CHAPTER IV. DEGREE ONE CIRCLE MAPS

This chapter will describe one of the most classical dynamical problems. For more detailed presentations, the reader is referred to [Alseda, Llibre, Misiurewicz] or to [de Melo, van Strien].

§14. The Rotation Number.

First recall some standard definitions. The notation $\mathbb{R}/\mathbb{Z}$ will be used for the circle of real numbers modulo $\mathbb{Z}$. The image of $x \in \mathbb{R}$ under the natural projection map from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$ will be written as $\xi = (x \bmod \mathbb{Z})$, and called the *residue class* of x modulo $\mathbb{Z}$. Note that $\mathbb{R}/\mathbb{Z}$ can be identified with the unit circle in the complex plane under the correspondence

$$\xi \;\leftrightarrow\; e^{2\pi i \xi} .$$

Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be any *circle map*, that is any continuous function from the circle to itself. By a *lift* of f we will mean a map $F : \mathbb{R} \to \mathbb{R}$ from the real line to itself which satisfies the identity $(F(x) \bmod \mathbb{Z}) = f(x \bmod \mathbb{Z})$, so that the following diagram commutes

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{R}/\mathbb{Z} & \xrightarrow{f} & \mathbb{R}/\mathbb{Z} . \end{array}$$

Here both vertical arrows map x to $(x \bmod \mathbb{Z})$.

Lemma 14.1. *Every circle map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ possesses a lift $F : \mathbb{R} \to \mathbb{R}$, which is unique up to the addition of a constant integer to the function F. Such a lift necessarily satisfies the identity*

$$F(x+1) \;=\; F(x) + d ,$$

where d is a constant integer called the **degree** *of f.*

The proof will be left to the reader. □

This chapter will concentrate on circle maps of degree $+1$. (For circle maps of degree $d \neq 1$, see §4E or §13B, as well as Problem 4-e.) In fact the cases where f is a degree one homeomorphism, so that F is strictly monotone, are of particular interest. However, we first study the more general case where F is not required to be monotone.

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§14A. Basic Properties. Let f be a circle map of degree $+1$, and let $F : \mathbb{R} \to \mathbb{R}$ be any lift of f. Then F commutes with the unit translation, $F \circ T = T \circ F$ where $T(x) = x + 1$. It follows that F commutes with all integer translations:

$$F(x+n) \;=\; F(x) + n \qquad \text{for every} \qquad n \in \mathbb{Z} . \tag{14 : 1}$$

Let $F : x_0 \mapsto x_1 \mapsto x_2 \mapsto \cdots$ be an arbitrary orbit under such a map F.

Definition. If the limit

$$\mathbf{tn}(F\,,\,x_0) \;=\; \mathbf{tn}(F\,,\,x_0+1) \;=\; \lim_{n\to\infty} \frac{x_n - x_0}{n} \;=\; \lim_{n\to\infty} \frac{x_n}{n} \;\in\; \mathbb{R}$$

exists, then this limit will be called the *translation number of the orbit*, or the translation number of F at the point x_0, or at $(x \bmod \mathbb{Z})$. If $\mathbf{tn}(F, x_0)$ exists and is independent of x_0, then it will be called simply the *translation number* of the map F, written $\mathbf{tn}(F)$.

If $\mathbf{tn}(F, x_0)$ exists, then it is easy to check that the translation number at x_0 for an arbitrary lift $x \mapsto F(x) + m$ of the degree one circle map f is given by

$$\mathbf{tn}(F + m\,,\, x_0) \;=\; \mathbf{tn}(F\,,\, x_0) + m\;. \qquad (14:2)$$

Hence the image of $\mathbf{tn}(F\,,\, x_0)$ in $\mathbb{R}/\mathbb{Z}$ depends only on the circle map f and not on the particular choice of lift. By definition, this image is called the Poincaré *rotation number* $\mathbf{rn}(f\,,\, \xi_0) \in \mathbb{R}/\mathbb{Z}$ for the orbit $\xi_0 \mapsto \xi_1 \mapsto \cdots$ of this circle map, where $\xi_i = (x_i \bmod \mathbb{Z})$. Similarly, if $\mathbf{tn}(F, x_0)$ exists and is independent of x_0, then the *rotation number* of the degree one circle map is a well defined element $\mathbf{rn}(f) \in \mathbb{R}/\mathbb{Z}$. As an elementary example, for any constant $c \in \mathbb{R}/\mathbb{Z}$ the rigid rotation $r^c(\xi) = \xi + c$ clearly has rotation number $\mathbf{rn}(r^c) = c$.

However, in general the limit $\mathbf{tn}(F, x)$ may fail to exist. Hence we introduce the notations

$$\mathbf{tn}^-(F\,,\, x_0) \;=\; \liminf_{n\to\infty} \frac{x_n - x_0}{n}\;, \qquad \mathbf{tn}^+(F\,,\, x_0) \;=\; \limsup_{n\to\infty} \frac{x_n - x_0}{n}\;,$$

where $x_n = F^{\circ n}(x_0)$. These will be called the *lower* and *upper* translation numbers for the given orbit. The smallest and largest values will be denoted by

$$\mathbf{tn}^-(F) \;=\; \inf_{x\in\mathbb{R}} \mathbf{tn}^-(F, x)\;, \qquad \mathbf{tn}^+(F) \;=\; \sup_{x\in\mathbb{R}} \mathbf{tn}^+(F, x)\;,$$

and the closed interval

$$\mathbf{TI}(F) \;=\; [\mathbf{tn}^-(F)\,,\, \mathbf{tn}^+(F)]$$

will be called the *translation interval* for the map F. Evidently this interval $\mathbf{TI}(F)$ reduces to a single point if and only if the translation number $\mathbf{tn}(F, x)$ is well defined and independent of x.

