Homework IX

More On Graphs
Due Nov. 18th

As usual, think about all problems, solve at least two of the problems. Your answers have to be justified and clear.

Recall the definitions we had for the last set of problems with this new definition:

- A *tree* is a connected graph with the minimum number of edges, i.e. removing any edge turns it into a disconnected graph. Recall that we showed in class that a tree has to be simple and it cannot have a *circuit*: a closed path.

1. Prove the converse of what we showed in class: if a connected graph doesn't have any circuit then it is a tree.
2. Show that every tree has at least one vertex of degree one. These are called *leaves* of the tree. (*Hint:* You can show that with induction. An easier way is to consider the longest path in the tree and show that the initial and terminal points of that path have degree one.)
3. Do the 6th problem in the last set of homework again. In our new terms, show that a tree with n vertices has $n-1$ edges.
4. For each statement state whether it is true or false. If it is false give a precise counter example. Note that when we remove an edge we keep everything else but when we remove a vertex we remove all the adjacent edges as well.
   (a) In a tree removing any vertex makes it disconnected.
   (b) In a tree removing any edge makes it disconnected.
   (c) If removing any edge from a graph disconnects it then it is a tree.
   (d) Every connected graph with n vertices and $n-1$ edges is a tree.
5. Prove that every tree with maximum degree k has at least k leaves.
6. Given natural numbers $d_1, d_2, \cdots, d_n$, with $d_1 + d_2 + \cdots + d_n = 2n-2$ can you construct a tree such that degrees of its vertices are $d_1, d_2, \cdots, d_n$.
7. Prove that we can color each vertex of a tree either black or white in a way that any two adjacent vertices have different colors.