

Problem Set IV

ELEMENTARY COUNTING

Due Feb. 26th

Think about all the problems and try to come up with ideas to solve them and write those. Write a complete solution for at least two of the problems. The solutions have to be clear and convincing for a skeptical classmate.

For the following problems you need to know these facts:

- The number of permutations of n different objects is:

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1$$

with the convention $0! = 1$. *Example:* There are $10! = 3628800$ different ways to seat 10 students in a row.

- If we have n_1 identical objects of type 1, n_2 identical objects of type 2, \dots and finally n_k identical objects of type k , the number of permutations of all these $n_1 + n_2 + \cdots + n_k$ objects is

$$\frac{(n_1 + n_2 + \cdots + n_k)!}{n_1!n_2!\cdots n_k!}.$$

Example: Having 2 red, 3 green and 5 blue balls, there are $(2 + 3 + 5)!/(2!3!5!) = 2520$ different ways to arrange them in a row.

1. Simplify the following expressions:

$$n!(n+1) \quad \frac{n!}{(n-1)!} \quad \frac{(n+1)!}{n(n+1)}$$

and also

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \quad \binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$$

when $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

2. We saw in class that n students have $n!$ different ways to sit in a row. Now, in how many ways they can sit around a round table? Note that different ways obtained by rotating them around the table will be considered the same.
3. For each of the following examples, find the number of different words you can obtain by permuting the letters. (A word doesn't have to make sense.)

“CLOSENESS” “VECTOR” “TRUST” “CARAVAN”

4. How many ways are there to pick two cards from a standard 52-card deck such that the first card is a spade and the second card is not an ace?
5. On a 5×8 board, a rook is placed on the lower left corner of the board. Assume that the rook can only move horizontally to right or vertically upward. In how many different ways, it can reach the upper right corner of the board? (*Hint:* Every possible path corresponds to a word with seven R's, when it moves a unit rightward, and four U's, when it moves upward. For example the word RUURRRURRUR corresponds to the path determined by moving a unit right, two units up, three units right, one unit up, two units right, one unit up and finally one unit right to get to the upper right corner.)
6. (a) Six boxes are numbered 1 through 6. How many ways are there to put 20 identical balls into these boxes?
(b) Consider the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 20$. How many nonnegative integer solutions this equation has?