Chapter I. DISCURSIVE INTRODUCTION.

§1. Chaotic Dynamics: Some History.

Dynamics (or *Dynamical Systems*), the study of how physical or mathematical systems evolve with time, is a subject which has developed through the collective efforts of mathematicians and of scientists in many different fields. Not only its applications but also its origins lie in many different branches of science. Although the emphasis in these notes will be on simple mathematical models, it is important to realize that the field has deep empirical roots. This preliminary chapter will describe several topics from Dynamics in an informal manner, with precise definitions often postponed to later sections.

§1A. Celestial Mechanics. A typical difficult dynamic problem, and one which has been studied since prehistoric times, is the study of the motions of the sun, moon and planets. This study has always been important for practical reasons, for example navigation and timekeeping, and often also for religious or astrological reasons.

The modern theory begins in the year 1590 with the publication of Galileo's studies of motion. Basing his work on careful experiments with pendulums and with falling bodies, Galileo contradicted several classical beliefs. Particularly important was his principle of *inertia*:[1]

> "··· a ship, for instance, having once received some impetus through the tranquil sea, would move continually around our globe without ever stopping; and placed at rest it would perpetually remain at rest, if in the first case all extrinsic impediments could be removed, and in the second case no external cause of motion were added."

In 1609 and 1619, Johann Kepler carried out what is perhaps the most impressive piece of detective work in the entire history of science.[2] By painstaking analysis of a series of careful astronomical observations, he put together three simple laws which describe the orbits of the planets. However, his understanding of these laws was very different from the modern one, since it was based on Aristotle's theories of motion: He believed that the sun must exert a sideways force which pushes the planet around their orbits.

It took some years for the scientific world of the seventeenth century to learn of these two very different pieces of work and to put them together. Kepler's third law states that the orbital period for a planet in a circular orbit of radius r is proportional to $r^{3/2}$. Writing the position vector at time t as $\vec{x} = (r\cos(\lambda t)\,,\, r\sin(\lambda t)\,,\, 0)$ with $\lambda = c/r^{3/2}$, we see that the acceleration vector $d^2\vec{x}/dt^2$ has length $r\lambda^2 = c^2/r^2$. Robert Hooke, in 1679, was perhaps the first to consider arguments of this type, combining Kepler's work with Galileo's concepts of inertia and uniform acceleration, to propose an inverse square law for gravitational attraction. However, the study of non-circular orbits under an inverse square force law leads to a mathematical problem which Hooke was unable to solve. Today we would describe it as the problem of solving the differential equation

$$\frac{d^2\vec{x}}{dt^2} = -c^2\,\vec{x}\,/\,\|\vec{x}\|^3\,.$$

[1] From a letter written in 1612. See [Drake].

[2] See for example [Peterson], [Ekeland].

Newton, in his Principia of 1686, was able to show[3] that the solutions of this equation are precisely the elliptical orbits which had been observed by Kepler. This gave strong support to his Law of Gravitation: *Any two bodies are attracted by a force which is proportional to the product of their masses, and inversely proportional to the square of their distance.* For an arbitrary collection of spherical bodies with positions $\vec{x}_i$ and masses m_i, this leads to the system of differential equations

$$\frac{d^2\,\vec{x}_i}{dt^2} \;=\; \sum_{j\neq i} G\, m_j\, \frac{\vec{x}_j - \vec{x}_i}{\|\vec{x}_j - \vec{x}_i\|^3}\;, \qquad (1:1)$$

where G is Newton's gravitational constant ($G = 6.67\times 10^{-8}\mathrm{cm}^3/(\mathrm{gram\,sec}^2)$). Although Newton was able to solve this pair of equations for the two body problem, the case of three or more bodies is much more difficult and has been an object of serious mathematical study ever since.

