

Problem Set IX

GRAPHS

Due Apr. 15th

As usual, think about all problems, write your ideas for solving them but write the solution for “three” of them carefully and mathematically.

For the following problems, you need to know these definitions:

- A *graph* consists of a finite set of points called *vertices* and a finite number of arcs called *edges* joining some of the vertices. In most problems, when we talk about a graph we mean a *simple* graph, that is a graph that between any two vertex there is at most one edge and there is no edge connecting a vertex to itself.
- The number of edges attached to each vertex is called the *degree* of that vertex.
- A graph is called *connected* if you can connect any two vertices by a sequence of edges.
- A path is a sequence of edges and vertices $v_0, e_1, v_1, e_2, v_3, \dots, v_{n-1}, e_n, v_n$ in a graph that e_i s are edges and v_j s are vertices and e_i connects v_{i-1} and v_i , $1 \leq i \leq n$ and $0 \leq j \leq n$. Also, we require that each edge and each vertex appears in the sequence at most once, except possibly that the initial and terminal vertices are the same, in which case we call it a *closed path*. In either case, the length of the path or the closed path is n . We usually denote the path only by its vertices v_0, v_1, \dots, v_n .
- A graph is called a *tree* if it is connected and contains no closed path in it.

1. In class, we defined a tree as a graph without any closed path. Show that the followings are equivalent definitions for a graph:

- (a) A graph is a tree if and only if for any two vertices, it has a unique path connecting them.
- (b) A tree is a connected graph with a minimum number of edges. (Removing any edge makes it disconnected.)

2. Prove the following statements:

- (a) A tree has at least two vertices of degree one. These are what we call the *leaves* of the graph. (*Hint*: You can use induction.)
- (b) In a tree the number of vertices is one plus the number of edges.

3. Given natural numbers d_1, d_2, \dots, d_n , with $d_1 + d_2 + \dots + d_n = 2n - 2$, can you construct a tree such that degrees of its vertices are d_1, d_2, \dots, d_n .
4. In NPBM (National Park for Bored Mathematicians), there is a lake with 7 islands. There are 1, 3 or 5 bridges leading to each island. Is it true that at least one of these bridges must lead to the shore of the lake?
5. In a country on planet Markar, there are 15 towns, each of which is connected by a road to at least 7 others. Prove that you can travel by roads from any town to any other town (possibly through intermediate towns).
6. In a tournament, n teams participate. Each two teams play exactly once and every game ends by a team winning and the other losing. Show that independent of the results, at the end, we can always put the teams in a row P_1, P_2, \dots, P_n , such that P_1 has won P_2 , P_2 has won P_3 and so on.
7. Let G be a graph in which every vertex has degree at least k , where k is an integer at least 2. Prove that G has a non-closed path of length at least k and a closed path of length at least $k + 1$.