Problem Set V

More Counting

Due Mar. 4th

Think about all the problems and try to come up with ideas to solve them and write those. Write a complete solution for at least two of the problems. The solutions have to be clear and convincing for a skeptical classmate.
In the beginning let's again recall some of the fact we described in class:

- The number of different ways to choose k objects from a set of n objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

  Here $0 \leq k \leq n$. $\binom{n}{k}$ is pronounced "n choose k."

  Example: There are $\binom{10}{3} = 10!/(3!7!) = 120$ different ways to form a team of 3 students out of a group of 10.

- The numbers $\binom{n}{k}$ form the so called *binomial coefficients.* The *Binomial Theorem* states that

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n.$$

  for any a, b. For $n = 2$ this reduces to the familiar formula $(a+b)^2 = a^2 + 2ab + b^2$, and for $n = 3$ to $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

1. (a) Ten points are marked on a plane so that no three of them are on the same straight line. How many triangles are there with vertices at these points?

   (b) Ten points are marked on a straight line, and eleven points are marked on another line parallel to the first. How many triangles are there with vertices at these points? How many parallelograms? How many rectangles?

2. (a) Using the Binomial Theorem, prove the following:

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = 0.$$

   (b) Prove that for any arbitrary set, the number of subsets with even elements is the same as the number of those with an odd number of elements.

3. Count the rectangles of all sizes formed using segments in a grid with m horizontal lines and n vertical lines. For example for $m = 3$ and $n = 2$, there are 3 of such rectangles.

4. Prove the following identity:

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

5. Given a positive integer n, prove that the number of quadruples of integers (a, b, c, d) such that $0 \le a \le b \le c \le d \le n$ is $\binom{n+4}{4}$.

6. Count the number of ways one can group 48 distinct people into 24 pairs. Can you find the answer in general case of $2n$ people to form n pairs? (The answer is 1 when $n = 1$ and is 3 when $n = 2$.)

7. Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine, with justification, the smallest integer n for which this is possible.