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§1B. Poincaré and Sensitive Dependence. The difficulty of the three body problem was brought to light very forcefully two hundred years later. To celebrate the sixtieth birthday of Oscar II, King of Sweden and Norway, in 1889, a mathematical contest was organized by Mittag-Leffler. One prize topic, proposed by Weierstrass, was to prove the convergence of certain power series arising in celestial mechanics.[4] Henri Poincaré, then a very young Professor in Paris, was awarded the prize for a paper which analyzed this problem. However, after his paper has been printed, but before it was circulated, a mistake was discovered. All of the printed copies were withdrawn. Poincaré went back to work, and completely rewrote the paper. In the revised version, he described the phenomenon which we now call *sensitive dependence on initial conditions.* (Compare §2C and §4D.) A few years later he explained it as follows:

> "A very small cause that escapes our notice determines a considerable effect that we cannot fail to see, and then we say that the effect is due to chance. If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation *approximately.* $\cdots$ It may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon."

Such sensitive dependence, combined with *recurrence* (orbits which repeatedly come back to the same small region), leads to the highly irregular, difficult to predict behavior which we call *chaotic* dynamics.

What happened next is extremely awkward for believers in the continuity of scientific progress. One of the most prominent mathematicians in the world had clearly announced a discovery which he believed to be not just an isolated curiosity but a fundamental phe-

[3] Compare [Arnol'd 1990] or [Milnor 1983].

[4] Compare [Peterson], [Barrow-Green].

nomenon which occurs throughout the natural world, and is important for all of science. However, the scientific world was not ready. Although Poincaré was honored, and his work was thoroughly studied, this particular discovery was almost totally ignored for seventy years.

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§1C. The Restricted 3-Body Problem: Continuous Versus Discrete Time. Consider three bodies which move according to the system of Newtonian differential equations (1), but suppose that one of these three has mass zero. Think for example of the first two bodies as sun and earth, and the third as an artificial satellite with negligible mass, ignoring all other bodies in the solar system. Then the third body will have no influence on the other two, which will move in elliptic orbits around their common center of gravity as described by Newton. Thus we can consider the position vectors $\vec{x}_1(t)$ and $\vec{x}_2(t)$ of the first two bodies with respect to the center of gravity as known periodic functions: If time is measured in years, then

$$\vec{x}_j(t) \;=\; \vec{x}_j(t+1) \qquad \text{for} \qquad j = 1\,,\,2\;.$$

Now, instead of studying three coupled differential equations, we can consider the single second order differential equation

$$\frac{d^2\,\vec{x}}{dt^2} \;=\; \sum_{j=1}^{2} G\,m_j\,\frac{\vec{x}_j - \vec{x}}{\|\vec{x}_j - \vec{x}\|^3}$$

in 3-space, with $\vec{x} = \vec{x}_3$, where $\vec{x}_1$ and $\vec{x}_2$ are known periodic functions of t . It is often convenient to write this as a pair of first order equations

$$\frac{d\vec{x}}{dt} \;=\; \vec{v}\,, \qquad \frac{d\vec{v}}{dt} \;=\; \sum_{j=1}^{2} G\,m_j\,\frac{\vec{x}_j - \vec{x}}{\|\vec{x}_j - \vec{x}\|^3} \tag{1 : 2}$$

where $\vec{v} = \vec{v}(t)$ is the velocity vector.

Poincaré took a further step. Instead of trying to follow the motion of the small body throughout the year, he pointed out that it would suffice, for studies of long time behavior, to keep track of its position $\vec{x}(t)$ and its velocity $\vec{v}(t)$ once a year. In other words, he saw that it was enough to restrict attention to integer values of the time t . Solving the differential equation (1 : 2) for the interval $n \le t \le n+1$, he noted that it is possible (in principle at least) to write

$$(\vec{x}(n+1)\,,\,\vec{v}(n+1)) \;=\; f(\vec{x}(n)\,,\,\vec{v}(n))\;,$$

where f is a well defined function of six real variables. (More precisely, f is well defined and smooth on an appropriate open subset of $.\,^6$.) Since the coefficients of (1 : 2) are periodic, this function f does not depend on the integer n . Hence, to study motion over a period of many years, we simply need to apply this same function f over and over again.

