MAT 303, Summer 2003

Final exam, July 10

Calculators not permitted. Primes and derivatives are respect to x for the first problem and respect to t in all other problems.

1. Find general solution of the following differential equations:

   (a) $x^2y^2 + 3y^2 - x^2y' = 0$

   Solution:
   One can see that this is a separable equation:

$$(x^2+3)y^2 = x^2y' \Rightarrow \frac{dy}{y^2} = \frac{x^2+3}{x^2}dx$$

$$\Rightarrow \int \frac{dy}{y^2} = \int \frac{x^2+3}{x^2}dx$$

$$\Rightarrow -\frac{1}{y} = x - \frac{3}{x} + C \Rightarrow y = \frac{x}{3 - x^2 - Cx}.$$

   (b) $xy' = 6y + x^4y^{2/3}$

   Solution:

$$xy' - 6y = x^4y^{2/3} \Rightarrow y' - \frac{6}{x}y = x^3y^{2/3}$$

   This is a Bernoulli equation. We substitute $u = y^{1-2/3} = y^{1/3}$

$$u = y^{1/3} \Rightarrow y = u^3 \Rightarrow y' = 3u^2u'$$

$$\Rightarrow 3u^2u' - \frac{6}{x}u^3 = x^3u^2 \Rightarrow u' - \frac{2}{x}u = \frac{x^3}{3}$$

   This is a linear equation and we have $\rho(x) = e^{-\int \frac{2}{x}dx} = e^{-lnx^2} = x^{-2}$. After multiplying the sides by $\rho(x)$

$$x^{-2}u' - 2x^{-3}u = \frac{x}{3} \Rightarrow (x^{-2}u)' = \frac{x}{3}$$

$$\Rightarrow \frac{u}{x^2} = \frac{x^2}{6} + C \Rightarrow y^{1/3} = u = \frac{x^4}{6} + Cx^2$$

$$\Rightarrow y = (\frac{x^4}{6} + Cx^2)^3.$$

(c) $(D-1)^2(D^2+4D+5)y=0$

Solution:
The characteristic equation is $(r-1)^2(r^2+4r+5)=0$. Therefore the roots are $r=1$ with multiplicity two and $r=-2\pm i$ with multiplicity one. Thus

$$y(x)=c_1e^x+c_2xe^x+c_3e^{-2x}\cos x+c_4e^{-2x}\sin x.$$

(d) $y'''-y''-y'+y=2e^x+3$

Solution:
The characteristic equation is

$$r^3-r^2-r+1=0\Rightarrow(r-1)^2(r+1)=0$$

$$\Rightarrow y_c=c_1e^x+c_2xe^x+c_3e^{-x}$$

is the solution to the homogenous equation. To find a particular solution for the nonhomogeneous equation we use the method of undetermined coefficients with $y_p=Ax^2e^x+B$. After substituting this into the equation one can see that $A=1/2$ and $B=3$ and we have $y_p=\frac{1}{2}x^2e^x+3$. Finally the general solution of the equation is

$$y=y_c+y_p=c_1e^x+c_2xe^x+c_3e^{-x}+\frac{1}{2}x^2e^x+3.$$

2. What is the inverse Laplace transform of $F(s) = \dfrac{e^{-3s}}{(s-2)^3}$.

Solution: If $G(s) = \frac{1}{(s-2)^3}$ and $H(s) = \frac{1}{s^3}$, we have

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{t^2}{2}$$

but

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{H(s-2)\} = e^{2t}h(t) = e^{2t}\frac{t^2}{2}$$

Finally

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{e^{-3s}G(s)\} = u_3(t)g(t-3)$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = u_3(t)e^{2t-6}\frac{(t-3)^2}{2}.$$

3. Solve the following initial value problem using Laplace transforms:

$$y'' + 2y' + y = e^{-t}\sin t, \quad y(0) = y'(0) = 0.$$

Solution:

