Time allowed: 2 hours 30 minutes. Name:

MAT 305 Calculus IV : Final Exam

Calculators not permitted. All problems are of equal value; attempt all eight.

1. (i) Show that the first order ODE

$$(2y - e^x) + xy' = 0$$

is *not* exact.
(ii) Find the equation that an integrating factor $\mu(x)$, which depends only on x, must satisfy to make the equation exact. Then solve for $\mu(x)$ and find an implicit solution y of the ODE.

2. (i) Find the value of r for which $y_1(x) = e^{rx}$ is a solution to the second order ODE

$$y'' - 6y' + 9y = 0.$$

(ii) Using $y_1(x)$ and reduction of order, find the general solution to the non-homogeneous ODE

$$y'' - 6y' + 9y = 2e^{3x}.$$

In other words, substitute a solution of the form $y(x) = v(x)y_1(x)$ and solve for $v(x)$.

3. Find the general solution to the second order non-homogeneous ODE

$$y'' + 4y = \frac{2}{\cos 2t}, \qquad 0 < t < \pi/4$$

by first solving the corresponding homogeneous ODE and then using variation of parameters.

4. Solve the second order ODE

$$y'' - xy' - 2y = 0$$

by substituting a series $y = \sum_{n=0}^{\infty} a_n x^n$, and finding two linearly independent series solutions (write the first four terms and the general term of each series). Use the ratio test to determine the radius of convergence of the two series.

NOTE: This will be a question about Euler's equation.

5. (i) Show that the second order ODE

$$4x^2y'' + (1 - 4x)y = 0$$

has a regular singular point at $x_0 = 0$.
(ii) By substituting a series $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, or otherwise, find the indicial equation for r and the exponents r_1 and r_2.
(iii) Find the series solution which corresponds to the larger exponent r_1 (write the first four terms and the general term).

6. Using the Laplace transform, solve the IVP

$$y'' + 2y' + 5y = \delta(t-4), \qquad y(0) = 0, \qquad y'(0)$$

where δ is the Dirac delta function.

7. (i) Consider the function $f(x) = L - x$, $0 < x < L$. Describe how to extend f to *both* an even and an odd function with period $2L$. Sketch three periods of each function.
(ii) Find the Fourier sine series of the odd extension of $f(x)$.

8. (i) Consider the heat equation

$$u_t = u_{xx}, \qquad 0 < x < 10, \qquad t > 0$$

with initial and boundary conditions

$$u(x, 0) = f(x) = 30, \qquad 0 < x < 10,$$

$$u(0, t) = u(10, t) = 0, \qquad t > 0.$$

Show that separation of variables, $u(x, t) = X(x)T(t)$, leads to a second order ODE for $X(x)$ and a first order ODE for $T(t)$, and write explicitly these ODEs.
(ii) Given that the fundamental solutions are

$$u_n(x, t) = \exp(-n^2\pi^2 t/100) \sin(n\pi x/10), \qquad n = 1, 2, \ldots$$

find the solution $u(x, t)$ to the heat equation, as a series.