More generally, consider any system of first order equations of the form

$$d\mathbf{x}/dt \;=\; \mathbf{w}(\mathbf{x}\,,\,t)\;,$$

with $\mathbf{x}\,,\,\mathbf{w}(\mathbf{x},t) \in .\,^d$, where the vector field $\mathbf{w}(\mathbf{x},t)$ is periodic,

$$\mathbf{w}(\mathbf{x}\,,\,t+1) \;=\; \mathbf{w}(\mathbf{x},t)\;.$$

(For the application to the restricted 3-body problem, we take $\mathbf{x}$ equal to the pair $(\vec{x}\,,\,\vec{v}) \in$ ^{6} .) We assume that this differential equation has solutions $\mathbf{x} = \mathbf{x}(t)$ for $0 \le t \le 1$ which depend uniquely and continuously[5] on the initial point $\mathbf{x}(0)$. Then, setting $\mathbf{x}(1) = f(\mathbf{x}(0))$, we can write $\mathbf{x}(n+1) = f(\mathbf{x}(n))$ for every integer value $t = n$ of the time. I will use the notation

$$f \;:\; \mathbf{x}(0) \;\mapsto\; \mathbf{x}(1) \;\mapsto\; \mathbf{x}(2) \;\mapsto\; \cdots$$

for the *orbit* (or the *trajectory*) with initial point $\mathbf{x}(0)$, or write

$$\mathbf{x}(n) \;=\; f^{\circ n}(\mathbf{x}(0))$$

where $f^{\circ n}$ stands for the n-fold composition $f \circ \cdots \circ f$ of f with itself.

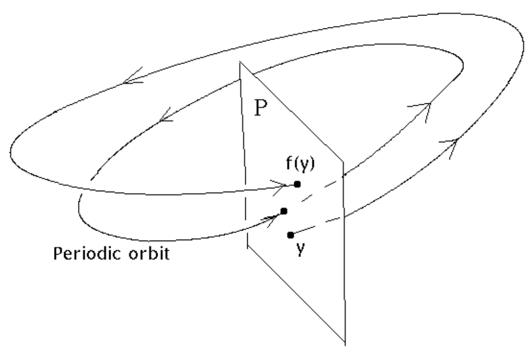

Figure 1. The first return map f to a Poincaré section P .

This basic construction of replacing a differential equation by an iterated mapping, and hence replacing the continuous time t by a discrete time $n \in \mathbf{Z}$, has had an important influence on subsequent developments in dynamics, since iterated mappings are often easier to understand. As another example of this reduction, consider an *autonomous* first order differential equation, that is one of the form

$$d\mathbf{x}/dt \;=\; \mathbf{w}(\mathbf{x}) \qquad \text{with} \qquad \mathbf{x}\,,\,\mathbf{w}(\mathbf{x}) \in\ ^{d}\,, \tag{1:3}$$

and suppose that we are given some periodic solution $\mathbf{x}(t) = \mathbf{x}(t+t_0)$ (or at least a solution which comes back very close to its initial position). Choose a $(d-1)$-dimensional plane P which is transverse to this periodic orbit at the point $\mathbf{x}(0)$. Then, starting at any point $\mathbf{y} \in P$ which belongs to a sufficiently small neighborhood of $\mathbf{x}(0)$, we can solve (2) with initial point $\mathbf{y}$ and follow the solution until in again hits the hyperplane P . This construction yields a mapping $f : U \to P$ where U is a neighborhood of $\mathbf{x}(0)$ in P . By definition, f is called the *first return map* to P associated to the differential equation (2). Evidently the study of iteration for this first return map is nearly equivalent to the study of solutions to the differential equation, so long as these solutions remain close to our chosen

[5] For *local* existence, uniqueness and continuity of solutions, we need that $\mathbf{w}(\mathbf{x}, t)$ is continuous in both variables and satisfies a Lipschitz condition of the form $\|\mathbf{w}(\mathbf{x}, t) - \mathbf{w}(\mathbf{y}, t)\| \le K\|\mathbf{x} - \mathbf{y}\|$. See for example [Hubbard and West] or [Graves].

periodic orbit. The only information which is lost is the length of time t which it takes to get from $\mathbf{x}(0)$ to $f(\mathbf{x}(0)) = \mathbf{x}(t)$.

