Problem Set XII

Maps and Cardinality

Due May 6th

As usual, think about all problems, write your ideas for solving them. *This time, you should do the first problem and at least one other one.* Again you can do more for extra credit and as always write the solutions carefully and mathematically.

As usual, let's recall some of the definitions and terms we saw in class:

- A function $f : A \to B$ is *injective* if for each $b \in B$, there is at most one $x \in A$ such that $f(x) = b$.
- A function $f : A \to B$ is *surjective* if for each $b \in B$, there is at least one $x \in A$ such that $f(x) = b$.
- A function $f : A \to B$ is a *bijection* if for every $b \in B$, there is exactly one $x \in A$ such that $f(x) = b$, or in other terms f is both injective and surjective.
- We say the sets A and B *have the same cardinality* if there is a one to one correspondence between their elements; more precisely, if there exists a bijection between them.
- An infinite set A is *countably infinite* (or *countable*) if it has the same cardinality with $\mathbb{N}$; otherwise A is *uncountably infinite* (or *uncountable*).

1. Determine which of the following statements are true. Give proofs for the true statements and counterexamples for the false statements.
   (a) Let A and B have the same cardinality and f is injective from A to B then it is a bijection.
   (b) If f is a map from A to itself, f is a bijection if and only if $f \circ f$ is a bijection.
   (c) When $f : \mathbb{R} \to \mathbb{R}$ is continuous, $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$ then f is a bijection.
   (d) A union of an infinite number of disjoint sets which are all infinite is uncountable.
2. Do the sets (a, b) and $[a, b]$ (open and closed intervals between a and b) have the same cardinality? ($a < b$ are real numbers.)
3. Can we have an uncountable number of disjoint *disks* in the plane? (A disk is a circle together with its interior.

4. Prove that the sets $[0, 1]$ and $[0, 1] \times [0, 1]$ have the same cardinality. [Hint: One nice way to do this is to use the decimal representations of the numbers in $[0, 1]$.]

5. If $f : A \to B$ and $g : B \to A$ are injective, prove that there exists a bijection $h : A \to B$, and hence A and B have the same cardinality. (Note that f and g are not necessarily bijections themselves.)

6. For any set A, let $P(A)$ be the set of subsets of A. Prove that A and $P(A)$ never have the same cardinality. [Hint: Assume there exists a bijection from A to $P(A)$. Use it to construct a subset of A which is not in the image of this map. This will be a contradiction!]

7. Use the above problem and show that $\mathbb{N}$ and $\mathbb{R}$ don't have the same cardinality and therefore $\mathbb{R}$ is uncountable. [Hint: You need to show that $\mathbb{R}$ has the same cardinality as $P(\mathbb{N})$.]

8. Does there exist an uncountable set of subsets of $\mathbb{N}$ that every two of them have finite intersection?