MAT/CSE 371: PROBLEM SET 4
SOLUTIONS TO SELECTED PROBLEMS

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These solutions are provided courtesy of Raymond Cassella (the last one also by Robert Rothenberg).

2.2.3 Show that

$$\{\forall x(\alpha \to \beta), \forall x\alpha\} \models \forall x\beta.$$

Let $\mathfrak{A}$ be any structure for the language,
and $s : V \to |\mathfrak{A}|$ any function such that

$$\models_{\mathfrak{A}} \forall x(\alpha \to \beta)\ [s] \tag{1}$$

$$\models_{\mathfrak{A}} \forall x\alpha\ [s]. \tag{2}$$

We need to show that

$$\models_{\mathfrak{A}} \forall x\beta\ [s],$$

that is, for every $d \in |\mathfrak{A}|$, we need $\models_{\mathfrak{A}} \beta\ [s(x|d)]$,
so let $d \in |\mathfrak{A}|$ be given.

By (1), we have $\models_{\mathfrak{A}} (\alpha \to \beta)\ [s(x|d)]$,
and by (2), we have $\models_{\mathfrak{A}} \alpha\ [s(x|d)]$.

By definition of $\models_{\mathfrak{A}}$, these two together imply
$\models_{\mathfrak{A}} \beta\ [s(x|d)]$, as desired.

2.2.21 Let y be a variable which does not occur free in α,
and let $\mathfrak{A}$ be any fixed structure for the language.
I claim that

$$\models_{\mathfrak{A}} \exists! x\alpha \quad \Leftrightarrow \quad \models_{\mathfrak{A}} \exists y \forall x(\alpha \leftrightarrow x = y) \tag{3}$$

First suppose that

$$\models_{\mathfrak{A}} \exists! x\alpha, \tag{4}$$

and let $s : V \to |\mathfrak{A}|$.
Then there is exactly one $d \in |\mathfrak{A}|$ such that $\models_{\mathfrak{A}} \alpha\ [s(x|d)]$.
Then for the first direction of (3) it suffices to show that

$$\models_{\mathfrak{A}} (\alpha \leftrightarrow x = y)[s(x|a)(y|d)], \quad \text{for any } a \in |\mathfrak{A}|. \tag{5}$$

But this is clear:

$$\models_{\mathfrak{A}} \alpha\ [s(x|a)(y|d)] \quad\Leftrightarrow\quad \models_{\mathfrak{A}} \alpha\ [s(x|a)] \quad\Leftrightarrow\quad a = d \quad\Leftrightarrow\quad \models_{\mathfrak{A}} (x{=}y)\ [s(x|a)(y|d)].$$

For the converse, suppose that there is some $d \in |\mathfrak{A}|$ for which (5) holds. Then for any $a \in |\mathfrak{A}|$, we have

$$\models_{\mathfrak{A}} \alpha\ [s(x|a)] \quad\Leftrightarrow\quad \models_{\mathfrak{A}} \alpha\ [s(x|a)(y|d)] \quad\Leftrightarrow\quad \models_{\mathfrak{A}} x{=}y\ [s(x|a)(y|d)] \quad\Leftrightarrow\quad a = d,$$

which gives (4).

A Prove

(6) $$\models \exists x(Qx \to \forall x Qx)$$

Let $\mathfrak{A}$ be any structure for the language, and $s : V \to |\mathfrak{A}|$ any function. There are two cases.

Case 1: $\models_{\mathfrak{A}} \forall x Qx\ [s]$. Let $d \in |\mathfrak{A}|$ be arbitrary. Then $\models_{\mathfrak{A}} \forall x Qx\ [s(x|d)]$, since x does not appear free in $\forall x Qx$. We therefore have

(7) $$\models_{\mathfrak{A}} (Qx \to \forall x Qx)\ [s(x|d)],$$

verifying (6).

Case 2: $\not\models_{\mathfrak{A}} \forall x Qx\ [s]$. Then there must be some $d \in |\mathfrak{A}|$ which for which $\not\models_{\mathfrak{A}} Qx\ [s(x|d)]$, making (7) true for this choice of d, verifying (6).

B

$$\begin{aligned}\models_{\mathfrak{A}} (\alpha \vee \beta)\ [s] &\Leftrightarrow \models_{\mathfrak{A}} ((\neg\alpha) \to \beta)\ [s]\\ &\Leftrightarrow \not\models_{\mathfrak{A}} (\neg\alpha)\ [s] \text{ or } \models_{\mathfrak{A}} \beta\ [s]\\ &\Leftrightarrow \models_{\mathfrak{A}} \alpha\ [s] \text{ or } \models_{\mathfrak{A}} \beta\ [s]\end{aligned}$$

C We have three formulas:

(8) $$\forall v_1 f v_0 v_1 = v_0$$

(9) $$\exists v_0 \forall v_1 f v_0 v_1 = v_1$$

(10) $$\exists v_0 (P v_0 \wedge (\forall v_1 P f v_0 v_1))$$

Define a structure $\mathfrak{A}$ by $|\mathfrak{A}| = \{0\}$, then $f^{\mathfrak{A}} : |\mathfrak{A}|^2 \to |\mathfrak{A}|$ must be given by $f^{\mathfrak{A}}(0,0) = 0$. Take $P^{\mathfrak{A}} = \{0\} \subset |\mathfrak{A}|^1$. I claim that $\mathfrak{A}$ is a model for all three formulas above.

