Problem Set XI

Continuity

Due Apr. 29th

As usual, think about all problems, write your ideas for solving them but write the solution for two of them carefully and mathematically. *You can do more for extera credit.*

In class, we discussed real numbers and *continuous* functions on real numbers. We also discussed the following theorem:

Theorem 1 (The Intermediate-Value Theorem) *Let f be a continuous function on the closed interval $[a, b]$. If $f(a) \geq 0$ and $f(b) \leq 0$, then there is a number $c \in [a, b]$ such that $f(c) = 0$.*

1. For a continuous function $f : [a, b] \to \mathbb{R}$, show that the set $S = \{x \in [a, b] | f(x) = x\}$ of fixed points of f is a *closed* subset of $\mathbb{R}$, i.e. if $x_n \in S$ and $x_n \to x$, then $x \in S$.

2. A real-valued continous function satisfies for all real x and y the functional equation

$$f(\sqrt{x^2 + y^2}) = f(x)f(y).$$

   Prove that $f(x) = [f(1)]^{x^2}$. [Hint: First prove the theorem for all numbers of the form $2^{n/2}$ where n is an integer. Then prove the theorem for all numbers of the form $m/2^n$, m an integer, n a nonnegative integer.]

3. Let $f, g : [0, 1] \to [0, 1]$ are continuous and $f \circ g = g \circ f$, prove that there exists a point $x \in [0, 1]$ such that $f(x) = g(x) = x$. Recall that $f \circ g$ is a function that $f \circ g(x) = f(g(x))$; so the assumption means that for any $x \in [0, 1]$, $f(g(x)) = g(f(x))$. [Hint: You may need to use problem 1.]

4. A rock climber starts to climb a mountain at 7:00 A.M on Saturday and gets to the top at 5:00 P.M. He camps on top and climbs back down on Sunday, starting at 7:00 A.M.and getting back to his original starting point at 5:00 P.M. Show that at some time of day on Sunday he was at the same elevation as he was at that time on Saturday. [Hint: Use the intermediate-value theorem!]

5. Prove that the trigonometric polynomial

$$a_0 + a_1 \cos x + \cdots + a_n \cos nx,$$

   where the coefficients are all real and $|a_0| + |a_1| + \cdots + |a_{n-1}| \leq a_n$, has at least $2n$ zeros in the interval $[0, 2\pi]$.

6. Consider a circular wire. If the temperature does not change abruptly from point to point then some pair of opposite points on the circle have the same temperature. [Hint: Construct a continuous function on $\mathbb{R}$ associated to this and use the intermediate-value theorem to prove the claim.]

7. There are containers of gas along a circular track. The total amount of gas is exactly enough to fuel a car once around the track. Prove that there is some starting place from which the car can complete the trip without running out of gas.

8. Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for any real number a, the equation $f(x) = a$ has exactly 3 solutions?