Homework VIII

Graphs

Due Nov. 11th

As usual, think about all problems, solve at least two of the problems. Your answers have to be justified and clear. The *extra problem* will be considered an extra credit.

For the following problems, you need to know these definitions:

- A *graph* consists of a finite set of points called *vertices* and a finite number of arcs called *edges* joining some of the vertices.
- The number of edges attached to each vertex is called the *degree* of that vertex.
- A graph is called *connected* if you can connect any two vertices by a sequence of edges.
- A graph is called *simple* if between two vertices we cannot have more than one edge and there is no edge connecting a vertex to itself (a *loop*).

1. In a graph, show that the sum of the degrees of all the vertices is twice the number of edges.
2. There are 30 students in a class. Can it happen that 9 of them have 3 friends each (in the class), eleven have 4 friends each, and ten have 5 friends each? (*Hint:* Consider the graph in which vertices represent the students and connect any two friends by an edge and use the previous problem.)
3. Can you draw 9 straight line segments in the plane, each of which intersecting exactly 3 others? (*Hint:* Assume you can, construct a graph by putting a vertex for each line and connecting two of them with an edge if the corresponding lines intersect and get a contradiction.)
4. In a league with two divisions of 11 teams each, is it possible to schedule a season with each team playing seven games in its division and four games against teams in the other division?
5. In "Homework II", we had the following problem:

   In a tournament n teams participate. Each two teams play exactly once and every game ends by a team winning and the other losing. Show that independent of the results, at the end, we can always put the teams in a row $P_1, P_2, \ldots, P_n$, such that P_1 has won P_2, P_2 has won P_3 and so on.

   Now consider a *directed graph* for this tournament, again use induction and see if you can do it. (A *directed graph* is a graph when we consider each edge with a direction, drawing it as an arrow pointing from one vertex to another.)
6. What is the minimum number of edges in a connected graph with n vertices? Prove your claim.

7. In a country on planet Markar, there are 15 towns, each of which is connected by a road to at least 7 otehrs. Prove that you can travel by road from any town to any other town (possibly through intermediate towns).

Extra problem. Try to prove the generalization of the last problem: If a simple graph has n vertices and each vertex has degree at least $n/2$, then the graph must be connected.