Problem Set X

Baby Geometry

Due Apr. 22nd

As usual, think about all problems, write your ideas for solving them but write the solution for two of them carefully and mathematically.

You may need to know the following facts from elementary geometry which we talked about in class.

- Among all paths connecting two points A and B in the plane, the straight line segment AB has the smallest possible length. We denote this length by $|AB|$.
- As a corollary, we have the so-called *triangle inequality*: For any three points A, B and C in the plane,
$$|AB| \leq |AC| + |CB|.$$
- A region R in the plane is called *convex* if for any two points A and B in R, the entire line segment AB is contained in R. For example, a triangle is always convex.

1. (a) Show that among any five points in the plane, no three of which are collinear, four of them make vertices of a convex quadrilateral.

   (b) If we have four convex sets in the plane such that every three of them have at least one point in common, then prove that the intersection of all of them is nonempty.

2. Inside a convex quadrilateral $ABCD$, find a point O such that the sum of lengths $|OA| + |OB| + |OC| + |OD|$ is the smallest.

3. Prove that in every convex pentagon, the sum of the lengths of all five diagonals is greater than the perimeter but is less than twice the perimeter.

4. Let P be a point in the plane of triangle ABC such that the segments PA, PB, and PC are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to PA. Prove that $\angle BAC$ is acute.

5. Consider a point O on the hypotenuse BC of a right triangle ABC. Draw perpendiculars OE and OF to AB and AC, respectively. Determine the position of O for which the length $|EF|$ is minimum.

6. Let $A_1A_2A_3$ be a triangle and let ω_1 be a circle in its plane passing through A_1 and A_2. Suppose there exist circles $\omega_2, \omega_3, \ldots, \omega_7$ such that for $k = 2, 3, \ldots, 7, \omega_k$ is externally tangent to ω_{k-1} and passes through A_k and A_{k+1}, where $A_{n+3} = A_n$ for all $n \geq 1$. Prove that $\omega_7 = \omega_1$.

7. There are a few islands in a 100-meter wide river, whose perimeters add up to 800 meters. The banks of the river are parallel lines. Gabriel claims that wherever he starts, he is able to cross the river by rowing at most 300 meters. Should one believe this?