We can bound the size of the translation interval as follows. The difference

$$\delta(x) \;=\; F(x) - x$$

will be called the *displacement* of x under F. Since $F(x+1) = F(x)+1$, this displacement is a periodic function, $\delta(x+1) = \delta(x)$. Hence $\delta(x)$ attains a finite maximum value and a finite minimum value. I will write briefly

$$\max\delta \;=\; \max_x\; \delta(x) \qquad \text{and} \qquad \min\delta \;=\; \min_x\; \delta(x)\;.$$

Note that the translation number can be defined as

$$\mathbf{tn}(F, x_0) \;=\; \lim_{n\to\infty} \frac{\delta(x_0) + \delta(x_1) + \cdots + \delta(x_{n-1})}{n}$$

when this limit exists. It can be described intuitively as the *average displacement* of a point on the orbit of x_0.

Lemma 14.2. *The inequalities*

$$\min\delta \;\le\; \mathbf{tn}^-(F) \;\le \mathbf{tn}^+(F) \;\le\; \max\delta$$

are always satisfied. Hence the translation interval $\mathbf{TI}(F) \subset [\min\delta\,,\, \max\delta]$ *is a (possibly degenerate) finite interval of real numbers.*

Proof. Since

$$\frac{x_n - x_0}{n} \;=\; \frac{\delta(x_0) + \delta(x_1) + \cdots + \delta(x_{n-1})}{n} \;\le\; \max_x \delta(x) \;,$$

we have $\mathbf{tn}^+(F\,,\,x_0) \;\le\; \max\,\delta$, and therefore $\mathbf{tn}^+(F) \;\le\; \max\,\delta$, with similar inequalities for the lower translation number. $\square$

Corollary 14.3. *For any such* F *, we have either:*

$$\begin{array}{lllll} 0 \;<\; \delta(x) & \text{for all} \;\; x & \iff & 0 < \mathbf{tn}^-(F)\;, & \text{or} \\ 0 \;=\; \delta(x_0) & \text{for some} \;\; x_0 & \iff & 0 \in \mathbf{TI}(F)\;, & \text{or else} \\ 0 \;>\; \delta(x) & \text{for all} \;\; x & \iff & 0 > \mathbf{tn}^+(F)\;. & \end{array}$$

Furthermore,

$$\begin{array}{llll} \textit{if} & \min\,\delta = 0\;, & \textit{then} & \mathbf{tn}^-(F) = 0\;, \quad \textit{and} \\ \textit{if} & \max\,\delta = 0\;, & \textit{then} & \mathbf{tn}^+(F) = 0\;. \end{array}$$

The proof is immediate. For example if $\min\,\delta = 0$, then $0 \le \mathbf{tn}^-(F)$ by 14.2, and equality must hold since F has a fixed point. Compare Figure 45(b). $\square$

Lemma 14.4. *The translation numbers for the* q*-fold iterate of* F *at a point are given by*

$$\mathbf{tn}^-(F^{\circ q}\,,\,x_0) \;=\; q\,\mathbf{tn}^-(F\,,\,x_0) \qquad \textit{and} \qquad \mathbf{tn}^+(F^{\circ q}\,,\,x_0) \;=\; q\,\mathbf{tn}^+(F\,,\,x_0)\;,$$

or in other words

$$\mathbf{TI}(F^{\circ q}) \;=\; q\,\mathbf{TI}(F)\;.$$

Proof. For fixed $q \ge 1$, note that any positive integer n can be written as $kq + \ell$ with $0 \le \ell < q$. Thus, as $n \to \infty$ and hence $k \to \infty$, we have an asymptotic equality $n \sim kq$. However, the difference $x_n - x_{kq}$ clearly remains bounded, hence the difference between the ratio $(x_{kq} - x_0)/(kq) \sim (x_{kq} - x_0)/n$ and the ratio $(x_n - x_0)/n$ tends to zero. Therefore these two ratios have the same *lim sup* and the same *lim inf* as $n \to \infty$. The conclusion follows easily. $\square$

Note: If F is a homeomorphism, then this result holds also for negative values of q (Problem 14-b).

As an example, suppose that $f^{\circ q}(\xi) = \xi$ is a periodic point for the degree one circle map f . Lifting $\xi \in \mathbb{R}/\mathbb{Z}$ to $x \in \mathbb{R}$, and lifting f to the map $F : \mathbb{R} \to \mathbb{R}$, it follows that $F^{\circ q}(x) = x + p$ for some integer p . We will say briefly that x is periodic (mod $\mathbb{Z}$).

Lemma 14.5. *If* x_0 *is periodic* (mod $\mathbb{Z}$) *, with* $F^{\circ q}(x_0) = x_0 + p$ *, then the translation number* $\mathbf{tn}(F, x_0)$ *at* x_0 *is well defined and is equal to the rational number* p/q *.*

(Here p/q need not be a fraction in lowest terms. Compare 14.7.)

Proof of 14.5. It is easy to check that $\mathbf{tn}(F^{\circ q}\,,\,x_0)$ is well defined and equal to $p \in \mathbb{Z}$, so it follows from 14.4 that $\mathbf{tn}(F, x_0)$ is also well defined and is equal to p/q . $\square$

We can now extend the Corollary 14.3 as follows. Let

$$\delta_q(x) \;=\; F^{\circ q}(x) \,-\, x\;.$$

Theorem 14.6. *Given integers* $q>0$ *and* p *, we have either:*

$$\begin{array}{llll} p < \delta_q(x) & \text{for all} \quad x & \Longleftrightarrow & p/q < \mathbf{tn}^-(F)\ , \qquad \text{or} \\ p = \delta_q(x_0) & \text{for some} \quad x_0 & \Longleftrightarrow & p/q \in \mathbf{TI}(F)\ , \qquad \text{or else} \\ p > \delta_q(x) & \text{for all} \quad x & \Longleftrightarrow & p/q > \mathbf{tn}^+(F)\ . \end{array}$$

