Solutions for QUIZ I

June 9th 2003

1. Find general solutions of the following differential equations:

$$x^2y' + 2xy = 5y^4$$

This is a Bernoulli Equation. Substitute:

$$u = y^{-3} \Rightarrow y = u^{-1/3} \Rightarrow y' = -1/3u^{-4/3}u'$$

Now we have:

$$-\frac{x^2}{3}u^{-4/3}u' + 2xu^{-1/3} = 5u^{-4/3}$$

$$\Rightarrow u' - \frac{6}{x}u = -\frac{15}{x^2}$$

This is linear now.

$$\rho(x) = e^{\int -\frac{6}{x}dx} = x^{-6}$$

Multiply both sides of the equation by $\rho(x)$:

$$x^{-6}u' - 6x^{-7}u = -15x^{-8}$$

$$\Rightarrow (x^{-6}u)' = -15x^{-8} \Rightarrow x^{-6}u = 15/7x^{-7} + C$$

$$\Rightarrow u = \frac{15}{7x} + Cx^6 \Rightarrow y = \frac{1}{\sqrt[3]{\frac{15}{7x} + Cx^6}}.$$

$$(1 + 2xy)\frac{dy}{dx} = 1 + y^2$$

(**Hint:** Regard y as the independent variable)

As it is suggested, consider y to be the variable and solve the equation for x as a function of y:

$$1 + 2xy = (1 + y^2)\frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} - \frac{2y}{1+y^2}x = \frac{1}{1+y^2}$$

This is linear and we have:

$$\rho(y) = e^{\int -\frac{2y}{1+y^2}dy} = e^{-\ln(1+y^2)} = \frac{1}{1+y^2}$$

After multiplying the sides of the equation by $\rho(y)$:

$$\frac{1}{1+y^2}\frac{dx}{dy} - \frac{2y}{(1+y^2)^2}x = \frac{1}{(1+y^2)^2}$$

$$\Rightarrow \frac{d}{dy}(\frac{1}{1+y^2}x) = \frac{1}{(1+y^2)^2}$$

$$\Rightarrow \frac{x}{1+y^2} = \int \frac{1}{(1+y^2)^2}dy$$

Substitute $z = \arctan y \Rightarrow dz = \frac{dy}{1+y^2}$ and $\frac{1}{1+y^2} = \cos^2 z$.

$$= \int \cos^2 z dz = \int (\frac{1}{2} + \frac{1}{2}\cos 2z)dz$$

$$= \frac{1}{2}z + \frac{1}{4}\sin 2z + C = \frac{1}{2}\arctan y + \frac{1}{4}\sin(2\arctan y) + C$$

But $\sin(2\arctan y) = \frac{2y}{1+y^2}$,

$$\Rightarrow \frac{x}{1+y^2} = \frac{1}{2}\arctan y + \frac{y}{2(1+y^2)} + C$$

$$\Rightarrow x = \frac{1}{2}(1+y^2)\arctan y + \frac{1}{2}y + C(1+y^2).$$

2. Write a differential equation of the form $\frac{dy}{dx} = f(x,y)$ for function g such that the tangent line to the graph of g at (x,y) passes through the point $(-y,x)$. Solve this equation!

If $y = y(x)$ is a solution for the problem, the slope of the tangent line for the graph of $y(x)$ at (x,y) is $y'(x)$. On the other hand, since this is a line passing through (x,y) and $(-y,x)$, its slope is $\frac{y-x}{x-(-y)} = \frac{y-x}{y+x}$. Therefore, we have:

$$y' = \frac{y-x}{y+x}$$

This is a homogenous equation. Substitute $u = y/x$:

$$y = xu \Rightarrow y' = xu' + u \quad \text{and} \quad \frac{y-x}{y+x} = \frac{u-1}{u+1}$$

$$\Rightarrow xu' + u = \frac{u-1}{u+1}$$

$$\Rightarrow xu' = -\frac{u^2+1}{u+1}$$

$$\Rightarrow \frac{u+1}{u^2+1}du = -\frac{dx}{x}$$

$$\Rightarrow -\ln|x| + C = \int \frac{u+1}{u^2+1}du = \int \frac{u}{u^2+1}du + \int \frac{1}{u^2+1}du = 1/2\ln(u^2+1) + \arctan u$$

$$\Rightarrow -\ln|x| + C = 1/2\ln(y^2/x^2+1) + \arctan(y/x)$$

$$\Rightarrow \frac{A}{x} = e^{\arctan(y/x)}\sqrt{y^2/x^2+1}$$

$$\Rightarrow e^{\arctan(y/x)}\sqrt{y^2+x^2} = A.$$