# Problem Set IX

## Graphs

*Due Apr. 15th*

1. In class, we defined a tree as a connected graph without any closed path. Show that the followings are equivalent definitions for a tree:

   (a) A connected graph is a tree if and only if for any two vertices, it has a unique path connecting them.

   (b) A tree is a connected graph with a minimum number of edges. (Removing any edge makes it disconnected.)

   **Solution:**

   (a) Of course, if the graph has a closed path in it and we take two vertices on this closed path, there exists at least two different paths connecting them. (Just on the closed path, we have two different paths connecting them.) On the other hand, assume there are two different paths connecting two vertices. Take the union of the two and remove all edges that are in both of them. One can show that what remains is a union of closed paths and some isolated vertices and therefore the graph is not a tree.

   (b) If a connected graph is not a tree then it has a closed path. One can see that removing one edge from this closed path does not disconnect the graph. On the other hand for a tree take any edge $[v_0, v_1]$. Part (a) shows that the only path connecting $v_0$ and $v_1$ is the one consisted of this edge alone. Therefore if we remove it, there will be no path connecting the two vertices and the graph becomes disconnected.

2. Prove the following statements:

   (a) A tree with more than one vertex has at least two vertices of degree one. These are what we call the *leaves* of the graph. (*Hint:* You can use induction.)

   (b) In a tree the number of vertices is one plus the number of edges.

   **Solution:**

   (a) The only tree with exactly two vertices is an edge alone and apparently it has two vertices of degree one. Now if we know the statement for all trees with less than $n$ vertices, let's prove it for a tree $T$ with $n$ vertices. Take an edge $e = [v_0, v_1]$ and look at $T - e$.

   **Claim:** *$T - e$ is a graph with two connected components and each component is a tree.*

*Proof.* Because of part (b) in last problem $T - e$ is disconnected, so it has at least two connected components. And it has exactly two because for every vertex $v$ the unique path connecting it to $v_0$ either contains the edge $e$ or not. If it does not then $v$ will be connected to $v_0$ in $T - e$ and if it does then $e$ has to be its last edge, which means that it is connected to $v_1$ in $T - e$. So, in $T - e$ everything is either connected to $v_0$ or to $v_1$ and we have two connected components. Each of these is a tree because if it had a closed path then it would mean that $T$ has a closed path too, which is impossible. $\square$

Let's name these components $T_0$ and $T_1$, which respectively contain $v_0$ and $v_1$. The number of vertices in each of them is less than $n$; so we can use the induction hypothesis. If one of them, say $T_0$, has exactly one vertex ($T_0 = \{v_0\}$) then $T_1$ has at least 2 and therefore has a vertex $v$ of degree one which is not $v_1$. But then $v_0$ and $v$ have degree one in $T$ and we are done. If each of them has at least two vertices then by induction hypothesis, $T_0$ has a vertex other than $v_0$ with degree one and $T_1$ also has a vertex other than $v_1$ of degree one. These two have degree one in $T$ as well and we have proved the statement for this case too.

(b) Again we can use induction and we also will use part (a). When the number of vertices is 1 it is obvious. Now assume $T$ has $n > 1$ vertices. By part (a), it has a vertex $v$ with degree one. With the same kind of proof as above, one can show that after removing $v$ and the unique edge adjacent to it we get a tree with $n-1$ vertices. By induction hypothesis, this new tree has $n - 2$ edges, therefore $T$ has $n - 1$ edges and we are done.

3. Given natural numbers $d_1, d_2, \ldots, d_n$, with $d_1 + d_2 + \cdots + d_n = 2n - 2$, can you construct a tree such that degrees of its vertices are $d_1, d_2, \ldots, d_n$.