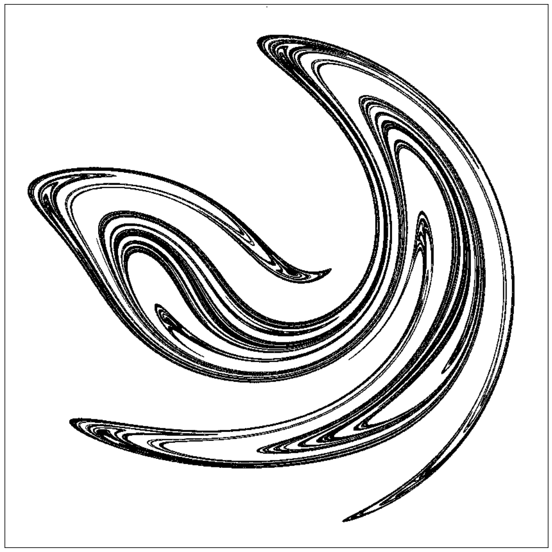

Figure 2. One of the attractors studied by Ueda. This picture in the $(x, \dot{x})$ *plane is obtained by repeatedly following the solution curves for the Duffing equation*

$$\ddot{x} + k\dot{x} + x^3 \ = \ B\cos t$$

for time $0 \le t \le 2\pi$ *, where* $\dot{x} = dx/dt$ *. Here* $k = .05$, $B = 7.5$ *, and* $(x, \dot{x}) \in [1.5, 3.5] \times [-5, 6]$.

§**1D. Chaotic Attractors: Ueda, Lorenz, and Hénon.** In the early 1960's the scientific world belatedly began to appreciate Poincaré's ideas about sensitive dependence. One of the first to observe this phenomenon was Yoshisuke Ueda, a young Electrical Engineer in Kyoto, who was part of a team studying periodically forced non-linear oscillations using an analogue computer. However, since his supervisors had fixed preconceived ideas on the subject, he was not allowed to report on this until much later. Here are three quotations:

> " As I watched my professor preparing the report without a mention of the $\cdots$ phenomenon, but rather with the smooth curves of the quasi-periodic oscillation, I was quite impressed by his technique of report writing."

> " [Another professor] admonished me personally: 'What you saw was simply the essence of quasi-periodic oscillations. You are too young to make conceptual observations.' [However] the existence of random oscillations (chaos) was so obvious in my mind, that the negative comment did not crush me."

> " People call chaos a new phenomenon, but it has always been around. There's nothing new about it — only people did not notice it."

At about the same time, Edward Lorenz, a meteorologist, studied a system of three first order differential equations

$$\dot{x} \ = \ a(y - x)\,, \qquad \dot{y} \ = \ c\,x - y - x\,z\,, \qquad \dot{z} \ = \ x\,y - b\,z\ ,$$

where x, y, z are quantities which describe the state of a circulating fluid. (The three constants are taken typically to be $a = 10$, $b = 8/3$, $c = 28$.) The solution curves seemed to oscillate in an irregular fashion. Furthermore, he was quite surprised by the accidental discovery that these curves depend in an extremely sensitive way on the precise initial conditions.