Furthermore,

$$\begin{array}{lll} \text{if} \quad \min \delta_q = p\ , & \text{then} \quad \mathbf{tn}^-(F) = p/q\ , & \text{and} \\ \text{if} \quad \max \delta_q = p\ , & \text{then} \quad \mathbf{tn}^+(F) = p/q\ . & \end{array}$$

Proof. This follows immediately from 14.4 and 14.5, making use of (14 : 2). □

Remark 14.7. It follows from 14.6 that any rational number p/q in the translation interval can actually be realized as a well defined translation number $\mathbf{tn}(F, x_0)$ for some choice of x_0 . In fact the same statement is true for irrational points in the translation interval. Compare 16.3 below. A map F with rational translation number p/q may well have points which not periodic (mod $\mathbb{Z}$), and may even have "non-primitive" periodic points, where the period is some multiple, strictly greater than q . In other words, the associated map $G(x) = F^{\circ q}(x) - p$ may have periodic points of period >1 . However, in that case, it follows from 14.6 (and is easy to check directly) that G must also have a fixed point:

> **Assertion.** *If* p/q *is a fraction in lowest terms, then* p/q *belongs to the translation interval* $\mathbf{TI}(F)$ *if and only if the map* F *admits an orbit which is periodic* (mod $\mathbb{Z}$) *with period exactly* q *and with translation number* p/q *.*

As an example, consider the map $F(x) = x - \sin(2\pi x)/2$ of Figure 45(a). Then F maps the interval $[-1/2\,,\,1/2]$ onto itself (but not homeomorphically). It follows easily that the translation number $\mathbf{tn}(F)$ is well defined and equal to 0 . In this case, the points $-1/4 \leftrightarrow 1/4$ form a period two orbit with translation number $0/2 = 0$, while $x=0$ is a fixed point with the same translation number $0/1 = 0$.

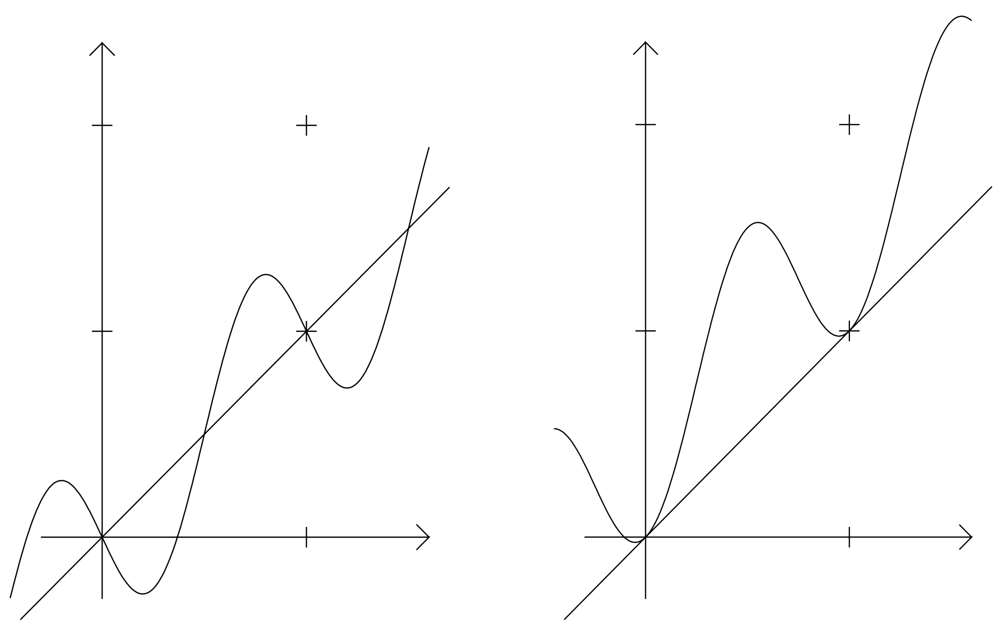


Figure 45. (**a**) *Graph of* $x - \sin(2\pi x)/2$. (**b**) *Graph of* $x + (1 - \cos(2\pi x))/2$.

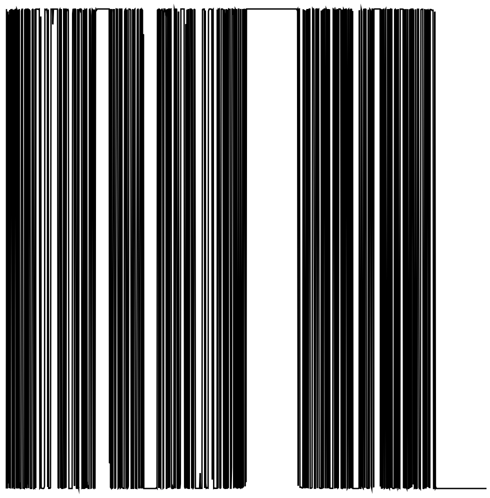
Figure 45(b) shows a similar looking example with quite different dynamic behavior. Let

$$F(x) \;=\; x + (1-\cos(2\pi x))/2\ .$$

Then again $F(0) = 0$ and hence $\mathbf{tn}(F, 0) = 0$. However, for this map we have $F : 0.5 \mapsto 1.5 \mapsto 2.5 \mapsto \cdots$, hence $\mathbf{tn}(F, 0.5) = 1$. Since the displacement,

$$\delta(x) \;=\; F(x) - x \;=\; (1-\cos(2\pi x))/2$$

has minimum zero and maximum $+1$, it follows from 14.2 that the translation interval $\mathbf{TI}(F)$ must be precisely equal to the unit interval $[0, 1]$. It follows that the associated degree one circle map f has periodic points with all possible rational rotation numbers. (However, most points of $\mathbb{R}/\mathbb{Z}$ belong to the (one-sided) attractive basin of $.5$ or 1, with associated rotation number $\mathbf{tn}(F, x)$ equal to one or zero respectively. The graph of $\mathbf{tn}^{\pm}(F, x)$ as a function of x is far from continuous. Compare Problem 14-c.)



