $-1\mapsto 8$ takes the form

$$\begin{array}{ccccccccc} ** & \mapsto & * & \mapsto & \odot & \qquad\qquad & * & \mapsto & \odot \\ & & 3 & & 6 & & & & 2 \end{array}$$

where the symbol $**$ stands for a critical point of multiplicity two. The multipliers at the two postcritical fixed points are $a^6=-27$ and $a^2=-3$ respectively.

Similarly we can study the Lattès map L/G_3 . In this case the three points of $\mathcal{T}$ which are fixed by G_3 all map to zero. Thus the three cone points of the orbifold $\mathcal{T}/G_3$ all map to one of the three. The corresponding diagram has the following form.

$$\begin{array}{ccccccccc} ** & \mapsto & \bullet & \mapsto & \odot & \longleftarrow\!\shortmid & \bullet & \longleftarrow\!\shortmid & ** \\ & & 3 & & 3 & & 3 & & \end{array}$$

If we put the critical points at zero and infinity, and the postcritical fixed point at $+1$ (compare [M4]), then this map takes the form

$$f(z) \;=\; \frac{z^3+\omega_3}{\omega_3 z^3+1}\;,$$

with critical orbits $0\mapsto\omega_3\mapsto 1 \circlearrowleft$, and $\infty\mapsto\overline{\omega}_3\mapsto 1\circlearrowleft$. In contrast to L/G_2 and L/G_6 , this cannot be represented as a map with real coefficients. In fact the invariant $a^3=-i\sqrt{27}$ is not a real number, so this f is not holomorphically conjugate to its complex conjugate or mirror image map. (For a similar example with $a^3=-8$ which occurs in the study of rational maps of the projective plane, see [BDM, §4 or §6].)

Note that f commutes with the involution $z\mapsto 1/z$. If we identify z with $1/z$ by introducing a new variable $w=z+1/z$, then we obtain a different model for L/G_6 , which is necessarily conformally conjugate to (17).

8.3. Flexible Lattès maps. Recall from §5 that there is just one connected family of flexible Lattès maps of degree a^2 for each even integer a , but that there are two distinct families of degree a^2 when a is odd. For $a^2=4$, the Schröder family (15), constructed by expressing $\mathrm{sn}^2(2t)$ as a rational function of $\mathrm{sn}^2(t)$, exhausts all of the possibilities. This family depends on a parameter $k^2\in\mathbb{C}\smallsetminus\{0,1\}$ and has postcritical set $\{0\,,\,1\,,\,\infty\,,\,1/k^2\}$, with all postcritical points mapping to the fixed point zero. Using the corresponding formula for $\mathrm{sn}^2(3t)$ and following Schröder's method, we obtain the family

$$f(z) \;=\; \frac{z\,(k^4z^4-6k^2z^2+4(k^2+1)z-3)^2}{(3k^4z^4-4k^2(k^2+1)z^3+6k^2z^2-1)^2} \tag{18}$$

of degree nine Lattès maps, with the same postcritical set $\{0\,,\,1\,,\,\infty\,,\,1/k^2\}$, but with all postcritical points fixed by f . Note that f commutes with the involution $z\mapsto 1/k^2z$ which permutes the postcritical points. Composing f with this involution, we obtain a different family

$$z\;\mapsto\;\frac{1}{k^2f(z)}\;=\;f\left(\frac{1}{k^2z}\right)\;=\;\frac{(3k^4z^4-4k^2(k^2+1)z^3+6k^2z^2-1)^2}{k^2z((k^4z^4-6k^2z^2+4(k^2+1)z-3)^2} \tag{19}$$

with the same postcritical set, but with all postcritical orbits of period two. Higher degree examples could be worked out by the same method. Presumably they look much like the degree four case for even degrees, and much like the degree nine case for odd degrees.

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