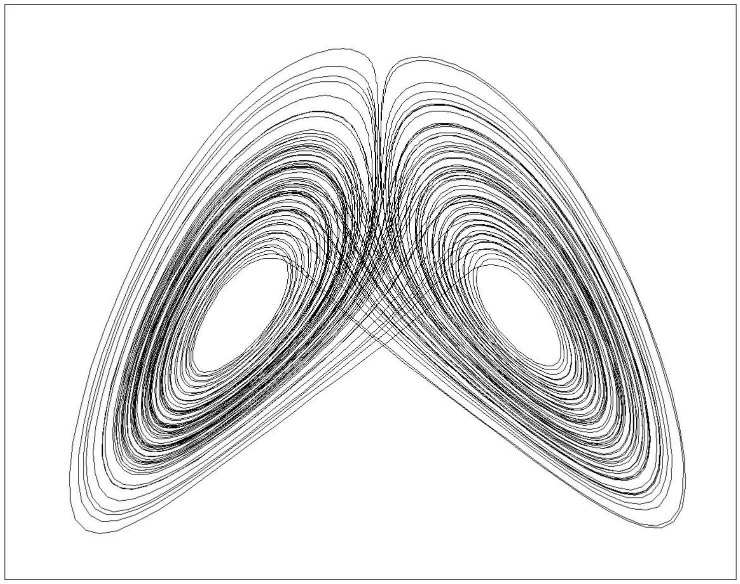

Figure 3. Part of an orbit in the Lorenz attractor, projected to a suitable plane.

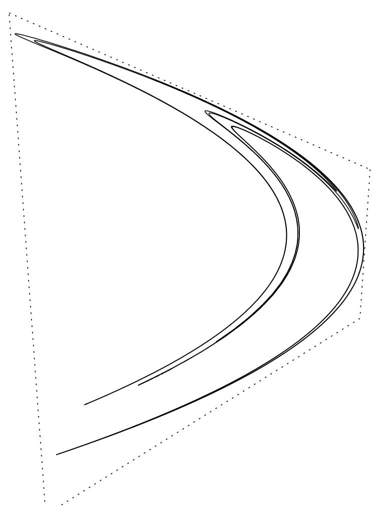

Figure 4. Attractor for the Hénon map $(x, y) \mapsto (1 - 1.4\,x^2 + y\,,\ 0.3\,x)$ *. All orbits which start within the dotted region must converge to this attractor, which is a compact connected set with connected complement.*

In the late 60's, M. Hénon, an astronomer, studied area preserving quadratic polynomial maps of the plane as a highly simplified model for the volume preserving transformations which typically occur in celestial mechanics. These maps display a strange mixture of chaotic and predictable behavior. A few years later, he tried to analyze the essential features of

Lorenz's differential equation and discovered that *area reducing* quadratic maps of the plane often display chaotic behavior which is similar to that observed by Lorenz, but much easier to study.

The Ueda, Lorenz, and the Hénon area reducing examples all seem to have the following basic properties. (For a similar example, but with explicit proofs, see §2E.)

(a) There is a compact set A, called an *attractor* with the property that all trajectories which start out close to A must inevitably converge towards A.

(b) The dynamics is *chaotic*, that is, even initial conditions which are extremely close to each other usually yield orbits which diverge dramatically. This leads to behavior which is highly irregular and difficult to predict.

(c) This attractor A is not a smooth object, but rather has an extremely complicated local geometry. Following Mandelbrot, such objects are called *fractals*.

Furthermore, the attractor A is mapped to itself by a complicated combination of stretching (in one direction) and compression (in another direction), often combined with folding, so that A fits back exactly onto itself. Although such objects are difficult to visualize and to analyze, it has since become clear that they are actually quite common, and play a fundamental role in dynamics.

Note that none of the three terms, "*chaotic*" or "*fractal*" or "*attractor*", has a precise definition which is generally accepted. Never-the-less, all three terms have clear approximate meanings, and are quite useful in practice. We will discuss possible definitions for these terms in later sections: See §4D, §5B.

Note. The term "*strange attractor*", coined by Ruelle and Takens, is often used in the literature as an alternative to "chaotic attractor". However, the usage is not fixed, and some authors use this term to mean "fractal attractor". (Compare [Anosov and Arnold, p. 37], [Grebogi et al], [Rasband].)

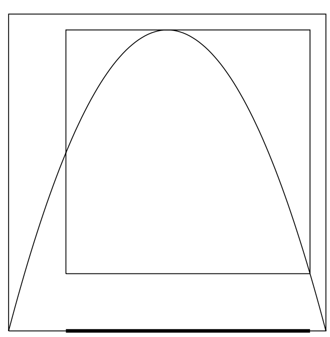

Figure 5. Graph of $f(x) = 3.8\,x\,(1-x)$ on the unit interval. The indicated subinterval $A = [.1805\,,\,.95]$ attracts nearly all orbits. The dynamics on A is believed to be chaotic, but A is certainly not a fractal.

