

# Problem Set II

## METHOD OF MATHEMATICAL INDUCTION

*Due Feb. 12th*

Think about all the problems and try to come up with ideas to solve them and write those. Write a complete solution for at least two of the problems. The solutions have to be clear and convincing for a skeptical classmate. Most of the problems can be solved using induction. Don't forget to follow the steps of the induction and write it precisely. If you have not handed the last assignment, don't forget to bring that too.

1. For  $n \in \mathbb{N}$ , prove that  $1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2$ .
2. If  $A$  is a set with  $n$  elements, use induction to show that it has  $2^n$  subsets (including itself and the empty set).
3. If  $A$  is a set with  $n$  elements and  $k \leq n$ , prove that  $A$  has exactly

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

$k$ -element subsets,  $n! = 1 \cdot 2 \cdot \dots \cdot n$ . (**Hint:** Fix  $k$  and prove it for  $n \geq k$ . you may need to show that  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ .)

4. Prove that for any  $n \geq 1$ ,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

5. "All horses are the same color." Someone claims that he can prove this by induction on the number of horses in a given set. Here is how: "Let  $P(n)$  be the statement that given any set of  $n$  horses, they all are the same color.  $P(1)$  is trivially true since if there is just one horse, then it is the same color as itself. Assume  $P(n)$  (the induction hypothesis). To prove  $P(n+1)$  (the induction step), take any set of  $n+1$  horses. By the induction hypothesis, horses 1 through  $n$  are the same color, and similarly, horses 2 through  $n+1$  are the same color. So horses 1 through  $n+1$  must be the same color as well. This proves  $P(n+1)$ ." What, if anything, is wrong with this reasoning?
6. In a tournament  $n$  teams participate. Each two teams play exactly once and no game finishes in a tie. Show that independent of the results, at the end, we can always put the teams in a row  $P_1, P_2, \dots, P_n$ , such that  $P_1$  has won  $P_2$ ,  $P_2$  has won  $P_3$ ,  $\dots$ , and finally  $P_{n-1}$  has won  $P_n$ .