Time allowed: 80 minutes. Name:

MAT 305 Calculus IV : Midterm 2

Calculators not permitted. All problems are of equal value; attempt all five.

1. Find the general solution to the first order ODE

$$y' - y = e^x$$

by substituting a series $y = \sum_{n=0}^{\infty} a_n x^n$ about $x_0 = 0$, finding the recurrence relation for a_n, and solving to find an expression for the general term a_n in terms of a_0. What is the radius of convergence of the solution?

2. (i) Find the singular points of the second order ODE

$$(1 - x^2)y'' + 6y = 0$$

and determine whether they are regular or irregular.
(ii) Substituting a series $y = \sum_{n=0}^{\infty} a_n x^n$ about the ordinary point $x_0 = 0$, show that one solution is a polynomial, and find the first four terms of the other solution.

3. (i) By substituting $y = x^r$ into the Euler equation

$$x^2y'' - 3xy' + 4y = 0, \qquad x > 0$$

find a value of r which gives a solution y_1.
(ii) Use reduction of order to find a second solution y_2.
(iii) Describe the behaviour of *both* solutions as $x \to 0$.

4. (i) Find the indicial equation and the exponents (at $x_0 = 0$) of Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad x > 0, \qquad \nu \in \mathbb{R}.$$

(ii) For Bessel's equation of order $\nu = 1/2$, substitute a series $\phi(x,r) = \sum_{n=0}^{\infty} a_n x^{n+r}$ about $x_0 = 0$, and find the recurrence relation.
(iii) Now find the general solution of Bessel's equation of order $\nu = 1/2$ by *either*

- finding a series for the larger exponent $r_1 = 1/2$, identifying the series in terms of elementary functions, and using reduction of order to find the second solution,
- *or* finding the series solution corresponding to the smaller exponent $r_2 = -1/2$ and showing that it breaks up into two linearly independent solutions.

5. Solve the IVP

$$y'' - 4y' + 3y = 0, \qquad y(0) = 0, \qquad y'(0) = 2,$$

by taking the Laplace transform, solving for $Y(s) = \mathcal{L}(y(t))$, and then inverting to find $y(t)$.