The three conditions (a), (b), and (c) are independent of each other; and each one is of independent interest. Here is a simple example of an attractor which is chaotic, but not fractal. (See also the beginning of §2E.) Consider the quadratic map

$$f(x) \;=\; 4\,c\,x\,(1-x)\;, \qquad (1:4)$$

sometimes called the *logistic map*, on the unit interval $[0,1]$, where c is some fixed parameter with $0 < c \le 1$. If $1/2 < c < 1$, then every orbit in the open interval $(0,1)$ is eventually absorbed by the subinterval $A = [f(c)\,,\, c]$. In fact, any point to the right of A immediately moves to the left of A, and then is pushed towards the right until it lands in A, where it remains thereafter. For many values of c, the dynamics on $[f(c)\,,\, c]$ is known to be chaotic. (Compare §2C.) The case $c = .95$, illustrated in Figure 5, is conjectured to be such a chaotic case. (Compare Figure 18 in §3B.)

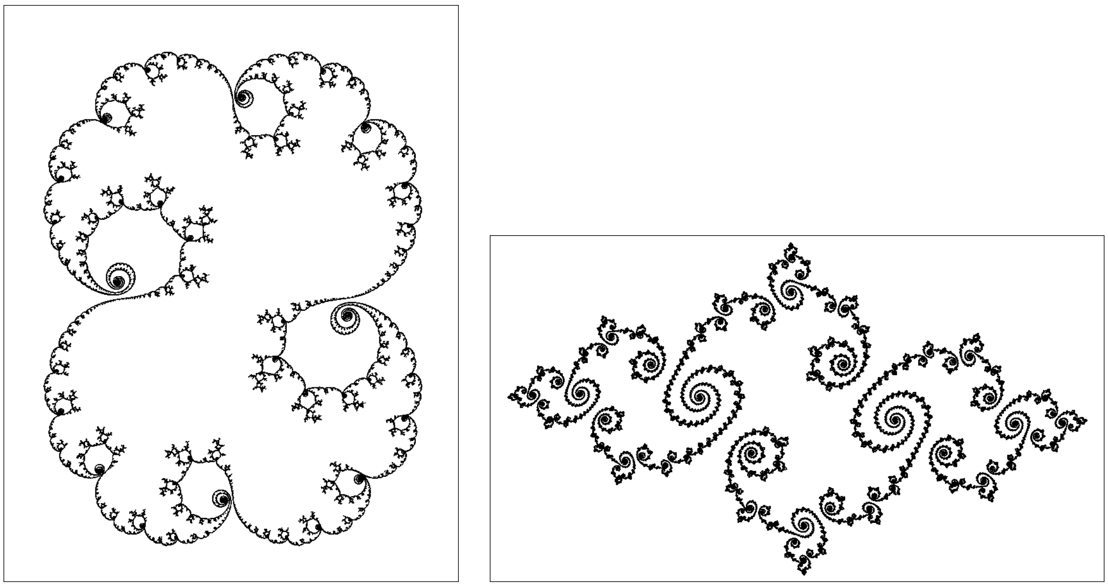

Figure 6. Two fractal "repellors" with chaotic dynamics. Left: the Julia set for the quadratic polynomial $z \mapsto z^2 + (.99 + .14i)\, z$ is a topological circle which divides the plane into a bounded region where all orbits converge to zero and an unbounded region where all orbits diverge to infinity. Right: the Julia set for $z \mapsto z^2 + (-.765 + .12i)$ is totally disconnected, homeomorphic to the Cantor middle third set.

There are are many examples of fractals with chaotic dynamics which are not attractors. One important example is the Smale horseshoe. An extremely interesting family of examples is provided by the *Julia sets* of rational mappings from the Riemann sphere $\mathbf{C} \cup \infty$ to itself. (Compare §4D, §4E, Problem 5-e, as well as [Milnor 1999].) Two examples are shown in Figure 6.

For examples of attractors which are fractal but not chaotic, see the discussion of the Feigenbaum attractor in §6 and the Denjoy counterexample in §8, as well as [Grebogi et al.].