For any $s : V \to |\mathfrak{A}|$, and any $a \in |\mathfrak{A}|$, $s(v_1|a) = s \equiv 0$, and we have

$$\overline{s}(f v_0 v_1) = f^{\mathfrak{A}}(s(v_0), s(v_1)) = 0 = \overline{s}(v_0)$$

which means $\models_{\mathfrak{A}} (fv_0v_1 = v_0)\ [s(v_1|a)]$, which proves $\mathfrak{A}$ is a model of (8).

In fact, we also have

$$\overline{s(v_0|0)(v_1|a)}(fv_0v_1) = f^{\mathfrak{A}}(s(v_0), s(v_1)) = 0 = \overline{s(v_0|0)(v_1|a)}(v_1),$$

and therefore $\models_{\mathfrak{A}} (fv_0v_1 = v_1)\ [s(v_0|0)(v_1|a)]$
for all $s : V \to |\mathfrak{A}|$, and $a \in |\mathfrak{A}|$,
which shows that $\mathfrak{A}$ models (9) as well.

We also have $0 \in P^{\mathfrak{A}}$, so that

$$\models_{\mathfrak{A}} Pv_0\ [s(v_0|0)],$$

and that

$$\overline{s}(fv_0v_1) = f^{\mathfrak{A}}(s(v_0), s(v_1)) = 0 \in P^{\mathfrak{A}},$$

which means $\models_{\mathfrak{A}} Pfv_0v_1\ [s(v_1|a)]$ for all $a = 0 \in |\mathfrak{A}|$, so that

$$\models_{\mathfrak{A}} \forall v_1 Pfv_0v_1\ [s(v_0|0)],$$

and therefore $\mathfrak{A}$ is a model for (10).

Define $\mathfrak{B}$ by taking $|\mathfrak{B}| = \{0, 1\}$,
define $f^{\mathfrak{B}} : |\mathfrak{B}|^2 \to |\mathfrak{B}|$ by $f(x, y) = 0$.
Let $P^{\mathfrak{B}} = \emptyset$.
I claim that $\mathfrak{B}$ is not a model for either (8), (9), or (10).

Let s be such that $s(v_0) = 1$. Then

$$\overline{s}(fv_0v_1) = f^{\mathfrak{B}}(s(v_0), s(v_1)) = 0 \neq 1 = s(v_0),$$

so that $\not\models_{\mathfrak{B}} fv_0v_1 = v_0\ [s]$,
so $\mathfrak{B}$ is not a model for (8).

Now let s be such that $s(v_1) = 1$. Then

$$\overline{s}(fv_0v_1) = f^{\mathfrak{B}}(s(v_0), s(v_1)) = 0 \neq 1 = s(v_1),$$

so that $\not\models_{\mathfrak{B}} fv_0v_1 = v_1\ [s]$,
so $\mathfrak{B}$ is not a model for (9).

For any $s : V \to |\mathfrak{B}|$, and any $b \in |\mathfrak{B}|$,
we have $s(v_0|b)(v_0) \notin P^{\mathfrak{B}}$, so that

$$\not\models_{\mathfrak{B}} Pv_0\ [s(v_0|b)] \quad \Rightarrow \quad \not\models_{\mathfrak{B}} Pv_0 \wedge (\forall v_1 Pfv_0f_1)\ [s(v_0|b)],$$

so that $\mathfrak{B}$ is not a model for (10).

C Here is an alternate set of models for the last question. Please fill in the proofs that these models satisfy (or don't satisfy) the given formulas.

Let P be a unary predicate symbol and f a binary function symbol. For each of the following, find a structure which satisfies the formula, and find another which does not.

(1) $\forall v_1 f v_0 v_1 = v_0$ is satisfied for the set of Natural Numbers where $f v_0 v_1 = v_0 \cdot v_1$ (multiplication) and $v_0 = 0$, and is not satisfied for the set of all people where $f v_0 v_1$ is the first-born child of v_0 and v_1 (since nobody is his own child).

(2) $\exists v_0 \forall v_1 f v_0 v_1 = v_1$ is satisfied for the set of Natural Numbers where f is multiplication and $v_0 = 1$, and is not satisfied for the set of all people where $f v_0 v_1$ is the first-born child of v_0 and v_1 (again, since nobody is his own child).

(3) $\exists v_0 (P v_0 \wedge (\forall v_1 P f v_0 v_1))$ is satisfied for the set of Natural Numbers where f is multiplication and Px means "x is a composite number", and is not satisfied for the set of all people where $f v_0 v_1$ is the first-born child of v_0 and v_1 and Px means "x is alive today" (since not all people v_1 have children who are alive today while, for instance, Julius Ceasar has no children who are alive today).