Graph of $x \mapsto \mathbf{tn}^{\pm}(F, x)$ *on the interval* $0 \le x \le 1$ *for the map* F *of Figure 45(b).*

Note that the *width*

$$w(f) \;=\; \mathbf{tn}^{+}(F) - \mathbf{tn}^{-}(F)$$

of the translation interval $\mathbf{TI}(F)$ depends only on the associated circle map f, and measures the extent to which the rotation number $\mathbf{rn}(f)$ is not uniquely defined. Evidently $w(f) > 0$ if and only f has periodic orbits with two different rotation numbers, and hence has periodic orbits with infinitely many different rotation numbers p/q, including all denominators $q \ge 1/w$. This width is also related to topological entropy:

Lemma 14.8. *If the width* $w(f)$ *of the translation interval* $\mathbf{TI}(F)$ *is strictly positive, then the topological entropy* $h_{\rm top}(f)$ *must also be strictly positive.*

Proof. First suppose that $w(f) \ge 2$. Choose points $x_{\rm max}$ and $x_{\rm min}$ where the displacement $\delta(x) = F(x) - x$ takes on its maximum and minimum values. Since $\delta(x)$ is non-constant with period 1, we may assume that $x_{\rm min} < x_{\rm max} < x_{\rm min} + 1$. Let I be the closed interval $[x_{\rm min}\,, x_{\rm max}] \subset \mathbb{R}$, and let $\widehat{I} \subset \mathbb{R}/\mathbb{Z}$ be the corresponding subset of the circle. Using 14.2, it follows that the image $F(I)$ has length greater than $w(f) \ge 2$. In other words, the image of $\widehat{I}$ under the circle map f winds at least twice around the circle.

It follows that we can find two disjoint compact subsets X_1, $X_2 \subset \widehat{I}$ so that $f(X_j) \supset \widehat{I}$ for $j = 1, 2$. It then follows from Problem 7-j that $h_{top}(f) \geq \log 2$.

Now suppose only that $w(f) > 0$. Choosing an integer q large enough so that $w(f) \geq 2/q$, we see by 14.4 that $w(f^{\circ q}) \geq 2$, and hence that $h_{\rm top}(f^{\circ q}) \geq \log 2$. Using 7.6(d), this implies that $h_{\rm top}(f) \geq (\log 2)/q > 0$, as required. (Note: For much sharper lower bounds on $h_{\rm top}(f)$, see [Alseda et al.].) □

We conclude the discussion of general degree one circle maps with an invariance proof. In fact we will show that the translation interval $\mathbf{TI}(F)$ of a lift of f is invariant, up to an additive integer, not only under orientation preserving topological conjugacy, but also under a more general "degree one semi-conjugacy". Here is a precise statement.

Lemma 14.9. *If f, f' and g are degree one circle maps, and if $g \circ f = f' \circ g$ so that the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \stackrel{f}{\longrightarrow} & \mathbb{R}/\mathbb{Z} \\ g\downarrow & & g\downarrow \\ \mathbb{R}/\mathbb{Z} & \stackrel{f'}{\longrightarrow} & \mathbb{R}/\mathbb{Z}\ , \end{array} \qquad (14:3)$$

then we can choose lifts F and F' so that $\mathbf{TI}(F) = \mathbf{TI}(F')$.

Proof. Choosing any lifts F, F' and G of f, f' and g, the difference

$$G(F(x)) - F'(G(x)$$

is an integer which depends continuously on x, and hence is constant. Adding this integer to F', we may assume that $G \circ F = F' \circ G$, so that the diagram

$$\begin{array}{ccc} \mathbb{R} & \stackrel{F}{\longrightarrow} & \mathbb{R} \\ G\downarrow & & G\downarrow \\ \mathbb{R} & \stackrel{F'}{\longrightarrow} & \mathbb{R}\ , \end{array}$$

is commutative. If $F : x_0 \mapsto x_1 \mapsto x_2 \mapsto \cdots$ is any orbit under F, note that

$$F' \ : \ G(x_0) \ \mapsto \ G(x_1) \ \mapsto \ G(x_2) \ \mapsto \ \cdots$$

is a corresponding orbit under F', and that any orbit under F' can be obtained in this way. Since the difference $|G(x_n) - x_n|$ remains bounded as $n \to \infty$, it follows that $\mathbf{tn}^{\pm}(F, x_0) = \mathbf{tn}^{\pm}(F', G(x_0))$ for every x_0 and every choice of $\pm$. The conclusion follows. □

§**14B. Monotone Maps and Cyclic order.** Now suppose that F is a *monotone* map, in the sense that:

$$x < y \qquad \Longrightarrow \qquad F(x) \leq F(y)\ .$$

We then say that f is a *monotone circle map*. In this case, the dynamics becomes much simpler and more tractable.

Theorem 14.10. *If $F : \mathbb{R} \to \mathbb{R}$ is monotone with $F(x+1) = F(x) + 1$, then the translation number $\mathbf{tn}(F) \in \mathbb{R}$ is well defined. (In other words, the translation interval $\mathbf{TI}(F)$ reduces to a single point.) If $\mathbf{tn}(F)$ is irrational, then the associated circle map f has no periodic orbit (compare §14C), while if $\mathbf{tn}(F) = p/q$, expressed as a fraction in lowest terms, then f has at least one orbit of period q and has no orbit of any other period.*

In particular, if f is any orientation preserving *circle homeomorphism*, then the degree is $+1$, and any lift F is *strictly monotone*,

$$x < y \qquad \Longrightarrow \qquad F(x) < F(y)\ .$$

Hence the rotation number $\mathbf{rn}(f) \in \mathbb{R}/\mathbb{Z}$ is certainly well defined.

