MATHEMATICAL LOGIC

SUNY AT STONY BROOK-SPRING 2003-MAT/CSE 371
COURSE NOTES
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1. Overview

1. Text:
   Herbert Enderton, *A Mathematical Introduction to Logic*
2. Class will be divided into three main sections:
   (a) Predicate Logic
   (b) First Order Logic
   (c) Undecidability
3. This course has several goals
   (a) Make precise the framework in which we work whenever we do mathematics
   (b) Mathematics is a form of deductive thought, so more generally what we would like to do is build a model for deductive thought.
   (c) Approach questions of solvability/computability
4. What does it mean to have a model?
   - Given a real-life object, a **model** is something which accurately represents some of the features of that object while ignoring others.
   - Whether or not the model serves its purpose depends on the choice of properties which we choose to represent.

   Example: Suppose I showed you a 2 inch diameter gray ball and said that it is a model airplane. It *is* a model, but for most purposes it is not a very good one. It is not a good model for a 747 airplane, because even though they both have the same color that is not a property that one is usually looking to preserve in a model airplane.
5. The real-life objects that we care about in this class are logical deductions.

   To quote an overused example (modified slightly):

Date: 1/21/2003.

All women are mortal.
Hypatia is a woman.
Therefore, Hypatia is mortal.

The deduction of the third line from the first two does not depend on any fact about Hypatia (Like she was head of the Platonist school at Alexandria around 400 AD) or even what the meaning of mortal is. We should note that it does however matter what we mean by **all**.

6. We shall study logical deductions like these; for the most part ignoring their content and focusing on their form. The vagueness of this previous sentence is something we want to get rid of by dealing with mathematical models.
7. Some basic questions we will study include:
   - What does it mean for one sentence to **follow logically** from another?
   - If a sentence does **follow logically** what methods of **proof** are necessary to establish this.
   - Must all declarations be either **True** or **False**?
8. Here is a brief overview of the 3 main parts of this course:
   - **Predicate Logic** This will be our first mathematical model of logic. It is not a very powerful system, but it will provide a good *toy model* to play with. It will provide an introduction to working (and proving things) in a formal system. Moreover, many of the theorems we want to prove in the second part have similar, but technically simpler, analogues here.
   - **First Order Logic** This is our main model of mathematical reasoning and closely corresponds to what we use as mathematicians. For one thing the language is rich enough to encode statements like the intermediate value theorem for continuous functions:
     $\forall f \forall a \forall b \forall N (((fb > N) \wedge (N > fa)) \rightarrow \exists c((a < c) \wedge (c < b) \wedge (fc = N)))$
     We will show various facts about such formal systems involving provability, enumerability, computability, and axiomizations. We will give definitions of what it should mean for something to be **reasonably computable**.
     Depending on time/interest I would like to include some discussion of Turing machines and the relation of mathematical logic to computer science.
   - **Undecidability** Time will dictate how much we can discuss this subject. Ideally we will cover:

- Gödel Incompleteness Theorem. Roughly this states that given any reasonably complex language and any decidable set of axioms then there exist statements in the language which cannot be proved or disproved (from these axioms). For example, the well ordering principle can not be proved from the Zermelo-Fraenkel axioms for set theory.
- Formal Number Theory. We will formalize number theory and show that this language is sufficiently complex in the sense needed for Gödel's Theorem.
- Prove the Incompleteness Theorem

2. Predicate Logic

1. What is a Formal Language? Generally it means we have:
   (a) An **Alphabet**. This is the collection of symbols we have to work with. For example some of the symbols of Predicate Logic are: $(,), \neg, \rightarrow, A_1, A_2, A_3$
   (b) Rules for forming **grammatically correct** finite sequences of symbols. For example $(A_1 \leftrightarrow (\neg A_2))$ is grammatically correct sentence, but $(A_1(\neg) \leftrightarrow$ is not.
   (c) **Allowable translations** between the formal language and english. (i.e. $A_1, A_2, \ldots$ are translated as declarative sentences, binary operatives translate as some form of conjunctive, etc)

   Note: It is only in the "allowable translations" that we give any meaning to the formal language. We can consider and work with formal languages without giving them any meaning (this is what a computer does). Of course it is in these translations that we relate the formal language to the rest of mathematics, to deductive reasoning, to questions of artificial intelligence, etc ...