MAT/CSE 371: PROBLEM SET 6
SOLUTIONS TO SELECTED PROBLEMS

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Most of these solutions are provided courtesy of Raymond Cassella.

- §2.5 #8

Create an extension $\mathcal{L}'$ of the original language $\mathcal{L}$
by adding the constant symbols c and d, and define

$$\sigma_k = \begin{cases} c \neq d & k = 0, \\ \neg Pcd & k = 1, \\ \forall v_1 \ldots \forall v_{k-1} \neg (Pcv_1 \wedge Pv_1v_2 \wedge \ldots \wedge Pv_{k-2}v_{k-1} \wedge Pv_{k-1}d) & k > 1. \end{cases}$$

Now take $\Gamma = \operatorname{Th}\mathfrak{A} \cup \{\sigma_k | k \in \mathbb{N}\}$,
where $\operatorname{Th}\mathfrak{A}$ is the set of all sentences true in $\mathfrak{A}$.

Claim: Γ is finitely satisfiable.
Proof: For any finite $\Gamma_0 \subset \Gamma$, there is some $n \in \mathbb{N}$ such that

$$\Gamma_0 \subset \operatorname{Th}\mathfrak{A} \cup \{\sigma_k | k < n\}.$$

Now create an extension $\mathfrak{B}$ of $\mathfrak{A}$ by taking $|\mathfrak{B}| = |\mathfrak{A}|$, $P^{\mathfrak{B}} = P^{\mathfrak{A}}$,
and $c^{\mathfrak{B}} = 0, d^{\mathfrak{B}} = n$. Since $\mathfrak{B}$ is an extension of $\mathfrak{A}$, it must satisfy $\operatorname{Th}\mathfrak{A}$.

Let $k < n$, and suppose for some $s : V \to |\mathfrak{B}|$ we had

$$\models_{\mathfrak{B}} Pcv_1 \wedge Pv_1v_2 \wedge \ldots \wedge Pv_{k-2}v_{k-1} \wedge Pv_{k-1}d[s].$$

Then we would have

$$|0 - s(v_1)| = |s(v_1) - s(v_2)| = \ldots = |s(v_{k-2}) - s(v_{k-1})| = |s(v_{k-1}) - n| = 1,$$

which clearly violates the triangle inequality for $k < n$.
Since s was arbitrary, it follows that $\models_{\mathfrak{B}} \sigma_k$. Therefore $\mathfrak{B}$ satisfies every element of Γ_0. □

By the compactness theorem, there is some structure $\mathfrak{C}$ which satisfies every member of Γ.
We then have that $\mathfrak{C} \equiv \mathfrak{A}$, for

$$\models_{\mathfrak{A}} \sigma \quad \Rightarrow \quad \sigma \in \operatorname{Th}\mathfrak{A} \quad \Rightarrow \quad \models_{\mathfrak{C}} \sigma$$

and

$$\not\models_{\mathfrak{A}} \sigma \quad \Rightarrow \quad \models_{\mathfrak{A}} \neg\sigma \quad \Rightarrow \quad (\neg\sigma) \in \operatorname{Th}\mathfrak{A} \quad \Rightarrow \quad \models_{\mathfrak{C}} \neg\sigma \quad \Rightarrow \quad \not\models_{\mathfrak{C}} \sigma.$$

Further, $\mathfrak{C}$ is not connected, because each σ_k rules out the existence of a path of length k
between $c^{\mathfrak{C}}$ and $d^{\mathfrak{C}} \in |\mathfrak{C}|$.

- Soundess Theorem $\Leftrightarrow$ Corollary 25E

$\Rightarrow$ Suppose Γ is satisfiable.
Let $\mathfrak{A}$ be a structure satisfying Γ,
and φ be any formula such that $\Gamma \vdash \varphi$.
By the soundness theorem, we therefore have $\Gamma \models \varphi$,
so in particular $\models_{\mathfrak{A}} \varphi$ and $\not\models_{\mathfrak{A}} \neg\varphi$.
Then $\Gamma \not\models \neg\varphi$, and by the soundess theorem again,
$\Gamma \nvdash \neg\varphi$. It follows that Γ is consistent.

$\Leftarrow$ Suppose $\Gamma \vdash \varphi$.
Then we have $\Gamma; \neg\varphi \vdash \varphi$, and also $\Gamma; \neg\varphi \vdash \neg\varphi$,
so that $\Gamma; \neg\varphi$ is inconsistent, and therefore not satisfiable, by the corollary.
Thus any structure satisfying Γ cannot also satsify $\neg\varphi$, and must therefore satisfy φ, i.e., $\Gamma \models \varphi$.

An alternate Proof of $\Leftarrow$ If Γ is not satisfiable then the condition $\Gamma \models \phi$ is vacuous (since there are no models of Γ), so we are done. So we assume Γ is satisfiable. Now towards a contradiction we will assume $\Gamma \vdash \phi$ and $\Gamma \not\models \phi$. But $\Gamma \not\models \phi$ implies that there exist a structure $\mathfrak{A}$ for which $\Gamma \models_{\mathfrak{A}} \neg\phi$. Then in $\mathfrak{A}$ we have that $\Gamma \cup (\neg\phi)$ is satisfiable. Since we are assuming the first statement we then also know that $\Gamma \cup (\neg\phi)$ is consistent. But, by hypothesis $\Gamma \vdash \phi$ and thus $\Gamma \cup (\neg\phi) \vdash \phi$. We also have $\neg\phi \vdash \neg\phi$ which tells us that $\Gamma \cup (\neg\phi) \vdash \neg\phi$. So $\Gamma \cup (\neg\phi)$ is supposed to be consistent, but from it we can deduce both ϕ and $\neg\phi$, this is a contradiction, thus proving the theorem.