

MAT/CSE 371: PROBLEM SET 1
SOLUTIONS TO SELECTED PROBLEMS

INSTRUCTOR: JASON BEHRSTOCK

- (1) p. 19 problem 2
- (2) p. 27, problem 3
- (3) p. 27, problem 4

Solution:

Here is one of many proofs. Let us make a table which lists all possible values of Σ, α, β (for Σ , the value T means that all formulas in Σ have value T ; value F for Σ means that at least one of the formulas in Σ has value F). We also list for each such combination values of $\Sigma; \alpha$ and $(\alpha \rightarrow \beta)$.

	Σ	α	β	$\Sigma; \alpha$	$\alpha \rightarrow \beta$
1	T	T	T	T	T
2	T	T	F	T	F
3	T	F	T	F	T
4	T	F	F	F	T
5	F	T	T	F	T
6	F	T	F	F	F
7	F	F	T	F	T
8	F	F	F	F	T

Then $\Sigma; \alpha \models \beta$ means that whenever $\Sigma; \alpha$ has value T , β also has value T . In other words, it means that the combination of values in row 2 can not happen for any values of A_i .

Now $\Sigma \models (\alpha \rightarrow \beta)$ means that whenever Σ has value T , so does $(\alpha \rightarrow \beta)$. Again, looking at the table above, we see that it means that combination of values in row 2 can not happen for any values of A_i . So these two conditions are equivalent.

This completes the proof of part (a); part (b) is completely parallel.

- (4) p. 28, problem 8

Solution:

First, one needs to understand what exactly is being asked. Let us illustrate this by an example. Let

$$\varphi = (A_1 \wedge \neg A_2)$$

and $\alpha_1 = (A_2 \vee A_3), \alpha_2 = A_5$ so that

$$\varphi^* = (\alpha_1 \wedge \neg \alpha_2) = ((A_2 \vee A_3) \wedge \neg A_5).$$

Next, if we choose some values $v(A_i)$ so that $v(\alpha_1) = T, v(\alpha_2) = F$, then

(1)
$$v(\varphi^*) = T \wedge \neg F = T$$

Now choose values for A_1, A_2 as follows: $u(A_1) = v(\alpha_1) = T, u(A_2) = v(\alpha_2) = F$. Then

(2)
$$u(\varphi) = T \wedge \neg F = T$$

which agrees with the value $v(\varphi^*)$. It is clear why it happens: both $v(\varphi^*)$ and $u(\varphi)$ are given by $T \wedge \neg F$. The difference is that in the first case, T appeared as value of α_1 and F as value of α_2 , while in the second case, T appeared as value of A_1 and F as value of A_2 .

But of course when we calculate the value of $T \wedge \neg F$, it does not matter where this T and F came from.

Now to a formal proof. It goes by induction in φ . Namely, we need to show that

- (a) The statement is true when φ is an atomic formula, i.e. $\varphi = A_i$ for some i .
- (b) If the statement is true for φ_1, φ_2 , then it is also true for $(\varphi_1 \wedge \varphi_2), (\varphi_1 \vee \varphi_2), (\varphi_1 \rightarrow \varphi_2), (\varphi_1 \leftrightarrow \varphi_2), (\neg \varphi_1)$.

The first statement follows from definitions: if $\varphi = A_i$, then $\varphi^* = \alpha_i$, and $u(\varphi) = u(A_i) = v(\alpha_i) = v(\varphi^*)$ (given).

The second follows immediately. For example, for $\varphi_1 \wedge \varphi_2$ we have

$$\begin{aligned} u(\varphi_1 \wedge \varphi_2) &= u(\varphi_1) \wedge u(\varphi_2) = v(\varphi_1^*) \wedge v(\varphi_2^*) && \text{(by induction assumption)} \\ &= v((\varphi_1 \wedge \varphi_2)^*). \end{aligned}$$

This proves part (a); part (b) is immediate corollary of (a).

- (5) p. 28, problem 10

Solution: Part (b): let A_1, A_2, A_3, \dots be predicate symbols. then let $\alpha_1 = A_1, \alpha_2 = (A_1 \wedge A_2), \alpha_3 = (\alpha_2 \wedge A_3), \dots, \alpha_n = (\alpha_{n-1} \wedge A_n), \dots$

Then let $\Sigma = \cup_{1 \leq i \leq n} \alpha_i$. Note that for any k , we have $\alpha_k \models \alpha_j$ for all $j \leq n$.

For a contradiction let's suppose we have a set $\Sigma' \subset \Sigma$ which is equivalent and independent. Let $\alpha_i, \alpha_j \in \Sigma'$. With out loss of generality we suppose $i < j$. But then we know $\alpha_j \models \alpha_i$. So Σ' must only contain one sentence. But since any sentence has only finitely predicate symbols and Σ has infinitely many sentences with (independent) predicate symbols we have a contradiction.

- (6) p. 29, problem 12

- (7) p. 29, problem 14

- (8) p. 33, problem 2

- (9) p. 42, problem 2