Proof of 14.10. A period q orbit for f cuts the circle up into q non-overlapping closed intervals, at least if $q > 1$, and these can be numbered as $I_1, \ldots, I_q$ so that f maps each I_j by a monotone map onto I_{j+1} (taking subscripts modulo q). Since it is easy to check that a monotone interval map cannot have any periodic points other than fixed points, it follows that f cannot have any periodic points with period other than q. However, if the translation interval for F were a non-degenerate interval, then it would follow from 14.6 that f had points of arbitrarily high period. □

Alternative Proof. First note that the displacement $\delta(x) = F(x) - x$ satisfies

$$\max_x \delta(x) \ - \ \min_x \delta(x) \ < \ 1 \qquad (14:4)$$

in the monotone case. For if $\delta(x) = F(x) - x$ takes its maximum at $x_{\max}$ and its minimum at $x_{\min}$, where we may assume that $x_{\max} < x_{\min} < x_{\max} + 1$, then

$$F(x_{\max}) = x_{\max} + \delta(x_{\max}) \ \le \ F(x_{\min}) = x_{\min} + \delta(x_{\min})$$

by monotonicity, hence $\delta(x_{\max}) - \delta(x_{\min}) \ \le \ x_{\min} - x_{\max} \ < \ 1$. Using Lemma 14.2, it follows that $\mathbf{tn}^+(F, x_0) - \mathbf{tn}^-(F, x_0) \ < \ 1$. Applying this inequality to $F^{\circ q}$ and using Lemma 14.4, it follows that

$$\mathbf{tn}^+(F, x_0) - \mathbf{tn}^-(F, x_0) \ < \ 1/q$$

for every q. Thus the upper and lower rotation numbers are equal, so that $\mathbf{tn}(F, x_0)$ is defined. Now consider a second orbit $F : y_0 \mapsto y_1 \mapsto y_2 \mapsto \cdots$, where we may assume that $x_0 < y_0 < x_0 + 1$. It follows inductively that $x_n \le y_n \le x_n + 1$. Therefore the two sequences $(x_n - x_0)/n$ and $(y_n - y_0)/n$ have the same limit. □

Comparing rotation numbers. We next describe an explicit method for computing and comparing rotation numbers. Given a monotone degree one circle map $f : \mathbb{R}/Z \to \mathbb{R}/\mathbb{Z}$ there is a unique lift $F : \mathbb{R} \to \mathbb{R}$ satisfying $0 \le F(0) < 1$. Let $I_0 = [0, \widehat{x}_{-1})$ be the half-open interval consisting of points $x \in [0,1]$ for which $f(x) < 1$, and let $I_1 = [\widehat{x}_{-1}, 1]$ be the (possibly degenerate) complementary interval consisting of points $x \in [0,1]$ for which $f(x) \ge 1$. Define the discontinous lift $\widehat{F} : [0,1] \to [0,1)$ by the formula

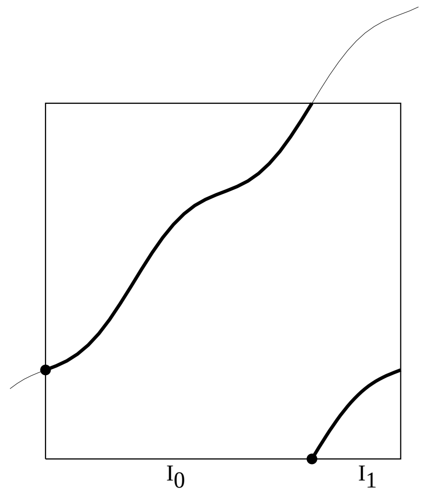
$$\widehat{F}(x) = F(x) - s_F(x)\ , \qquad \text{where} \qquad s_F(x) \ = \ \begin{cases} 0 & \text{for } x \in I_0\,, \\ 1 & \text{for } x \in I_1\,. \end{cases}$$

Following the orbit $\widehat{x}_{-1} \mapsto 0 \mapsto \widehat{x}_1 \mapsto \widehat{x}_2 \mapsto \cdots$ under $\widehat{F}$, we obtain an infinite sequence of bits

$$\vec{b} \ = \ \vec{b}(F) \ = \ (b_1, b_2, b_3, \ldots)$$

by setting $b_k = b_k(F)$ equal to $s_F(\widehat{x}_k)$. It is not difficult to check that the translation number of F can be computed as the limiting average of these bits,

$$\mathbf{tn}(F) \ = \ \lim_{m\to\infty} \frac{1}{m}\Big(b_1 + \cdots + b_m\Big)\ . \qquad (14:5)$$



Graphs of $\widehat{F}$ *(emphasized), and of* F *.*

In fact, a straightforward induction shows that

$$F^{\circ m}(0) \;=\; b_1 + \cdots + b_m + \widehat{x}_m \;, \qquad \text{with} \quad 0 \le \widehat{x}_m < 1 \;,$$

hence

$$b_1 + \cdots + b_m \;\le\; F^{\circ m}(0) \;<\; b_1 + \cdots + b_m + 1 \;.$$

Combining this inequality with 14.6, it follows that

$$\frac{b_1 + \cdots + b_m}{m} \;\le\; \mathbf{tn}(F) \;\le\; \frac{b_1 + \cdots + b_m + 1}{m} \;, \qquad (14:6)$$

and the formula (14 : 5) certainly follows.

