MATHEMATICAL LOGIC

SUNY AT STONY BROOK-SPRING 2003-MAT/CSE 371
COURSE NOTES
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1. Overview

1. Text:
   Herbert Enderton, *A Mathematical Introduction to Logic*
2. Class will be divided into three main sections:
   1. (a) Predicate Logic
   2. (b) First Order Logic
   3. (c) Undecidability
3. This course has several goals
   1. (a) Make precise the framework in which we work whenever we do mathematics
   2. (b) Mathematics is a form of deductive thought, so more generally what we would like to do is build a model for deductive thought.
   3. (c) Approach questions of solvability/computability
4. What does it mean to have a model?
   - Given a real-life object, a **model** is something which accurately represents some of the features of that object while ignoring others.
   - Whether or not the model serves its purpose depends on the choice of properties which we choose to represent.

   Example: Suppose I showed you a 2 inch diameter gray ball and said that it is a model airplane. It *is* a model, but for most purposes it is not a very good one. It is not a good model for a 747 airplane, because even though they both have the same color that is not a property that one is usually looking to preserve in a model airplane.
5. The real-life objects that we care about in this class are logical deductions.

   To quote an overused example (modified slightly):

Date: 1/21/2003.

All women are mortal.
Hypatia is a woman.
Therefore, Hypatia is mortal.

The deduction of the third line from the first two does not depend on any fact about Hypatia (Like she was head of the Platonist school at Alexandria around 400 AD) or even what the meaning of mortal is. We should note that it does however matter what we mean by **all**.

(6) We shall study logical deductions like these; for the most part ignoring their content and focusing on their form. The vagueness of this previous sentence is something we want to get rid of by dealing with mathematical models.

(7) Some basic questions we will study include:
  - What does it mean for one sentence to **follow logically** from another?
  - If a sentence does **follow logically** what methods of **proof** are necessary to establish this.
  - Must all declarations be either **True** or **False**?

(8) Here is a brief overview of the 3 main parts of this course:
  - **Predicate Logic** This will be our first mathematical model of logic. It is not a very powerful system, but it will provide a good *toy model* to play with. It will provide an introduction to working (and proving things) in a formal system. Moreover, many of the theorems we want to prove in the second part have similar, but technically simpler, analogues here.
  - **First Order Logic** This is our main model of mathematical reasoning and closely corresponds to what we use as mathematicians. For one thing the language is rich enough to encode statements like the intermediate value theorem for continuous functions:
    $\forall a \forall b \forall N(((fb > N) \wedge (N > fa)) \rightarrow \exists c((a < c) \wedge (c < b) \wedge (fc = N)))$
    We will show various facts about such formal systems involving provability, enumerability, computability, and axiomizations. We will give definitions of what it should mean for something to be **reasonably computable**.
    Depending on time/interest I would like to include some discussion of Turing machines and the relation of mathematical logic to computer science.
  - **Undecidability** Time will dictate how much we can discuss this subject. Ideally we will cover:

  - Gödel Incompleteness Theorem. Roughly this states that given any reasonably complex language and any decidable set of axioms then there exist statements in the language which cannot be proved or disproved (from these axioms). For example, the well ordering principle can not be proved from the Zermelo-Fraenkel axioms for set theory.
  - Formal Number Theory. We will formalize number theory and show that this language is sufficiently complex in the sense needed for Gödel's Theorem.
  - Prove the Incompleteness Theorem

2. Predicate Logic

(1) What is a Formal Language? Generally it means we have:
  (a) An **Alphabet**. This is the collection of symbols we have to work with. For example some of the symbols of Predicate Logic are: $(,), \neg, \rightarrow, A_1, A_2, A_3$
  (b) Rules for forming **grammatically correct** finite sequences of symbols. For example $(A_1 \leftrightarrow (\neg A_2))$ is grammatically correct sentence, but $(A_1(\neg) \leftrightarrow$ is not.
  (c) **Allowable translations** between the formal language and english. (i.e. $A_1, A_2, \ldots$ are translated as declarative sentences, binary operatives translate as some form of conjunctive, etc)

  Note: It is only in the "allowable translations" that we give any meaning to the formal language. We can consider and work with formal languages without giving them any meaning (this is what a computer does). Of course it is in these translations that we relate the formal language to the rest of mathematics, to deductive reasoning, to questions of artificial intelligence, etc...

