MAT/CSE 371: PROBLEM SET 2
SOLUTIONS TO SELECTED PROBLEMS

INSTRUCTOR:JASON BEHRSTOCK

Most of these solutions are provided courtesy of Raymond Cassella.

(1) p. 53 problem 6

(2) p. 54 problem 12

Solution: We claim (claim 0): $\{\wedge, \top, \bot\}$ is not complete, because one cannot express anything equivalent to $(\neg \mathcal{A})$ using these connectives. That is, any unary function expressible through them is either a constant $\top, \bot$, or the identity $\mathcal{A}$.

First one needs to check this for the propositional symbols (and for connectives which are 0-ary). Let β and γ, each be either $A, \top$, or $\bot$. By the induction principle, to prove our above claim, it now suffices to check this claim holds under the formula building operations, (i.e under combining β and γ by $\wedge$). That is we need to check that each of these gives us a sentence that is tautologically equivalent to either $\top$, $\bot$, or $\mathcal{A}$.

Now there are nine ($=3^2$, you should check why this is the right number) cases for $\alpha = (\beta \wedge \gamma)$.

Claim1: If either β or γ is $\bot$, then α is tautologically equivalent to $\bot$ as well (5 cases here!).

Claim 2: Now suppose neither β or γ is $\bot$. If either (or both) of β and γ is $\mathcal{A}$, then α is equivalent to $\mathcal{A}$ (3 cases here).

Claim 3: The remaining possibility is that both β and γ are $\top$. In which case so is α (1 case).

For a complete argument you need a argument to prove each of the three claims above. There are several ways to do this, providing truth tables is one...

By the induction principle claim 0 is true for all wffs using these connectives, thereby showing what we want.

(3) p. 65 problem 4

Solution:

For $i \in \mathbb{N}$, and $j \in \{1,2,3,4\}$, let $\mathcal{A}_i^j$ represent the statement "C_i is colored color j". Define

$$\alpha_i^k = \bigvee_{1 \leq j < k} (\neg \mathcal{A}_i^j) \bigvee_{k=j} \mathcal{A}_i^j \bigvee_{k<j\leq 4} (\neg \mathcal{A}_i^j)$$

A truth assigment to the $\{\mathcal{A}_i^j\}$ satisfies α_i^k iff country i is colored exactly one color (color k). So if country i is colored k(i) (here we think of k as a function of i) then the for each i exactly one of the sentences $\alpha_i^1, \alpha_i^2, \alpha_i^3, \alpha_i^4$ is satisfied, in particular the the one $\alpha_i^{k(i)}$ is satisfied. Then let:

$$\Sigma_1 = \{\alpha_1^{k(1)}, \alpha_2^{k(2)}, \alpha_3^{k(3)} \ldots\}$$

A truth assigment to the $\{\mathcal{A}_i^j\}$ satisfies Σ_1 iff each country is colored exactly one color. Now let Σ_2 be the set containing, for each pair of adjacent countries C_i and C_j, the wff

$$\bigwedge_{k=1}^{4} \neg(\mathcal{A}_i^k \wedge \mathcal{A}_j^k),$$

so a truth assignment satisfies Σ_2 iff no adjacent countries have the same color.

Any finite subset $K \subset \Sigma_1 \cup \Sigma_2$ must involve finitely many of the countries, and the content of the Four Color Theorem is precisely that K is satisfiable, i.e., there exists a coloring such that each country is colored exactly one color, and if two countries are adjacent, their colors differ.

$\Sigma_1 \cup \Sigma_2$ is therefore finitely satisfiable, so by the compactness theorem it is satisfiable. A truth assigment satisfying every member of $\Sigma_1 \cup \Sigma_2$ corresponds to a four coloring of the countries, so the (infinite) map can be four colored.

(4) p. 66 problem 8

Solution: Suppose Σ is decidable. Then it is also semidecidable, and therefore effectively enumerable.

Also, since we have an effective procedure which, given an expression α, will decide whether or not $\alpha \in \Sigma$, we can form a new procedure which gives the opposite output of the first procedure, so that Σ^c is also decidable, and therefore effectively enumerable.

Now suppose Σ and Σ^c are effectively enumerable (semidecidable).

We then have an effective procedure P_1 which, given an expression ϵ, produces the answer "yes" in finite time whenever $\epsilon \in \Sigma$,

and another effective procedure P_2 for Σ^c, which returns "yes" whenever $\epsilon \notin \Sigma$.

Given an expression ϵ, we follow a new procedure P.

We first follow P_1 for 1 minute, then P_2 for 1 minute, then P_1 for 2 minutes, then P_2 for 2 minutes, then P_1 for 3 minutes, then P_2 for 3 minutes, etc., until we get a positive response from either P_1 or P_2.

If P_1 produces "yes", then $\epsilon \in \Sigma$, so P returns "yes".

If P_2 produces "yes", then $\epsilon \in \Sigma^c$, so P returns "no".

For any ϵ, we have exactly one of $\epsilon \in \Sigma$ or $\epsilon \in \Sigma^c$, so exactly one of P_1 or P_2 will give "yes" after a finite amount of time.

Therefore P returns "yes" iff $\epsilon \in \Sigma$, and "no" iff $\epsilon \notin \Sigma$, so that Σ is decidable.

(5) Prove $\mathbb{Z} \times \mathbb{Z}$ is countable

Solution: I will talk about this in class and give a few different solutions.