

Homework VI

Elementary Number Theory

Due Oct. 28th

As usual, think about all problems, solve the first problem and at least one other problem and write them carefully.

You need to know the following facts for this set of problems. In what follows, all the numbers are integers, by which we mean an element of the set $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

- We say that n divides m , or that m is divisible by n , if $m = nk$ for some integer k . In this case, we write $n|m$. n is also called a *divisor* of m . For example, $3|12$, $15|135$, $1|n$ and $n|n$ for all integers n .
- An integer $p \geq 2$ is called a *prime* if p and 1 are the only divisors of p . For example, 2, 11, 43 are primes while 35 is not, since both 5 and 7 divide 35. Note that 2 is the only even integer which is also prime.
- **Division Algorithm.** If a and b are arbitrary integers, $b > 0$, there are unique integers q , *quotient*, and r , *remainder*, such that

$$a = qb + r, \quad 0 \leq r < b.$$

q and r are uniquely determined by n and k . It is easy to see that n is divisible by k if and only if the remainder is zero.

- **Congruence.** Given integers n , m and k , we say that n is *congruent to m modulo k* if n and m have the same remainder when we divide them by k . In this case we write $n \equiv m \pmod{k}$. For example, $27 \equiv 0 \pmod{3}$, $17 \equiv 5 \pmod{4}$ and $12 \equiv 1002 \pmod{11}$.

Equivalently, $n \equiv m \pmod{k}$ if and only if k divides $n - m$. The relation \equiv has the following property:

If $n \equiv m \pmod{k}$ and $n' \equiv m' \pmod{k}$, then

$$n + n' \equiv m + m' \pmod{k}$$

and also

$$nn' \equiv mm' \pmod{k}.$$

1. Let $N = 43 \times 23 - 15 \times 17 + 32 \times 21$, determine:

- (a) the parity of N ;
- (b) the units digit of N ;
- (c) the remainder when N is divided by 7.

(Of course, you should do this without actually computing N and explain how you did it.)

2. In class, we saw different criteria for determining the remainder of an integer when it is divided by 3 or 7. The following algorithm determines the remainder of an integer n when it is divided by 11. Prove that this algorithm works:

Consider n in its decimal form. Alternatively put + and - in between digits of n from right to left and compute the result. n and the obtained number have the same remainder when divided by 11. For example, consider 238153 and compute $-2 + 3 - 8 + 1 - 5 + 3 = -8$. Remainder of -8 when divided by 11 is 3, since $-8 = (-1) \times 11 + 3$ (As you see the remainder for a negative number is defined just the same as a positive number), and it is not hard to see that remainder of 238153 when divided by 11 is the same: $238153 = 2165 \times 11 + 3$.

3. Prove that if two natural numbers have the same number of copies of each digit in their decimal representations, then they differ by a multiple of 9.
4. What is the rightmost digit of 4^{33} ? What about 7^{33} ? (**Hint:** Consider congruence modulo 10.)
5. Prove that for any integer n , $n(n + 1)(n + 3)$ is divisible by 6.
6. (a) If n is not divisible by 3 can $5n$ be divisible by 3?
(b) This time if n is not divisible by 4 can $10n$ be divisible by 4?
7. If p is a prime number $p > 3$ prove that $p^2 - 1$ is divisible by 24.