After taking Laplace transform of both sides of the equation we have

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = \mathcal{L}\{e^{-t}\sin t\}$$

when $\mathcal{L}\{y(t)\} = Y(s)$. Also we can see that $\mathcal{L}\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$. This with the fact that $y(0) = y'(0) = 0$ give

$$(s^2 + 2s + 1)Y(s) = \frac{1}{s^2 + 2s + 2}$$

$$\Rightarrow Y(s) = \frac{1}{(s^2 + 2s + 2)(s^2 + 2s + 1)} = \frac{1}{(s^2 + 2s + 2)(s + 1)^2}$$

Using partial fractions we have

$$\frac{1}{(s^2 + 2s + 2)(s + 1)^2} = \frac{1}{(s + 1)^2} - \frac{1}{s^2 + 2s + 2}$$

But

$$\mathcal{L}^{-1}\{\frac{1}{(s + 1)^2}\} = e^{-t}t$$

and

$$\mathcal{L}^{-1}\{\frac{1}{s^2 + 2s + 2}\} = \mathcal{L}^{-1}\{\frac{1}{(s + 1)^2 + 1}\} = e^{-t}\sin t$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{-t}t - e^{-t}\sin t.$$

4. A 5 kg body is attached to a vertically suspended spring with spring constant $k = 500\ N/m$. We assume there is no friction or resistance for the system. If the body is pulled down 1 m below the springs original length and then released from rest at time $t = 0$,

   (a) find its position function $x(t)$. ($g = 9.8\ m/s^2$)

   (b) What is the period and frequency for this motion?

Solution:

The forces on the object are the gravitational force and the force from the spring. If we assume that upward is the positive direction and $x(t)$ shows change of length of the spring, then the total force at time t will be $F = -mg - kx(t)$. Therefore

$$mx'' = -mg - kx \Rightarrow x'' + \frac{k}{m}x = -g$$

$$\Rightarrow x'' + 100x = -9.8$$

This is a nonhomogeneous second-order linear equation. The general solution for the homogenous equation is $x_c = c_1 \cos 10t + c_2 \sin 10t$. Using the method of undetermined coefficients $x_p = A$ and after substituting $A = -0.098$ and we have

$$x(t) = c_1 \cos 10t + c_2 \sin 10t - 0.098$$

But $x(0) = -1$ and $x'(0) = 0$

$$\Rightarrow c_1 - 0.098 = -1 \text{ and } c_2 = 0$$

$$\Rightarrow x(t) = -0.902 \cos 10t - 0.098$$

It is clear from above that $T = \frac{2\pi}{10} = \frac{\pi}{5}$ and $\nu = \frac{1}{T} = \frac{5}{\pi}$.

5. Solve the system of differential equations

$$\begin{cases} x' = 3x - y + t \\ y' = 5x - 3y \end{cases}$$

where $x(0) = y(0) = 0$.

Solution:

We can write the system as

$$\begin{cases} (D-3)x + y = t \\ -5x + (D+3)y = 0 \end{cases}$$

The operational determinant is $(D-3)(D+3)+5 = D^2 - 4$ and we have

$$(D^2 - 4)x = (D+3)t = 3t + 1$$

This is a nonhomogeneous equation and we have

$$x_c = c_1 e^{2t} + c_2 e^{-2t} \text{ and } x_p = -\frac{3}{4}t - \frac{1}{4}$$

$$\Rightarrow x(t) = c_1 e^{2t} + c_2 e^{-2t} - \frac{3}{4}t - \frac{1}{4}$$

On the other hand using the first equation we have

$$y = 3x - x' + t = c_1 e^{2t} + 5c_2 e^{-2t} - \frac{5}{4}t$$

But

$$x(0) = c_1 + c_2 - \frac{1}{4} = 0 \text{ and } y(0) = c_1 + 5c_2 = 0$$

$$\Rightarrow c_1 = \frac{5}{16} \text{ and } c_2 = -\frac{1}{16}$$

$$\Rightarrow \begin{cases} x(t) = \frac{5}{16}e^{2t} - \frac{1}{16}e^{-2t} - \frac{3}{4}t - \frac{1}{4} \\ \\ y(t) = \frac{5}{16}e^{2t} - \frac{5}{16}e^{-2t} - \frac{5}{4}t \end{cases}$$