Problem Set III

Pigeonhole Principle

Due Feb. 19th

Think about all the problems and try to come up with ideas to solve them and write those. Write a complete solution for at least two of the problems. The solutions have to be clear and convincing for a skeptical classmate.
But first, let's recall the simple theorem, we discussed in class:

Theorem 1 *If $kn + 1$ objects (for $k \geq 1$) are distributed among n boxes, one of the boxes will contain at least $k + 1$ objects.*

For most of the problems, you don't need even to refer to this proof. In some cases a proof by contradiction may work, i.e. assume the conclusion is false and get a contradiction.

1. Prove that for any set of five points in the interior of a square of side length one, there is at least one pair of points whose distance is less than $\sqrt{2}/2$.

2. Let S be a subset of $\{1, 2, \ldots, 3n\}$ with size $2n + 1$. Prove that S must contain three consecutive numbers. Show that the same conclusion for subsets of size $2n$ is wrong by giving a counter example.

3. Suppose that each square of a 4×7 chessboard is colored either black or white. Prove that in any such coloring the board must contain a rectangle (with sides parallel to the sides of the board), whose corners are centers of four squares with the same color.

4. Given five lattice points in the plane (any point whose coordinates are integers), show that the mid point of one of the segments joining two of these points is a lattice point too. (**Hint:** Consider different cases for the parity of coordinates of the points.)

5. Suppose that the numbers 1 through 10 appear in some order around a circle. Prove that some set of three consecutive numbers sums to at least 17.

6. Given any subset of the set $\{1, 2, \ldots, 99\}$ with at least ten elements, show that there are two disjoint nonempty subsets of the set with equal sum of their elements. (**Hint:** Consider how many different subsets this set has and then what are the possible values for sum of the elements of any such subset. Use this to find two, not necessarily disjoint, subsets with that property and then use them to find two disjoint one.)

7. A 6×6 checkerboard with 36 squares is tiled by 18 dominoes: 1×2 pieces. Prove that every such tiling can be cut between some pair of adjacent rows or columns without cutting any domino.