MAT/CSE 371: PROBLEM SET 5
SOLUTIONS TO SELECTED PROBLEMS

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These solutions are provided courtesy of Raymond Cassella.

2.4.3a Define

$$\Delta = \{\alpha \,|\, \overline{v}(\alpha) = T \Leftrightarrow \models_{\mathfrak{A}} \alpha\,[s]\};$$

we show inductively that Δ contains all formulas.
It clearly contains all prime formulas, by definition of v.

Now suppose $\alpha \in \Delta$. Then we have

$$\overline{v}(\neg\alpha) = T \quad \Leftrightarrow \quad \overline{v}(\alpha) = F \quad \Leftrightarrow \quad \not\models_{\mathfrak{A}} \alpha\ [s] \quad \Leftrightarrow \quad \models_{\mathfrak{A}} \neg\alpha\ [s],$$

so $\neg\alpha \in \Delta$, and if $\beta \in \Delta$, then

$$\begin{aligned} \overline{v}(\alpha \to \beta) \quad &\Leftrightarrow \quad \overline{v}(\beta) = T \text{ or } \overline{v}(\alpha) = F \\ &\Leftrightarrow \quad \models_{\mathfrak{A}} \beta\ [s] \text{ or } \not\models_{\mathfrak{A}} \alpha\ [s] \\ &\Leftrightarrow \quad \models_{\mathfrak{A}} (\alpha \to \beta)\ [s], \end{aligned}$$

so $(\alpha \to \beta) \in \Delta$.

2.4.3b Suppose Γ tautologically implies φ; we must to show $\Gamma \models \varphi$.

Let $\mathfrak{A}$ be a structure, and $s : V \to |\mathfrak{A}|$, such that for all $\alpha \in \Gamma$, $\models_{\mathfrak{A}} \alpha\ [s]$; we must show $\models_{\mathfrak{A}} \varphi\ [s]$.

Defining v as above, for every $\alpha \in \Gamma$, we have $\models_{\mathfrak{A}} \alpha\ [s]$, and therefore $\overline{v}(\alpha) = T$.
Since Γ tautologically implies φ, we must have $\overline{v}(\varphi) = T$ as well, and therefore $\models_{\mathfrak{A}} \varphi\ [s]$, as desired.

2.4.5 It will follow from the below claim that we may take $f(n) = 3n$.

Claim: If $\langle \alpha_1, \ldots, \alpha_n \rangle$ is a deduction from Γ,
then there is a deduction $\langle \beta_1, \ldots, \beta_N \rangle$,
with $N \leq 3n$, such that $\{\forall x \alpha_i\} \subset \{\beta_i\}$.

Proof: Induction on n.

Base case $n = 0$ is trivial.

Suppose the claim is true for n, and let
$\langle \alpha_1, \ldots, \alpha_n, \varphi \rangle$ be a deduction from Γ.
Then $\langle \alpha_1, \ldots, \alpha_n \rangle$ is also a deduction,
so take the deduction $\langle \beta_1, \ldots, \beta_N \rangle$ given by the induction hypothesis.

Now, there are three cases.

Case 1: $\varphi \in \Lambda$.
Then φ is a generalization of some logical axiom,
so $\forall x\varphi \in \Lambda$ is as well.
We may therefore let $\beta_{N+1} = \forall x\varphi$, and
$\langle \beta_1, \ldots, \beta_{N+1} \rangle$, with length $N + 1 \leq 3n + 1 < 3(n + 1)$,
satisfies the claim.

Case 2: $\varphi \in \Gamma$.
Let $\beta_{N+1} = \varphi \in \Gamma$.
By assumption x does not occur free in Γ,
so that $\beta_{N+2} = (\varphi \to \forall x\varphi) \in \Lambda$ is in axiom group 4.
By modus ponens, we may take $\beta_{N+3} = \forall x\varphi$.
Then $\langle \beta_1, \ldots, \beta_{N+3} \rangle$ satisfies the claim,
with $N + 3 \leq 3(n + 1)$.

Case 3: There are some $i, j \leq n$ such that $\alpha_j = \alpha_i \to \varphi$.
Then there are some $k, l \leq N$ such that $\beta_k = \forall x\alpha_i$,
$\beta_l = \forall x(\alpha_i \to \varphi)$.
Take $\beta_{N+1} = (\forall x(\alpha_i \to \varphi) \to (\forall x\alpha_i \to \forall x\varphi)) \in \Lambda$ by axiom group 3.
Then $\beta_{N+2} = \forall x\alpha_i \to \forall x\varphi$,
by modus ponens on β_l and β_{N+2}.
Now take $\beta_{N+3} = \forall x\varphi$, by modus ponens on β_k and β_{N+2}.
Thus $\langle \beta_1, \ldots, \beta_{N+1} \rangle$ satisfies the claim, with $N + 3 \leq 3(n + 1)$.

2.4.7a To show that $\vdash \exists x(Px \to \forall xPx)$,
it suffices, by reductio ad absurdum, to show that
$\Gamma = \{\forall x\neg(Px \to \forall xPx)\}$ is inconsistent.
We show that $\Gamma \vdash \forall xPx$ and $\Gamma \vdash \neg\forall xPx$.

$\Gamma \vdash \neg(Px \to \forall xPx)$ by axiom group 2 and modus ponens.
Since $\neg(Px \to \forall xPx)$ tautologically implies Px,
by rule T we have $\Gamma \vdash Px$,
and by generalization we have $\Gamma \vdash \forall xPx$.

$\neg(Px \to \forall xPx)$ also tautologically implies $\neg\forall xPx$,
so applying rule T again we obtain $\Gamma \vdash \neg\forall xPx$.

2.4.7b $\Gamma = \{Qx, \forall y(Qy \to \forall zPz)\}$.

By algebraic variance, $\forall y(Qy \to \forall zPz) \vdash \forall x(Qx \to \forall zPz)$.

By axiom group 2 and modus ponens, $\Gamma \vdash (Qx \to \forall zPz)$.

By modus ponens again, $\Gamma \vdash \forall zPz$.

Then by algebraic variance and modus ponens, $\Gamma \vdash \forall xPx$.

2.4.11 By generalization, it is sufficient to show $\vdash (x = y \to y = x \to x = z)$.

By deduction, it is sufficient to show $(x = y) \vdash (y = z \to x = z)$.

For this we use the following deduction:

α_1	$x = y$	$\in \Gamma$
α_2	$x = y \to (x = x \to y = x)$	Ax6
α_3	$x = x \to y = x$	MP1, 2
α_4	$x = x$	Ax5
α_5	$y = x$	MP3, 4
α_6	$y = x \to (y = x \to x = z)$	Ax6
α_7	$y = z \to x = z$	MP5, 6