   **Solution:** Yes, we do it inductively. If $n = 1$, we have just one number $d_1 = 0$ and of course we have the desired tree. Also when $n = 2$, we have to have $d_1 = d_2 = 1$ and it is clear what the tree is. Now, assume $n > 2$ and we can solve the problem for $n - 1$. Assume $d_i$s are in descending order $d_1 \geq d_2 \geq \cdots d_n$. It is clear that they all cannot be bigger than one because then the sum would be at least $2n$. So $d_n = 1$. On the other hand $d_1 > 1$, otherwise they all had to be equal to 1 and the sum would be $n$ which is strictly less than $2n - 2$ for $n > 2$. Now take the set $d_1 - 1, d_2, \ldots, d_{n-1}$ obtained by removing $d_n$ and subtracting one from $d_1$. We can easily see that this new set satisfies all the properties in the problem for $n - 1$. Therefore by induction hypothesis, we can construct the tree for this set of numbers and the first vertex has degree $d_1 - 1$. Now just put an $n$th vertex and connect it with one edge to that one. It is easy to see that this is a tree (in fact, it is enough to show it is connected since we know it has $n - 1$ edges) and the set of degrees for it is the one we wanted.

4. In NPBM (National Park for Bored Mathematicians), there is a lake with 7 islands. There are 1, 3 or 5 bridges leading to each island. Is it true that at least one of these bridges must lead to the shore of the lake?

   **Solution:** If the bridges were all connecting the islands to each other then take the graph with the islands as its vertices and for every bridge put an edge between the islands that it connects. The assumption will translate to the fact that degree of each vertex in this graph is either 1, 3 or 5. In any case, it is an odd number and since we have 7 islands the sum of all the degrees will be an odd number too. But this is impossible, because we showed in class that the sum of the degrees in any graph has to be an even number.

5. In a country on planet Markar, there are 15 towns, each of which is connected by a road to at least 7 others. Prove that you can travel by roads from any town to any other town (possibly through intermediate towns).

   **Solution:** Take the corresponding graph. The towns are the vertices and put an edge between two towns if there is a road going from one to the other. We need to prove that this graph is connected. If not then it has more than one connected components. One of these has at most 7 vertices. Now, a vertex in this component can be connected only to vertices in the same component. But there are at most 6 more vertices which contradicts the fact that every town is connected to 7 others.

6. In a tournament, $n$ teams participate. Each two teams play exactly once and every game ends by a team winning and the other losing. Show that independent of the results, at the end, we can always put the teams in a row $P_1, P_2, \ldots, P_n$, such that $P_1$ has won $P_2$, $P_2$ has won $P_3$ and so on.

   **Solution:** We do it by induction. For $n = 1$, it is clear. Assume $n > 1$ and we know the claim for a tournament with $n-1$ teams. Take a set of $n-1$ teams from the original set. If we ignore the games that the $n$th team has played, we have a tournament with $n-1$ teams. By induction hypothesis, we can put the $n-1$ teams in a row $P_1, P_2, \ldots, P_{n-1}$ that each team has won the team that follows it in the row. Now if the $n$th team has won $P_1$ then we can just put it in the beginning and we are done. If it has lost to $P_1$ and has won $P_2$ then we can put it between those two. If it has lost to $P_2$ and has won $P_3$ then we can put it between them and so on. In fact, either at some stage this $n$th team has lost to $P_i$ and has won $P_{i+1}$ and then we can put it between them or it has lost to every body including $P_{n-1}$ and we can put it at the end of the row.

7. Let $G$ be a simple graph in which every vertex has degree at least $k$, where $k$ is an integer at least 2. Prove that $G$ has a non-closed path of length at least $k$ and a closed path of length at least $k+1$.

   **Solution:** Let's prove the first part first. Take the longest non-closed path $v_0, v_1, \ldots, v_n$ in $G$. If $n \geq k$ then we are done. Otherwise the number of vertices $\{v_1, v_2, \ldots, v_n\}$ is less than $k$ and we know $v_0$ is connected to at least $k$ vertices. Therefore it is connected to a vertex other than these, but then we could add that vertex to our path and get a longer path. This contradiction proves the claim.

   For the second part, again take the longest path $v_0, v_1, \ldots, v_n$. Using the first part, $n \geq k$ and we know that there is no edge between $v_0$ and any vertex outside the set $\{v_1, v_2, \ldots, v_n\}$. But since its subset $\{v_1, v_2, \ldots, v_{k-1}\}$ has less than $k$ elements and $v_0$ is connected to $k$ vertices, it has to be connected to a vertex $v_l$ where $l \geq k$. But then the closed path $v_0, v_1, v_2, \ldots, v_l, v_0$ has length at least $k+1$.