Problem Set VI

ELEMENTARY NUMBER THEORY

Due Mar. 11th

Here, I am giving solutions of those problems that we couldn't finish solving in class.

1. In class, we saw different criteria for dtermining the remainder of an integer when divided by 3 or 7. We proved that for 3.

   (a) Try to prove it for 7. To remind you, this was the algorithm:

   Consider n and divide it by 50. Take the quotint and remainder and add them together. The obtained number will have the same remainder as n when divided by 7.

   (b) Now prove that the following criteria works for 11:

   Consider n in its decimal presentation. From right, take its first digit, subtract the second digit, add the third digit and so on until when you are done with all of its digits. The result will have the same remainder as n, when we divide them by 11. For example, consider 238153 and compute $3 - 5 + 1 - 8 + 3 - 2 = -8$. Remainder of -8 when divided by 11 is 3, since $-8 = (-1)(11) + 3$, so is the reminder of 238153: $238153 = 2165 \times 11 + 3$.

2. What is the rightmost digit of 4^{2004}? What about 7^{2004}?

3. If N is the product of k consecutive positive integers prove that N is divisible by $k!$.

4. Prove that for any set of n integers, there is a subset of them whose sum is divisible by n.

   Solution: Assume $\{a_1, a_2, \ldots, a_n\}$ is such a set. Take the following summs 0, a_1, $a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_{n-1}, a_1 + a_2 + \cdots + a_n$. These are $n + 1$ numbers. We know that there are only n possiblities for the remainder of a number modulo n: $0, 1, 2, \ldots, n - 1$. Therefore, at least two of these sums have the same remainder modulo n. If we subtract those two, it gives us a number divisible by n; but that is also sum of a number of elements of the set like $a_{i+1} + a_{i+2} + \cdots + a_{j-1} + a_j$ and therefore the set $\{a_{i+1}, a_{i+2}, \ldots, a_{j-1}, a_j\}$, has the desired property.

5. (a) Show that $2^{2x+1} + 1$ is divisible by 3.

   (b) Prove or disprove: $2^x \equiv 2^y \pmod n$ if $x \equiv y \pmod n$.

   Solution:

   (a) We know that $4 = 2^2 \equiv 1 \pmod 3$; thus $2^{2x} \equiv 1^x = 1 \pmod 3$ and $2^{2x+1} = 2 \times 2^{2x} \equiv 2 \times 1 = 2 \pmod 3$. This finally proves that $2^{2x+1} + 1 \equiv 2 + 1 \equiv 0 \pmod 3$ and we are done.

   (b) Take $x = 1, y = 4$ and $n = 3$. Then $1 \equiv 4 \pmod 3$ but $2^1 = 2 \not\equiv 16 = 2^4 \pmod 3$.

6. (a) Prove that there are infinitely many primes!

   (b) Prove that there are infinitely many primes of the form $4n + 3$ and infinitely many primes of the form $6n + 5$, where n is an integer.

   Solution:

   (a) This is the famous Euclid's proof: If there are only a finite number of primes, say: $p_1, p_2, \ldots, p_n$, then consider

   $$N = p_1 \cdot p_2 \cdots p_n + 1.$$

   It is clear that N cannot be divisible by any of the primes p_i and it is bigger than one therefore it has to be a prime itself. Contradiction!

   (b) We can use a similar argument as above: If the number of primes of the form $4n + 3$ is finite say: $p_1, p_2, \ldots, p_n$ than look at

   $$N = 4p_1 \cdot p_2 \cdots p_n - 1.$$

   N cannot be divisible by any p_i. Therefore, in its prime factorization all the primes that appear are of the form $4n + 1$; but a product of numbers of the form $4n + 1$ will be also of the form $4n + 1$. (If $a \equiv 1 \pmod 4$ and $b \equiv 1 \pmod 4$ then $ab \equiv 1 \pmod 4$.) This gives a contradiction N is of the form $4n + 3$.

   The same argument works for numbers of form $6n + 5$. Seeking a contradiction take

   $$N = 6p_1 \cdot p_2 \cdots p_n - 1,$$

   when $p_1, p_2, \ldots, p_n$ are all the primes of the form $6n + 5$. N will be a product of primes of the form $6n + 1$, but a product of such numbers will be of the form $6n + 1$ as well and we have a contradiction.

7. For any $n \geq 1$, prove that there exists an n-digit number with odd digits which is divisible by 5^n.

   Solution: We construct the numbers by induction: For $n = 1$, consider 5 and it has all the required properties. Now by induction hypothesis assume we have an n-digit number A, with odd digits which is divisible by 5^n. We want to use this and construct an $(n + 1)$-digit number with the desired properties for $n + 1$. I claim that one of the numbers $10^n + A$, $3 \times 10^n + A$, $5 \times 10^n + A$, $7 \times 10^n + A$ or $9 \times 10^n + A$ will work. These all have $n + 1$ digits and the digits are all odd. We only need to choose one of them which is divisible by 5^{n+1}. The point is that the set $\{1, 3, 5, 7, 9\}$ has all the possible reminders modulo 5. Since 2^n is relatively prime respect to 5, the set $\{2^n, 3 \times 2^n, 5 \times 2^n, 7 \times 2^n, 9 \times 2^n\}$ also has all the possible reminders modulo 5. Therefore sum of $A/5^n$ with one of those, say $k \times 2^n$, will be divisible by 5, where k is in $\{1, 3, 5, 7, 9\}$. We can see that therefore:

$$5|(k \times 2^n + \frac{A}{5^n}) \;\Rightarrow\; 5|\frac{k \times 10^n + A}{5^n} \;\Rightarrow\; 5^{n+1}|(k \times 10^n + A)$$

and $k \times 10^n + A$ is such an $(n + 1)$-digit number.