Quiz III

June 30

1. Solve the following differential equation with the initial conditions:

$$y^{(3)} + y' = 2 - \sin x, \qquad y(0) = 1, y'(0) = 1, y''(0) = -2.$$

Solution:
We first need to find the general solution for the homogenous equation, y_c. The characteristic equation is:

$$r^3 + r = 0 \Rightarrow r(r^2 + 1) = 0$$

$$\Rightarrow y_c = c_1 + c_2 \cos x + c_3 \sin x$$

Using the method of undetermined coefficients, since 0 and $\imath$ are both roots of the characteristic equation, a possible particular solution will be:

$$y_p = Ax + Bx \cos x + Cx \sin x$$

After applying the differential equation for this particular solution we get $A = 2$, $B = 0$ and $C = \frac{1}{2}$. Therefore the general solution for the equation will be:

$$y = c_1 + c_2 \cos x + c_3 \sin x + 2x + \frac{1}{2} x \sin x$$

But

$$y(0) = c_1 + c_2 = 1, \quad y'(0) = c_3 + 2 = 1, \quad y''(0) = -c_2 + 1 = -2$$

$$\Rightarrow c_1 = -2, c_2 = 3, c_3 = -1$$

$$\Rightarrow y = -2 + 3\cos x - \sin x + 2x + \frac{1}{2} x \sin x.$$

2. Use the method of variation of parameters to find a particular solution of the given differential equation.

$$y'' + y = \csc^2 x.$$

Solution:

$$y'' + y = 0 \Rightarrow y_1 = \cos x, \ y_2 = \sin x$$

$$\Rightarrow y_p = u_1 \cos x + u_2 \sin x$$

And the equations for u_1' and u_2' are:

$$u_1' y_1 + u_2' y_2 = 0 \Rightarrow u_1' \cos x + u_2' \sin x = 0$$

$$u_1' y_1' + u_2' y_2' = f(x) \Rightarrow -u_1' \sin x + u_2' \cos x = \csc^2 x$$

$$\Rightarrow u_2' = \frac{\cos x \csc^2 x}{\cos^2 x + \sin^2 x} = \frac{\cos x}{\sin^2 x}$$
$$\Rightarrow u_2 = \int \frac{\cos x}{\sin^2 x} dx$$

Using a substitution $v = \sin x$, we have:

$$u_2 = \int \frac{dv}{v^2} = -\frac{1}{v} = -\csc x$$

On the other hand, using one of the equations we have:

$$u_1' = \csc x \Rightarrow u_1 = \int \csc x dx = \int \frac{\sin x}{\sin^2 x} dx = \int \frac{\sin x}{1 - \cos^2 x}$$

This time using the substitution $v = \cos x$, we have:

$$u_1 = \int \frac{-dv}{1 - v^2} = -\frac{1}{2} \int (\frac{1}{1-v} + \frac{1}{1+v}) dv = \ln \sqrt{\frac{1-v}{1+v}} = \ln \frac{1 - \cos x}{|\sin x|}$$

$$\Rightarrow y_p = \cos x \ln \frac{1 - \cos x}{|\sin x|} - \sin x \csc x = \cos x \ln \frac{1 - \cos x}{|\sin x|} - 1.$$

3. Determine the period and frequency of the simple harmonic motion of a body of mass 0.75 kg on the end of a spring with spring constant 48 N/m.

Solution:
The differential equation in general is $mx'' + cx' + kx = f(t)$. Here $m = 0.75$, $c = 0$, $k = 48$ and $f(t) = 0$. Therefore:

$$0.75x'' + 48x = 0 \Rightarrow x'' + 64x = 0$$

The general solution of this equation is:

$$x = A \cos 8t + B \sin 8t$$

for constants A and B. These function have the period $T = 2\pi/8 = \pi/4$ and the frequency is $\nu = \frac{1}{T} = \frac{4}{\pi}$.