MAT305 PRACTICE MIDTERM SOLUTIONS
SECOND MIDTERM

1. Making the substitution $y' = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$, and substituting $e^x = \sum x^n/n!$, we get the following local expression for the differential equation in a neighborhood of 0:

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - a_n x^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

For this to be satisfied, the coefficients a_n must satisfy the following identity:

$$(n+1)a_{n+1} - a_n = \frac{1}{n!}$$

$$a_{n+1} = \frac{1}{(n+1)!} + \frac{a_n}{n+1}$$

Writing out the first few terms, we have $a_1 = 1 + a_0$, $a_2 = 1/2 + (1+a_0)/2 = (2+a_0)/2$, $a_3 = (3+a_0)/6$ etc. In general, we *guess* the general formula to be $a_n = (n+a_0)/n!$ (this can be proved by induction: if $a_n = (n+a_0)/n!$, then by the recursive identity

$$a_n + 1 = \frac{1}{(n+1)!} + \frac{n+a_0}{n!}\cdot\frac{1}{n+1} = \frac{1+n+a_0}{(n+1)!})$$

Using the ratio test:

$$\frac{a_{n+1}x^{n+1}}{a_n x^n} = x\cdot\frac{(n+1)+a_0}{(n+1)!}\cdot\frac{n!}{n+a_0}$$

$$= \left[x\cdot\frac{(n+1)+a_0}{n+a_0}\right]\cdot\frac{1}{n+1}$$

As $n\to\infty$, the bracketed factor tends to x, whereas $\frac{1}{n+1}\to 0$, thus the ratio of consecutive terms goes to 0 for all x, and the series converges for all x.

2. (i) The singular points correspond to the roods of the y'' term, or in this case, the roots of $1-x^2$, ± 1. Since all coefficients are polynomial, the singular points are regular if

$$\lim_{x\to\pm1}(x-\pm1)\frac{0}{1-x^2}$$

and

$$\lim_{x\to\pm1}(x-\pm1)^2\frac{6}{1-x^2}$$

exist. Since this is the case for both $+1$ and -1, both singular points are regular.

(ii) Substituting $y'' = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$ we get the following local representation of the differential equation:

$$(1-x^2)\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + 6a_n x^n = 0$$

Or rather

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - n(n-1)x^n + 6a_nx^n = 0$$

From which we get the recursive identity:

$$(n+1)(n+2)a_{n+2} + (6 - n(n+1))a_n = 0$$
$$(n+1)(n+2)a_{n+2} = (n+3)(n-2)a_n$$

Thus the even coefficients depend only on a_0, while the odd coefficients depend only on a_1. Substituting $n = 2$, we see that $a_4 = 0$, and therefore all a_{2k}, $k > 1$ must vanish. In other words, $y_1 = a_0(1 - 1/3x^2)$, (as we see $a_2 = -1/3$).

Computing the first three odd identities, we have: $6a_3 = -4a_1$, $20a_5 = 6a_3$ and $30a_7 = 24a_5$, which we can then solve to see $a_3 = -2/3a_1$, $a_5 = -1/5a_1$ and $a_7 = -1/6a_1$. Thus $y_2 = a_1(x - 2/3x^3 - 1/5x^5 - 1/6x^7 + \cdots)$.

3. Making the substitution, we have

$$(r(r-1) - 3r + 4)x^r = 0 \implies \quad (r-2)^2x^r = 0$$

Therefore we have one solution $y_1 = x^2$.

Now, to use the method of reduction of order, we substitute $y_2(x) = v(x)y_1(x) = vx^2$. In this manner, the differential equation becomes:

$$x^2(v''x^2 + 4v'x + 2v) - 3x(v'x^2 + 2vx) + 4vx^2 = 0$$
$$v''x^4 + 4v'x^3 + 2vx^2 - 3v'x^2 - 5vx^2 + 4vx^2 = 0$$
$$v''x^4 + v'x^3 = 0$$

so that we have reduced to a first order differential equation that we can solve for v'.

$$\frac{v''}{v'} = \frac{-1}{x}$$
$$\ln(v') = -\ln(x)$$
$$v' = \frac{1}{x}$$
$$v = \ln(x)$$

So that our second solution is $y_2 = \ln(x) \cdot x^2$. Using L'Hopital's rule we see that both solutions approach 0 as $x \to 0$.

4. (i) The indicial equation $F(r) = r(r-1) + p_0r + q_0$ is, in this case, $= r(r-1) + r - v^2 = (r+v)(r-v)$, and so has exponents $\pm v$.

(ii),(iii) see pg. 299 in the book.

5. Taking the Laplace transform of the differential equation, we have:

$$[s^2\mathcal{L}(y) - sy(0) - y'(0)] - 4[s\mathcal{L}(y) - y(0)] + 3\mathcal{L}(y) = 0$$

regrouping yields:

$$(s^2 - 4s + 3)\mathcal{L}(y) = y(0)(s-4) + y'(0)$$

So that we have:

$$\mathcal{L}(y) = \frac{2}{s^2 - 4s + 3} = \frac{2}{(s-1)(s-3)}$$

By splitting this fraction up using the method of partial fractions, we have:

$$\mathcal{L}(y) = \frac{1}{s-3} - \frac{1}{s-1} = \mathcal{L}(e^{3t}) - \mathcal{L}(e^{t})$$

So that we can easily invert this Laplace transform, and we have $y = e^{3t} - e^{y}$.