Problem Set VIII

More on Number Theory

Due Mar. 25th

As usual, think about all problems, write your ideas for solving them but write the solution for two of them carefully and mathematically.
You can do more for extra credit.

1. Let n be a positive integer such that $n+1$ is divisible by 24. Prove that the sum of all the divisors of n is divisible by 24.

2. Prove that any two consecutive Fibonacci numbers F_n, F_{n+1}, $n > 2$, are relatively prime, i.e. $\gcd(F_n, F_{n+1}) = 1$. The Fibonacci numbers are defined recursively by:
$$F_0 = 1,\ F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0.$$

3. (a) If $x^3 + y^3 = z^3$ has a solution in integers x, y, z, show that one of the three must be a multiple of 7.
   (b) If n is a positive integer greater than 1 such that $2^n + n^2$ is prime, show that $n \equiv 3 (\text{mod } 6)$.

4. (a) Prove that if one of the numbers $2^n - 1$ and $2^n + 1$ is prime, $n > 2$, then the other number is not.
   (b) What is the largest number N for which you can say that $n^5 - 5n^3 + 4n$ is divisible by N for every integer n?

5. Given an integer n, show that an integer can always be found which contains only the digits 0 and 1 (in the decimal representation) and which is divisible by n.

6. A certain locker room contains n lockers numbered $1, 2, 3, \ldots, n$ and all are originally locked. An attendant performs a sequence of operations $T_1, T_2, \ldots, T_n$, whereby with the operation T_k, $1 \leq k \leq n$, the condition of being locked or unlocked is changed for all those lockers and only those lockers whose numbers are multiple of k. He finishes these n operations and some doors will remain locked and some unlocked at the end. Do this for a few examples and guess which ones will remain locked and which ones remain unlocked. Prove your guess mathematically.