A better way of actually computing $\mathbf{tn}(F)$ would be to compare the symbol sequence $\overrightarrow{b}(F)$ with the symbol sequences for various rigid rotations $x \mapsto x + c$, making use of the following. Let $G : \mathbb{R} \to \mathbb{R}$ be another such monotone map satisfying $0 \le G(0) < 1$ and $G(x+1) = G(x) + 1$.

Lemma 14.11. *If* $\overrightarrow{b}(F) < \overrightarrow{b}(G)$, *using the lexicographical order for infinite symbol sequences, then* $0 \le \mathbf{tn}(F) \le \mathbf{tn}(G) \le 1$, *with strict inequalities whenever these translation numbers are irrational.*

Proof. The hypothesis that $\overrightarrow{b}(F) < \overrightarrow{b}(G)$ means that there exists an integer m with $b_k(F) = b_k(G)$ for $k < m$, but $b_m(F) < b_m(G)$. It follows that

$$b_1(G) + \cdots + b_m(G) \;=\; b_1(F) + \cdots + b_m(F) + 1 \;.$$

Using the inequality (14 : 6), it follows that

$$\mathbf{tn}(F) \;\le\; \frac{b_1(G) + \cdots + b_m(G)}{m} \;\le\; \mathbf{tn}(G) \;,$$

and the conclusion follows. □

Cyclic Order. In order to understand rotation numbers from a more intrinsic point of view, we will need the following. Let $(\xi_1, \cdots, \xi_n)$ and $(\eta_1, \cdots, \eta_n)$ be finite sequences of (not necessarily distinct) points on the circle $\mathbb{R}/\mathbb{Z}$.

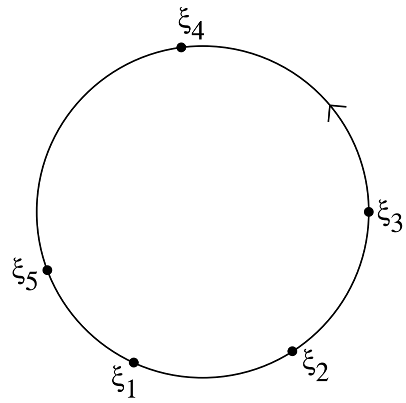


Figure 46. Five points in positive cyclic order. If these are the points $\xi_1 \mapsto \xi_2 \mapsto \cdots$ *of a period 5 orbit, then this orbit has rotation number* $1/5 \pmod{\mathbb{Z}}$ *.*

Definition. Two such finite sequences of the same length have the same *cyclic order type* if and only if there is an orientation preserving homeomorphism of the circle which carries ξ_j to η_j for every j .

In the case of n *distinct* points on the circle, it is easy to check that there are exactly $(n-1)!$ possible cyclic order types. In particular, for a pair of distinct points there is only one cyclic order type; but for three distinct points there are two possible types, which we can describe as "positive" or "negative" cyclic order. In fact for any n there is always one distinguished type: By definition, the points $\xi_1 , \xi_2 , \cdots , \xi_n$ are in *positive cyclic order* if we can choose representatives $x_j \in \mathbb{R}$ with $\xi_j = (x_j \bmod \mathbb{Z})$ so that

$$x_1 \;<\; x_2 \;<\; \cdots \;<\; x_n \;<\; x_1 + 1 \;.$$

Note that this condition is invariant under cyclic permutation of the ξ_j .

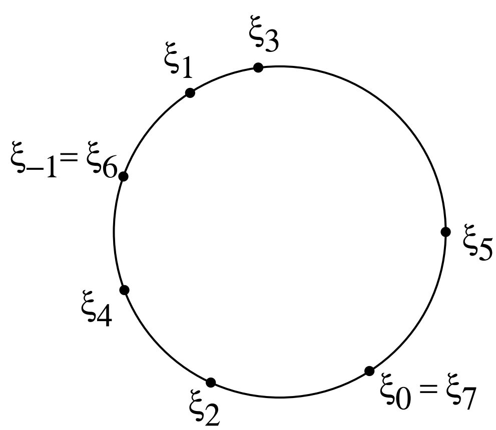


Figure 47. A period 7 *orbit with rotation number* $3/7$ *(modulo* $\mathbb{Z}$*). Note that each* ξ_{i+1} *is the third orbit point counterclockwise from* ξ_i *.*

Now consider two maps f and g from $\mathbb{R}/\mathbb{Z}$ to itself, and let

$$f : \xi_0 \mapsto \xi_1 \mapsto \xi_2 \mapsto \cdots \quad \text{and} \quad g : \eta_0 \mapsto \eta_1 \mapsto \eta_2 \mapsto \cdots$$

be orbits under these maps. We will say that the orbits $\{\xi_i\}$ and $\{\eta_i\}$ have the same *cyclic order type* if for each finite n there exists an orientation preserving homeomorphism of the circle which carries ξ_j onto η_j for $j \le n$. We will prove the following.

Theorem 14.12. *Let f be a monotone degree one circle map. Then the rotation number $\mathbf{rn}(f)$ can be computed from the cyclic order type of an arbitrary orbit $f : \xi_0 \mapsto \xi_1 \mapsto \cdots$. This cyclic order type coincides with the cyclic order type for an orbit under the rigid rotation $\xi \mapsto \xi + c$, where $c = \mathbf{rn}(f)$, if and only if either*

(*a*) *the given orbit is periodic, or*

(*b*) *the rotation number is irrational.*

Proof. Using the projection $[0,1] \to \mathbb{R}/\mathbb{Z}$, the orbit $\widehat{x}_{-1} \mapsto 0 \mapsto \widehat{x}_1 \mapsto \cdots$ under $\widehat{F}$ corresponds to an orbit $\xi_{-1} \mapsto \xi_0 \mapsto \xi_1 \mapsto \cdots$ under f. Evidently the bit $b_k = b_k(F)$ takes the value zero whenever the three points ξ_{-1}, ξ_0, ξ_k are in positive cyclic order, or when $\xi_0 = \xi_k$, and the value one otherwise. Hence the rotation number of f can be computed from (14 : 5). (As examples, for the map shown in Figure 46, the sequence $\vec{b}$ takes the form $\overline{00010} = (0,0,0,1,0,0,0,0,1,0,\ldots)$, repeating periodically with period five, so that the limiting average (14 : 5) takes the value 1/5; while for Figure 47 the sequence is $\overline{0101010}$ with period seven, so that the limiting average takes the value 3/7.)