(2) Homework #1: Sec 1.1: #2. Sec 1.2 #3,4,8,10ab (c for extra credit), 12, 14. Sec 1.3 #2. Sec 1.4 #2 (just look at this last one)

(3) **The Language**
  (a) **Propositional Symbols** $A_1, A_2, A_3, \ldots$
  We call these the first propositional symbol, the second propositional symbol, etc. These are the parameters of our language. Their translations are not fixed.
  (b) **Logical Symbols** These symbols will have fixed interpretations. The connectives allow us to combine propositional

symbols, while the punctuation is to ensure that sentences can be read unambiguously.

   (i) **Sentential Connective Symbols**
   We list these symbols below, followed by their names, and then the english phrase whose essenence we are trying to capture:

$\neg$	$\vee$	$\wedge$	$\rightarrow$	$\leftrightarrow$
negation	*disjunction*	*conjunction*	*conditional*	*biconditional*
“not″	*“or″*	*“and″*	$if\_then\_$	$\_iff\_$

   (ii) **Punctuation** (,) These are called the left and right parenthesis.

(4) Definition: An **expression** is a finite sequence of symbols.

(5) (rough) The **well-formed formulae** (alternatively we say **wff**, or simply **formulae**) will be the gramatically correct expressions. For us this will mean that:
   (a) Every sentence symbol is a wff.
   (b) If α and β are wff, then so are: $\neg\alpha$, $\alpha \vee \beta$, $\alpha \wedge \beta$, $\alpha \rightarrow \beta$, and $\alpha \leftrightarrow \beta$
   (c) An expression is a wff iff it is forced to be one by (a) and (b).

(6) **Induction Principle**: If S is a set of wff containing all the sentence symbols and closed under all five formula building operations, then S is the set of all wffs.

(7) A **truth assignment** v for a set S of sentence symbols is a function $v : S \rightarrow \{F, T\}$.

(8) We want an extension $\bar{v}$ which satisfies the obvious conditions imposed by the desired meanings of the connectives. i.e. $\bar{v}((\alpha \leftrightarrow \beta)) = T$ iff $\bar{v}(\alpha) = \bar{v}(\beta)$ and is F otherwise. Similarly for the other connectives.

(9) **Theorem**: For any truth assignment v for a set S there exist a unique function $\bar{v} : \bar{S} \rightarrow \{F, T\}$ satisfying the properties of an entension.
Proof by a parsing algorithm.

(10) Definition: a truth assignment v **satisfies** ϕ iff $\bar{v}(\phi) = T$.
Whenever we write this we are tacitly assuming that all predicate symbols in ϕ are in $domain(v)$.

(11) Let Σ denote a set of wff. Let τ denote a wff. Definition: Σ **Tautologically Implies** τ (written $\Sigma \models \tau$) iff every truth assignment that satisfies every member of Σ also satisfies τ.
We think of Σ as hypothesis and τ as a conclusion.

(12) Example: If $\emptyset \models \tau$ then we say that τ is a **tautology**. For instance $\tau = ((A_1 \to A_2) \leftrightarrow ((\neg A_1) \vee A_2))$ is a tautology.

(13) Another Example: If Σ is never satisfied then it is vacuously true that for any τ we have $\Sigma \models \tau$. For instance let $\Sigma = \{A, (\neg A)\}$.

(14) Another Example: If $\Sigma = \{A, (A \to B)\}$, then the statement $\Sigma \models B$, is known as obtaining B through Modus Ponens.

(15) **Truth Tables** provide a way to see if $\{\sigma_1, \ldots, \sigma_k\} \models \tau$ by looking at all possible truth assignments for the predicate symbols in each of the σ_j and seeing if any of these assignments satisfy τ.

Remark: If there are n predicate symbols then the truth table is 2^n lines long.

Question (**P=NP ?**): Does there exist a method to check $\{\sigma_1, \ldots, \sigma_k\} \models \tau$ requiring only a polynomial (in n) number of calculations?

(16) Homework #2: Sec 1.5: #6,12. Sec 1.7 #4,8, and look at #5. Also prove $\mathbb{N} \times \mathbb{N}$ is countable (see chapter 0).

(17) Certain types of electrical circuits provide a physical model of Propositional Logic.

(18) **Compactness Theorem** Σ is satisfiable if and only if it is finitely satisfiable.

Proof: Build a maximal finitely satisfiable set and use it to build a truth function under in which Σ is satisfied.

(19) Explain the notions of Effective Procedure, Decidable, Effectively Enumnerable, ...