MAT 303 Calculus IV : Midterm: Summer 2003

1. Verify that the following differential equation is exact and then solve it for the given initial condition:

$$(\cos x - y)dx - (x+1)dy = 0, \qquad y(0) = -1.$$

Solution:

$$\frac{d}{dy}(\cos x - y) = -1 = \frac{d}{dx}(-x-1)$$

Therefore the equation is exact and the solution will be in the form $F(x,y) = C$, when:

$$F = \int (\cos x - y)dx + g(y) = \sin x - xy + g(y)$$

But $F_y = -x - 1$, thus

$$-x + g'(y) = -x - 1 \Rightarrow g'(y) = -1$$

$$\Rightarrow g(y) = -y \Rightarrow F = \sin x - xy - y$$

$$\Rightarrow \sin x - xy - y = C$$

$$\Rightarrow y = \frac{\sin x - C}{x+1}$$

is the general solution.

$$y(0) = -1 \Rightarrow C = -1$$

$$\Rightarrow y(x) = \frac{\sin x - 1}{x+1}.$$

2. What is the general solution for:

$$y^{(6)} + 18y^{(4)} + 81y'' = 0.$$

Solution:
The characteristic equation is:

$$r^6 + 18r^4 + 81r^2 = 0$$

$$\Rightarrow r^2(r^2+9)^2 = 0$$

The roots are 0 and $\pm 3i$, each with multiplicity two and the general solution will be:

$$y = A + Bx + C\cos 3x + Dx\cos 3x + E\sin 3x + Fx\sin 3x.$$

3. Consider the initial value problem

$$y' = -y, \qquad y(0) = 1$$

Use Euler's Method with step size 1/2 to approximate $y(1)$ by hand.

Solution:

$$x_0 = 0, \quad y_0 = 1, \quad h = 0.5$$

Step One:

$$y_1 = y_0 + f(x_0, y_0)h = 1 - (1)(0.5) = 0.5, \quad x_1 = x_0 + h = 0.5$$

Step Two:

$$y_2 = y_1 + f(x_1, y_1)h = 0.5 - (0.5)(0.5) = 0.25, \quad x_2 = x_1 + h = 1$$

$$\Rightarrow y(1) \simeq 0.25$$

4. A tank contains $100l$ of brine. Pure water enters the tank at a rate of $10l/min$. The solution is kept thoroughly mixed and drains from the tank at the same rate.

a) Find a differential equation for y(t), the amount of salt remained in the tank after t minutes.
b) Solve this differential equation if there is initially $15kg$ of dissolved salt in the tank.
c) Is there any equilibrium solution for this differential equation? Is it stable or unstable? Sketch a few typical solution curves qualitatively.

Solution:
a)

$$\frac{dy(t)}{dt} = \text{rate in} - \text{rate out}$$

But since there is pure water entering the tank, the "rate in" is zero. On the other hand, the concentration of the solution in the tank at time t is $y(t)/100\ kg/l$. Therefore the "rate out" is $10 \times \frac{y(t)}{100} = y(t)/10\ kg/min$ and

$$\frac{dy}{dt} = -y(t)/10$$

b) This is a separable equation and we can write:

$$\frac{dy}{y} = -\frac{dt}{10} \Rightarrow \int \frac{dy}{y} = -\int \frac{dt}{10}$$

$$\Rightarrow \ln y = -t/10 + C \Rightarrow y(t) = Ae^{-t/10}$$

By assumption, $y(0) = 15$, so $A = 15$ and we have

$$y(t) = 15e^{-t/10}$$

c) $y(t) \equiv 0$ is the only equilibrium solution and it is stable because $-y$ is negative for $y > 0$ and positive for $y < 0$.
(There has to be a picture also!)

5. Consider the differential equation

$$2x^2y'' - 5xy' + 3y = 0, \qquad x > 0.$$

a) Substitute $y = x^r$ into the equation and find two solutions.
b) Show that these two are linearly independent and find the general solution for the differential equation.
c) Solve it for initial conditions $y(1) = 0$, $y'(1) = 2.5$.

Solution:
a) If $y = x^r$ then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Therefore

$$2x^2y'' - 5xy' + 3y = 2r(r-1)x^r - 5rx^r + 3x^r = (2r^2 - 7r + 3)x^r$$

If this is zero, then $2r^2 - 7r + 3 = 0$ and $r = 3$ or $r = 1/2$. This shows that x^3 and $\sqrt{x}$ are solutions for this equation.
b) One can show these two are independent directly or by using their Wronskian:

$$W(x^3, \sqrt{x}) = \frac{1}{2}x^{5/2} - 3x^{5/2} = -\frac{5}{2}x^{5/2} \neq 0.$$

This is a second-order linear equation with coefficients which are all continous on $x > 0$. Therefore once we have two linearly independent solutions, all the solutions are linear combinations of those two and the general solution is:

$$y = c_1x^3 + c_2\sqrt{x}.$$

c)

$$y'(x) = 3c_1x^2 + \frac{c_2}{2\sqrt{x}}$$

$$y(1) = c_1 + c_2 = 0$$

$$y'(1) = 3c_1 + \frac{c_2}{2} = 2.5$$

After solving we get $c_1 = 1$ and $c_2 = -1$ and finally

$$y = x^3 - \sqrt{x}.$$