Now let F be any lift of f and let $F : x_0 \mapsto x_1 \mapsto \cdots$ be any orbit. According to 14.6 we have

$$\mathrm{sgn}(x_{n+q} - x_n - p) \;=\; \mathrm{sgn}(\mathbf{tn}(F) - p/q) \qquad (14:7)$$

for any triple of integers p and $n + q > n \ge 0$ with $p/q \ne \mathbf{tn}(F)$. If $\mathbf{tn}(F)$ is irrational, this means that the precise ordering of all of the integer translates of orbit points is completely determined by the translation number. Hence the cyclic order type of any orbit under f is also uniquely determined.

Remark. This argument works even for circle maps which are not monotone: *If the translation number $c = \mathbf{tn}(F)$ is well defined and irrational, then for any orbit the precise ordering of the collection of all integer translates of orbit points x_q is uniquely determined by c.*

Finally, to conclude the proof of 14.12, we must consider the case of a rational rotation number. If $\mathbf{rn}(f) \equiv p/q$ expressed as a fraction in lowest terms, then for any orbit of period precisely equal to q, the cyclic order type is again uniquely determined by (14 : 7). But if f is monotone, then according to 14.10, every periodic orbit has period precisely q, and the conclusion follows. □

§14C. Irrational Rotation Numbers.

The following result is well known in the monotone case, and is due to [Auslander and Katznelson] in the non-monotone case.

Theorem 14.13. *For a degree one circle map the following three conditions are equivalent:*

(*i*) *The rotation number $\mathbf{rn}(f)$ is well defined and irrational.*

(*ii*) *f has no periodic orbits.*

(iii) f is degree one monotonely semiconjugate to an irrational rotation $r^c(\xi) = \xi + c$. That is, there is a degree one monotone circle map g so that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \xrightarrow{f} & \mathbb{R}/\mathbb{Z} \\ g\downarrow & & g\downarrow \\ \mathbb{R}/\mathbb{Z} & \xrightarrow{r^c} & \mathbb{R}/\mathbb{Z} \end{array} ,$$

where the constant c is necessarily equal to the rotation number $\mathbf{rn}(f)$.

Here the map f is not necessarily monotone. (Compare 15.4.) The proof of 14.13 will be based on two elementary observations. The first is a classical assertion due to Kronecker.

Lemma 14.14. *The orbit $\xi_0 \mapsto \xi_1 \mapsto \cdots$ under the rotation $r^c(\xi) = \xi + c$ is everywhere dense in the circle $\mathbb{R}/\mathbb{Z}$ if and only if c is irrational.*

Proof. Let $0 \le \mathbf{d}(\xi, \eta) \le 1/2$ denote the length of the shorter path from ξ to η within the circle $\mathbb{R}/\mathbb{Z}$. For any $n > 1$, cut the circle $\mathbb{R}/\mathbb{Z}$ into n half-open intervals of length $1/n$, and consider the $n+1$ orbit points $\xi_0, \cdots, \xi_n$. At least two of the points, say ξ_i and ξ_j, must belong to the same interval, and hence have distance $\mathbf{d}(\xi_i, \xi_j) < 1/n$. Setting $h = |i - j|$, we see that $\mathbf{d}(\xi_0, \xi_h) < 1/n$, with $0 < h \le n$. If c is irrational, then $\xi_0 \ne \xi_h$, and it follows easily that the iterates $\xi_0, \xi_h, \xi_{2h}, \ldots$ come within distance $1/n$ of every point of the circle, where n can be arbitrarily large. On the other hand, if c is rational with denominator q, then clearly there can be only q distinct points in the orbit. □

Now let $I = [a, b] \subset \mathbb{R}$ be a compact interval.

Lemma 14.15. *If $\phi : I \to \mathbb{R}$ is a continuous map with $\phi(I) \supset I$, then I contains a fixed point.*

Proof of 14.15. Choose points $x_0 \in I$ and $x_1 \in I$ with $\phi(x_0) = a$ and $\phi(x_1) = b$. Then $\phi(x_0) \le x_0$ and $\phi(x_1) \ge x_1$, hence there must be at least one point x between x_0 and x_1 with $\phi(x) = x$. □

Proof of 14.13. The equivalence (i) $\iff$ (ii) follows immediately from 14.6, while the implication (iii) $\implies$ (i) follows from 14.9. Thus we need only show that (i), (ii) $\implies$ (iii).

Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f with translation number $\mathbf{tn}(F) = c \in \mathbb{R} \smallsetminus \mathbb{Q}$, and choose some orbit $F : x_0 \mapsto x_1 \mapsto \cdots$. According to Remark 14.12, the points $x_q - p \in \mathbb{R}$ are ordered in the same way as the points $q\,c - p$, associated with the translation $T^c(y) = y + c$. Furthermore, it follows from 14.14 that these points $q\,c - p$ are everywhere dense on the real line. It follows easily that there is one and only one map $G : \mathbb{R} \to \mathbb{R}$ which is monotone and satisfies $G(x_q - p) = q\,c - p$ for every pair of integers $q > 0$ and p. In fact we can set

$$G(x) \;=\; \sup\{q\,c - p \;;\; x_q - p \;\le\; x\} \;=\; \inf\{q\,c - p \;;\; x_q - p \;\ge\; x\} .$$

Evidently G is monotone and continuous, with $G(x+1) = G(x) + 1$. Thus for each fixed $y \in \mathbb{R}$ the pre-image

$$I_y \;=\; G^{-1}(y)$$

is either a point or a closed interval. These (possibly degenerate) intervals are disjoint, cover the real line, and are ordered along the real line in the same way as the subscripts y. We want to prove that F maps each I_y precisely onto the interval I_{y+c} . In fact, approximating y on the left (or the right) by numbers of the form $q\,c-p$, we see easily that F maps the left (or right) endpoint of I_y onto the corresponding endpoint of I_{y+c} . It certainly follows that $F(I_y)\supset I_{y+c}$, and hence $F^{\circ q}(I_y)\supset I_{y+q\,c}$. If $F(I_y)$ were strictly larger than I_{y+c} , then its image $G(F(I_y))$ would be a non-degenerate interval of real numbers. Hence it would contain some interior point of the form $z=y-n\,c-m$, where $n>0$ and m are integers. This would imply that $F(I_y)\supset I_z$. Since $F^{\circ n}(I_z)\supset I_{z+n\,c}=I_{y-m}$, this would imply that

$$F^{\circ n+1}(I_z)\ \supset\ F(I_{y-m})\ \supset\ I_{z-m}\ .$$

Therefore, by 14.15, the map $x\mapsto F^{\circ n+1}(x)+m$ would have a fixed point in the interval I_z . Hence the circle map f would have a corresponding periodic point $\xi=f^{n+1}(\xi)$. This contradicts the hypothesis that f has no periodic points, and completes the proof of 14.13. □

Remark. The map G which is constructed in this proof is unique up to an additive constant. In other words, the semiconjugacy g is unique up to post-composition with a rotation. The proof is not difficult. If we recall Weyl's Theorem 3.7, which asserts that every orbit under an irrational rotation is evenly distributed with respect to Lebesgue measure, then we can restate this result as follows.

> **Corollary 14.16.** *If f is a circle map without periodic points, then there is one and only one probability measure μ on the circle which is f-invariant, so that $\mu(S)=\mu(f^{-1}(S))$ for every measurable set S . Furthermore, every orbit is evenly distributed with respect to μ .*

Here we do not need to assume as a separate hypothesis that f has degree one, since a circle map with any degree $d\neq 1$ always has at least $|d-1|>0$ fixed points. (Compare Problem 4-e and the proof of 4.12.)

Proof of existence. For any interval $I\subset\mathbb{R}/\mathbb{Z}$, define $\mu(I)$ to be the length $\ell(g(I))$ of its image under the monotone semiconjugacy g . For example, if I maps to a single point under g , then $\mu(I)=0$. It is now easy to check that μ is the required f-invariant probability measure, and that every orbit is evenly distributed.

Proof of uniqueness. If μ is f-invariant, then the push forward $g_*(\mu)$ is invariant under the irrational rotation r^c . It then follows easily from 3.7 that $g_*(\mu)$ coincides with the Lebesgue measure ℓ on the circle. Now for any interval I the length $\ell(g(I))$ equals $(g_*\mu)(g(I))=\mu(g^{-1}g(I))$. This is equal to $\mu(I)$, since the difference $g^{-1}g(I)\smallsetminus I$ consists at most of two intervals $J\subset g^{-1}(a)$ and $J'\subset g^{-1}(b)$, where $g(I)=[a,b]$. Evidently $\mu(J)=\mu(J')=0$, and the conclusion follows. □

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§14D. Some Problems.

Problem 14-a. Using 14.6, prove that

$$\mathbf{tn}^+(F)\ =\ \lim_{q\to\infty}\frac{1}{q}\max_x\,\delta_q(x)\ ,\qquad \text{and that}\qquad \mathbf{tn}^-(F)\ =\ \lim_{q\to\infty}\frac{1}{q}\min_x\,\delta_q(x)\ .$$

Problem 14-b. If F is strictly monotone so that F^{-1} is also well defined and continuous, show that $\mathbf{tn}(F^{\circ q})=q\,\mathbf{tn}(F)$ not only for positive q , but also for negative q .

Problem 14-c. For the map F of Figure 45(b), show that given integers $n_k \gg 0$ we can find small numbers $\epsilon_k > 0$ so that

$$\begin{aligned} F^{\circ n_k}(\epsilon_k) &= .5 - \epsilon_{k+1} \quad \text{for } k \text{ even,} \\ F^{\circ n_k}(.5 - \epsilon_k) &= n_k + \epsilon_{k+1} \quad \text{for } k \text{ odd.} \end{aligned}$$

By suitable choice of the n_k show that the numbers $\mathbf{tn}^{\pm}(F, \epsilon_0)$ can take completely arbitrary values, subject only to the inequalities $0 \le \mathbf{tn}^{-}(F, \epsilon_0) \le \mathbf{tn}^{+}(F, \epsilon_0) \le 1$.

Problem 14-d. Show that any homeomorphism $F : \mathbb{R} \to \mathbb{R}$ without fixed points is topologically conjugate to the unit translation $T(x) = x + 1$. If F and G are fixed point free homeomorphisms with $F \circ G = G \circ F$, show that there is a unique line L through the origin in $\mathbb{R}^2$ so that if the integer pair (p, q) lies on one side of L then

$$F^{\circ q}(x) > G^{\circ p}(x) \qquad \text{for all} \quad x \ ,$$

while for (p, q) on the other side of L the opposite inequality holds. The slope of L can be described as the *translation number of* F *with respect to* G . Note that this is the reciprocal of the translation number of G with respect to F , and note that it changes sign if we replace F by F^{-1} or G by G^{-1} .